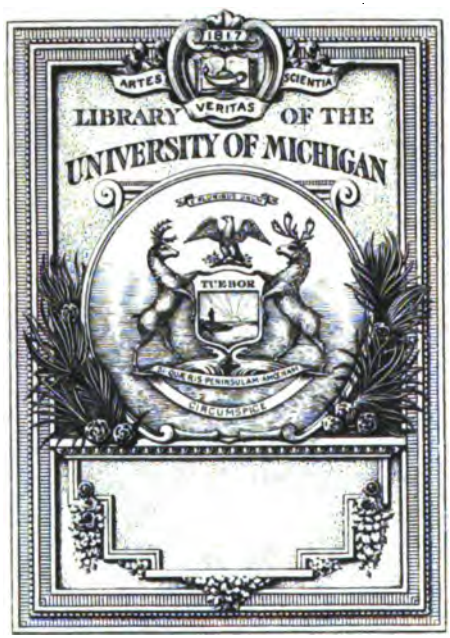
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DE PROGRESSIONIBVS ARCVVM CIRCVLARIVM, QVORVM TANGENTES SECVNDVM DATAM LEGEM PROCEDVNT.

DISQVISITIONES ANALYTICAE

MAXIME

AD

CALCVLVM INTEGRALEM ET DOCTRINAM SERIERVM

PERTINENTES.

AVCTORE

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VOLVMEN I.

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DE PROGRESSIONIBVS ARCVVM CIRCVLARIVM, QVORVM TANGENTES SECVNDVM DATAM LEGEM PROCEDVNT.

§. I. Plerarumque ferierum, quarum fummae hucusque inueftigatae funt, termini generales a quantitatibus tranfcendentibus liberi funt, quamquam fummae ipfae eiusmodi quantitatibus affectae reperiantur. Eae quidem feries, quae fecundum finus cofinusue angulorum multiplorum progrediuntur, ad vfitatum genus referri poffunt, quippe conftat, ex cognita fumma feriei $a + bx + cx^2 + dx^3 +$ etc. fponte confequi fummas ferierum: $a + bx \sin. \phi + cx^2 \sin. 2\phi + dx^3 \sin. 3\phi +$ etc. et $a + bx \cos. \phi + cx^2 \cos. 2\phi + dx^3 \cos. 3\phi +$ etc. Longe autem diuerfa eft ratio aliarum ferierum, quae alias quantitates tranfcendentes inuoluunt. Quarum fummatio grauioribus plerumque difficultatibus obuoluta vel nonnunquam vires analyfeos, quas hactenus quidem nacta eft, fuperare videtur.

§. II. Inter pauciffima eiusmodi ferierum, quoad terminum generalem tranfcendentium, fpecimina, memorabile extat illud, quod exhibuit L. EVLERVS in peculiari commentatione (*), qua feries arcuum circularium, quorum tangentes fecundum certam legem procedunt, contemplatus eft, easque fummandi methodum docuit, fimplicitate omnino confpicuam, at, ex ipfius inuentoris iudicio, indirectam, et ad cafus tantum faciliores reftrictam. Quid poft hunc laborem, cui, quantum equidem fciam, deinceps a Geometris nihil amplius additum fuit, in fummatione iftarum ferierum praeftandum reftet, vix clarius apparebit, ac fi ipfa viri fummi verba adiicere liceat, quibus commentatio laudata incipit. Quae ita fe habent: "Infinitas huiusmodi progreffiones exhiberi poffe, vel ex his exemplis liquet, quae olim propofui (**), fcilicet denotante π arcum duos angulos rectos metientem inueni, effe $\frac{\pi}{4} =$ A. tang. $\frac{1}{2} +$ A. tang. $\frac{1}{8} +$ A tang. $\frac{1}{18}$ + A.

(*) De progreffionibus arcuum circularium, quorum tangentes fecundum certam legem procedunt. *Nou. Commentar. Acad. Scient. Imper. Petropol. T. IX. (Petropoli 1764. 4.) pag. 40-52.*

(**) Primo harum ferierum mentionem fecit EVLERVS in Comment. De variis modis circuli quadraturam numeris proxime exprimendi. *Commentar. Vet. T. IX. p. 234.*

$+$ A. tang. $\frac{1}{32}$ $+$ A. tang. $\frac{1}{50}$ $+$ etc. quae feries arcuum in infinitum progreditur, tangente cuiusque indefinite exiftente $= \frac{1}{2xx}$, fimili modo eft $\frac{\pi}{4} =$ A. tang. $\frac{1}{3}$ $+$ A. tang. $\frac{1}{7}$ $+$ A. tang. $\frac{1}{13}$ $+$ A. tang. $\frac{1}{21}$ $+$ A. tang. $\frac{1}{31}$ $+$ etc. hac arcuum ferie pariter in infinitum continuata, cuius quisque terminus indefinite eft A. tang. $\frac{1}{xx+x+1}$. Tales autem feries eo magis videntur omni attentione dignae, quod nulla adhuc conftet methodus, earum fummam a priori inueniendi, atque etiam ipfi arcus omnes inter fe fint incommenfurabiles. Quin etiam ne expectare quidem licet methodum, cuius ope in genere huiusmodi ferierum, quamcunque legem tangentes fequantur, fumma inueftigari queat; fed potius, nifi haec lex certis conditionibus fit adftricta, nullo modo eae ad fummam reuocari poffe videntur, quae quidem arcu circulari exprimatur. Quamobrem in hoc negotio alia via non patet, nifi vt a pofteriori huiusmodi feries inueftigemus, quarum deinceps contemplatio fortaffe viam quamdam directam patefaciet; hincque modum exponam facilem ad quotcunque huiusmodi feries perueniendi, qui cum fimpliciffimis principiis innixus, ad tam ardua perducat, omnino mereri videtur, vt diligentius euoluatur." Quae EVLERI effata anfam mihi praebuerunt, vt curatius inquirerem, quo pacto ea, quae in hoc problematum genere adhuc defiderantur, fuppleri queant: cumque ad methodum, iftas feries fummandi, peruenerim directam ac late patentem, eam in hac commentatione euoluere conftitui, hunc quippe laborem Analyftis haud ingratum fore exiftimans: cum penitiori cognitione fatis ampli ferierum generis, quod poft prima elementa ab EVLERO delibata incultum hactenus iacuit, doctrinae ferierum aliqua acceffio contingere videatur.

§. III. De ordine, quem in pertractando hoc argumento feruabo, haec praemonenda funt.

A. Sectio I. Primo *formulas generales* proponam, ex principiis mere trigonometricis et algebraicis deductas, easque ad quamuis tangentium legem, fummasque vel finitas vel infinitas patentes. Iam quoad adplicationem harum formularum duo ferierum genera difcernenda videntur.

B. Sectio II. *Primum genus* eas complectitur feries, quarum fummatio ex formulis praedictis elici poteft, quin aliorum theorematum vel analyfeos infinitorum auxilio opus fit, quarumque fumma per Arcum exprimitur, cuius tangens algebraice eft affignabilis: quas ideo feries duplici hoc refpectu *algebraice fummabiles* appellabo. Ad hoc genus pertinent cunctae feries ab EVLERO fummatae. Quae nimirum feries, vt paucis dicam, quo fumma commentationis fupra laudatae redeat, duplicis funt *fpeciei:* a) *primae fpeciei* exempla compluria exhibuit EVLERVS, quae deinceps hac vna fummatione generaliore complexus eft: A. tang. $\frac{1}{L+M+N}$

$+$ A.

$+ \text{A.tang.}\, \frac{1}{4L+2M+N} + \text{A. tang.}\, \frac{1}{9L+3M+N} \ldots + \text{A. tang.}\, \frac{1}{Lx^2+Mx+N}$ $+ \text{etc.} = \text{A. tang.}\, \frac{2}{L+M}$, supposito $4LN = M^2 - L + 4$. b) *Alterius speciei* quatuor tantum exempla proposuit, haec nimirum: $\frac{\pi}{8} = \text{A. tang.}\, \frac{1}{2} - \text{A. tang.}\, \frac{1}{12} + \text{A. tang.}\, \frac{1}{70} - \text{A. tang.}\, \frac{1}{408} + \text{A. tang.}\, \frac{1}{2378} - \text{etc.}$ denominatoribus 2, 12, 70 etc. seriem recurrentem scalae bimembris 6, — 1 constituentibus, quippe $70 = 6 \cdot 12 - 2$; $408 = 6 \cdot 70 - 12$; $2378 = 6 \cdot 408 - 70$ etc. porro: $\frac{\pi}{12} = \text{A. tang.}\, \frac{1}{4} + \text{A. tang.}\, \frac{1}{64} + \text{A. tang.}\, \frac{1}{900} + \text{A. tang.}\, \frac{1}{12544} + \text{etc.}$ vbi denominatores sunt quadrata, quorum radices hanc progressionem constituunt:

1	2	3	4	5	. . . .	x
2	8	30	112	418	. .	$\frac{(2+\sqrt{3})^x - (2-\sqrt{3})^x}{\sqrt{3}}$

Reliquas binas series hoc loco omitto, cum pro illis legem tangentium haud expresserit EVLERVS: quae deinceps euoluetur. Cuius iam vtriusque speciei series generi nostro primo subsunt, nec tamen illud absoluunt. Ita quoad primam speciem (Cap. I.), vt vnum proferam exemplum, summationi ab EVLERO commemoratae (§.II.): $\frac{\pi}{4} = \text{A. tang.}\, \frac{1}{2} + \text{A. tang.}\, \frac{1}{8} \ldots + \text{A. tang.}\, \frac{1}{2xx} + \text{etc.}$ haec generalior (nec sub formula a. comprehensa) supponi potest: $\frac{(2r-1)\pi}{4} = \text{A. tang.}\, \frac{r^2}{2} + \text{A. tang.}\, \frac{r^2}{8} + \text{A. tang.}\, \frac{r^2}{18} \ldots + \text{A. tang.}\, \frac{r^2}{2xx} + \text{etc.}$ denotante r quemcunque numerum integrum. Alteram speciem amplius euoluendam duxi (Cap. II.), cum pro ea EVLERVS formulas generales haud exhibuerit, quas ex ipsius methodo, adhibitis fractionibus continuis, elicere difficilius videtur. Quae disquisitio perducit ad theoremata, ex forma simplici vna cum latiore ambitu aliquam commendationem habentia. Ita, vt in binis exemplis iam commemoratis (b) subsistam, primum ad hanc formam reuocatur: $\text{A. tang.}\, \frac{1}{A} - \text{A. tang.}\, \frac{1}{A(A^2+2)} + \text{A. tang.}\, \frac{1}{A(A^2+2)^2 - A} - \text{etc.} \pm \text{A. tang.}\, \frac{1}{z} \mp \text{etc.} = \frac{1}{2}\text{A. tang.}\, \frac{2}{A}$, denominatoribus in serie recurrente scalae A^2+2, -1 progredientibus; alterum ad hanc formam: $\text{A. tang.}\, \frac{1}{A} + \text{A. tang.}\, \frac{1}{B} + \text{etc.} + \text{A. tang.}\, \frac{1}{z} + \text{etc.} = \frac{1}{2}\text{A. sin.}\, \frac{2}{A}$; vbi denominatores sunt quadrata, quorum radices formant seriem termini generalis ζ (poſito $z = \zeta\zeta$) $= \dfrac{\left(\frac{A}{2} + \surd\left(\frac{A^2}{4} - 1\right)\right)^x - \left(\frac{A}{2} - \surd\left(\frac{A^2}{4} - 1\right)\right)^x}{\surd\left(A - \frac{4}{A}\right)}$;

ſiue, quod planius videtur, ſeriem recurrentem ſcalae $A, -1$; poſito $B = A^3$. Quae tamen ipſae formae alio reſpectu exhibent caſus tantum particulares ſummationum generaliorum.

C. Sect. III. Praeter hasce ſeries, quae ad Trigonometriam et Algebram vulgarem pertinere videntur, aliud ſupereſt ſerierum *genus*, quarum ſummatio altioris eſt indaginis, ac, niſi adhibitis theorematibus ex Trigonometria ſublimiore vel Analyſi infinitorum, abſolui nequit: quarumque ſumma exprimitur Arcu, cuius ipſa tangens quantitates tranſcendentes inuoluit; quae ideo ſeries duplici ratione *tranſcendenter ſummabiles* vocari poſſunt. Cuius generis quod hactenus ne ſpecimen quidem exhibitum fuerit, eo magis eſt, cur forte mirari poſſis, cum duorum problematum generalium ſolutio in poteſtate ſit. Quodſi nimirum fuerit $\frac{p}{q}$ functio quaecunque fracta par, vel x^{ti} numeri naturalis (x), vel x^{ti} imparis $(2x - 1)$, ſummabilis eſt ſeries infinita: $A.\,tang.\,\frac{a}{b} + A.\,tang.\,\frac{c}{d} + A.\,tang.\,\frac{e}{f} + \ldots + A.\,tang.\,\frac{p}{q} +$ etc. Nec minus ſummari poteſt ſeries ſignis alternantibus inſtructa: $A.\,tang.\,\frac{a}{b} - A.\,tang.\,\frac{c}{d} + A.\,tang.\,\frac{e}{f} \ldots \pm A.\,tang.\,\frac{p}{q} \mp$ etc. denotante $\frac{p}{q}$ functionem quamcunque fractam, modo ſit impar, x^{ti} numeri imparis (*). Quorum problematum eos praeſertim caſus euoluam, cum ſit $\frac{p}{q}$ pro ſignis ſeriei aequalibus $= \frac{a}{x^{2m} + b}$ vel $= \frac{a}{(2x - 1)^{2m} + b}$, pro ſignis inaequalibus $= \frac{a}{(2x - 1)^{2m} - 1}$, quorſum referendae ſunt v. c. hae binae ſeries:

$$A.\,tang.\,\frac{a}{1^4 + b} + A.\,tang.\,\frac{a}{2^4 + b} + A.\,tang.\,\frac{a}{3^4 + b} + \text{etc.}$$

et

(*) Denominationes functionis *paris* et *imparis* adhibuit EVLERVS (Introductio in Analyſin infinitorum, T. I. Lauſannae 1748. p. 12.) Functio par variabilis x ſiue integra ſiue fracta pares tantum ipſius x poteſtates inuoluit, functio impar integra impares tantum; at functio *fracta impar* eſt, vel functio integra par diuiſa per imparem, vel impar per parem. Poſito enim pro x, — x, functio par eundem valorem ſeruet, impar oppoſitum recipiat neceſſe eſt.

et A. tang. $\frac{a}{1^3}$ — A. tang. $\frac{a}{3^3}$ + A. tang. $\frac{a}{5^3}$ — A. tang. $\frac{a}{7^3}$ + etc.

Nec obferuatione indignum videtur, quod, denotante $\frac{p}{q}$ quamcunque functionem fractam variabilis x, A. tang. $\frac{p}{q}$ femper refolui queat in tot Arcus, quorum tangentes funt fractiones fimplices, veluti A. tang. $\frac{\alpha}{x+\beta}$ + A. tang. $\frac{\gamma}{x+\delta}$ + A. tang. $\frac{\varepsilon}{x+\zeta}$ + etc. ad quot gradus affurgit denominator q: quae quidem refolutio femper realis eft, fecus ac in fimili ipfarum functionum refolutione accidere conftat.

A. SECTIO PRIMA.

Formulae generales.

PROBLEMA I.

§. IV. *Propofita quacunque ferie quantitatum* t^{I}, t^{II}, t^{III} . . . t^{X} *inuenire expreffionem fummae Arcuum, quorum tangentes illis quantitatibus aequantur, vel* S. A. tang. t^{X} = A. tang. t^{I} + A. tang. t^{II} + . . . + A. tang. t^{X}.

Solutio.

1) Binorum Arcuum fumma ex regula trigonometrica eft = A. tang. $\frac{t^{I}+t^{II}}{1-t^{I}t^{II}}$.

Quibus iam in vnum Arcum collectis, fi tertius veluti alter additur, prodit fumma trium Arcuum = A. tang.

$$\frac{\frac{t^{I}+t^{II}}{1-t^{I}t^{II}}+t^{III}}{1-\frac{(t^{I}+t^{II})\,t^{III}}{1-t^{I}t^{II}}}$$

$$= \text{A. tang. } \frac{t^{I}+t^{II}+t^{III}-t^{I}t^{II}t^{III}}{1-t^{I}t^{II}-t^{I}t^{III}-t^{II}t^{III}}$$

Eadem ratione reperitur fumma quatuor arcuum, partibus rite ordinatis,

$$= \text{A. tang. } \frac{t^{I}+t^{II}+t^{III}+t^{IV}-t^{I}t^{II}t^{III}-t^{I}t^{II}t^{IV}-t^{I}t^{III}t^{IV}-t^{II}t^{III}t^{IV}}{1-t^{I}t^{II}-t^{I}t^{III}-t^{I}t^{IV}-t^{II}t^{III}-t^{II}t^{IV}-t^{III}t^{IV}+t^{I}t^{II}t^{III}t^{IV}}.$$

2) Ex hisce iam cafibus lex generalis haud obfcure fe prodit. Defignentur fummae Vnionum, Binionum, Ternionum etc. ex quantitatibus t^{I}, t^{II}, t^{III}, t^{IV} . . . t^{X} conflatarum, literis A, B, C etc. erit A. tang. t^{I} + A. tang. t^{II} + A. tang. t^{III} + . . + A. tang. t^{X} = A.

$= \text{A. tang.} \frac{A - C + E - G + \text{etc.}}{1 - B + D - F + H - \text{etc.}}$ In numeratore occurrunt combinationes secundum numeros impares, seu Con^{2r-1}tiones, in denominatore praeter vnitatem combinationes secundum numeros pares. Signa vtrinque alternantur.

3) Cuius legis vt demonstratio vniuersalis condatur, supponatur ea obtinere pro x Arcubus. Accedente x + 1^to^ Arcu erit

$$\text{A. tang. } t^{I} + \text{A. tang. } t^{II} + \ldots + \text{A. tang. } t^{x} + \text{A. tang. } t^{x+1}$$

$$= \text{A. tang.} \frac{\frac{A - C + E - G \ldots}{1 - B + D - F \ldots} + t^{x+1}}{1 - \frac{(A - C + E \ldots)}{1 - B + D \ldots} t^{x+1}}$$

$$= \text{A. tang.} \frac{A + t^{x+1} - C - Bt^{x+1} + E + Dt^{x+1} - G - Ft^{x+1} + \ldots}{1 - B - At^{x+1} + D + Ct^{x+1} - F - Et^{x+1} + \ldots}$$

Iam vulgo constat, quantitatibus combinandis accedente noua, accedere etiam combinationes nouas cuiusuis classis eas, quae oriuntur, dum noua quantitas iungitur combinationibus reliquarum quantitatum classis proxime inferioris. Inde, acceptis A^{I}, B^{I} C^{I} etc. eodem sensu respectu quantitatum $t^{I}, t^{II}, \ldots t^{x+1}$, ac A, B, C etc. quoad $t^{I}, t^{II}, \ldots t^{x}$ (2), sequentes habentur aequationes:

$$A + t^{x+1} = A^{I};\ B + t^{x+1} A = B^{I};\ C + t^{x+1} B = C^{I};\ D + t^{x+1} C = D^{I};\ \text{etc.}$$

Quibus adhibitis oritur summa Arcuum numero x + 1, $= \text{A. tang.} \frac{A^{I} - C^{I} + E^{I} - G^{I} + \text{etc.}}{1 - B^{I} + D^{I} - F^{I} + H^{I} - \text{etc.}}$

Quare si lex (2) obseruata, pro x Arcubus obtinet, obtinebit eadem pro x + 1 Arcubus: vnde eam vniuersalem esse, exempla (1) probant.

Hypothesis.

§. V. *Proposita serie quantitatum* $\alpha^{I}, \alpha^{II}, \alpha^{III}, \alpha^{IV}, \ldots \alpha^{x}$ *producta ex illarum binis, tribus* ... x *conflata designentur per* $P\alpha^{II}, P\alpha^{III} \ldots P\alpha^{x}$, *vt sit* $P\alpha^{x} = \alpha^{I} \alpha^{II} \alpha^{III} \ldots \alpha^{x}$.

Sicuti nimirum summa harum quantitatum exprimitur per $S\alpha^{x}$, i. e. praefixo signo summae termino generali vel vltimo, ita haud incommode productum exprimi videtur, praefixo signo producti eidem termino, qui iam *factorem generalem*, vti illic partem, refert.

Ex hac notatione sponte consequitur, esse $P(\alpha^{x}\beta^{x}) = P\alpha^{x} \cdot P\beta^{x}$, et $P\left(\frac{\alpha^{x}}{\beta^{x}}\right) = \frac{P\alpha^{x}}{P\beta^{x}}$, assumta alia serie, cuius terminus generalis est β^{x}.

THEO-

THEOREMA I.

§. VI. *Summa seriei:* A. tang. t^{I} + A. tang. t^{II} + ... + A. tang. t^{X} est

$$= \text{A. tang. } \sqrt{-1} \left\{ \frac{1 - \frac{(1+t^{I}\sqrt{-1})(1+t^{II}\sqrt{-1})\ldots(1+t^{X}\sqrt{-1})}{(1-t^{I}\sqrt{-1})(1-t^{II}\sqrt{-1})\ldots(1-t^{X}\sqrt{-1})}}{1 + \frac{(1+t^{I}\sqrt{-1})(1+t^{II}\sqrt{-1})\ldots(1+t^{X}\sqrt{-1})}{(1-t^{I}\sqrt{-1})(1-t^{II}\sqrt{-1})\ldots(1-t^{X}\sqrt{-1})}} \right\}$$

vel ex signo (§. V.)

$$\text{S. A. tang. } t^{X} = \text{A. tang. } \sqrt{-1} \left\{ \frac{1 - P\left(\frac{1+t^{X}\sqrt{-1}}{1-t^{X}\sqrt{-1}}\right)}{1 + P\left(\frac{1+t^{X}\sqrt{-1}}{1-t^{X}\sqrt{-1}}\right)} \right\}.$$

Demonstratio.

1) Seruatis literis, quae §. IV. adhibitae sunt, ex theoria aequationum constat esse:

$$\left(1-\frac{t^{I}}{z}\right)\left(1-\frac{t^{II}}{z}\right)\left(1-\frac{t^{III}}{z}\right)\cdots\left(1-\frac{t^{X}}{z}\right) = 1 - \frac{A}{z} + \frac{B}{z^2} - \frac{C}{z^3} + \text{etc.}$$

et posito + z pro — z,

$$\left(1+\frac{t^{I}}{z}\right)\left(1+\frac{t^{II}}{z}\right)\left(1+\frac{t^{III}}{z}\right)\cdots\left(1+\frac{t^{X}}{z}\right) = 1 + \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \text{etc.}$$

2) Quibus aequationibus inuicem additis prouenit

$$2\left(1 + \frac{B}{z^2} + \frac{D}{z^4} + \frac{F}{z^6} + \text{etc.}\right)$$

$$= \left(1-\frac{t^{I}}{z}\right)\left(1-\frac{t^{II}}{z}\right)\ldots\left(1-\frac{t^{X}}{z}\right) + \left(1+\frac{t^{I}}{z}\right)\left(1+\frac{t^{II}}{z}\right)\ldots\left(1+\frac{t^{X}}{z}\right)$$

Subtractio praebet:

$$2\left(\frac{A}{z} + \frac{C}{z^3} + \frac{E}{z^5} + \frac{G}{z^7} + \text{etc.}\right)$$

$$= \left(1+\frac{t^{I}}{z}\right)\left(1+\frac{t^{II}}{z}\right)\ldots\left(1+\frac{t^{X}}{z}\right) - \left(1-\frac{t^{I}}{z}\right)\left(1-\frac{t^{II}}{z}\right)\ldots\left(1-\frac{t^{X}}{z}\right)$$

3) Ponatur iam $z^2 = -1$, vel $z = \sqrt{-1}$, et erit $2(1 - B + D - F + \text{etc.})$ $= (1+t^{I}\sqrt{-1})(1+t^{II}\sqrt{-1}) \ldots (1+t^{X}\sqrt{-1}) + (1-t^{I}\sqrt{-1})$. $(1-t^{II}\sqrt{-1}) \ldots (1-t^{X}\sqrt{-1})$, et $\frac{2}{\sqrt{-1}} \cdot (A - C + E - G + \text{etc.}) =$ $(1-t^{I}\sqrt{-1})(1-t^{II}\sqrt{-1}) \ldots (1-t^{X}\sqrt{-1}) - (1+t^{I}\sqrt{-1})$ $(1+t^{II}\sqrt{-1}) \ldots (1+t^{X}\sqrt{-1})$. Quibus valoribus suppositis in expressione problematis praecedentis:

S. A.

$$S.\ A.\ tang.\ t^{X} = A.\ tang.\ \frac{A - C + E - etc.}{1 - B + D - F\ etc.}$$

prodit formula ipsa demonstranda.

Corollarium 1.

§. VII. Cum quaeuis quantitas imaginaria ad formam $M + N\sqrt{-1}$ reuocari queat, productum factorum imaginariorum $(1 + t^{I}\sqrt{-1})(1 + t^{II}\sqrt{-1}) \ldots (1 + t^{X}\sqrt{-1})$, eadem forma exhibeatur: quo posito erit, $\sqrt{-1}$ abeunte in $-\sqrt{-1}$, $(1 - t^{I}\sqrt{-1})(1 - t^{II}\sqrt{-1}) \ldots (1 - t^{X}\sqrt{-1}) = M - N\sqrt{-1}$. Quare aequabitur productum $P\,\frac{1 + t^{X}\sqrt{-1}}{1 - t^{X}\sqrt{-1}} = \frac{M + N\sqrt{-1}}{M - N\sqrt{-1}}$. Iam summa seriei $A.\ tang.\ t^{I} + A.\ tang.\ t^{II} + \ldots + A.\ tang.\ t^{X}$ erit

$$= A.\ tang.\ \sqrt{-1}\left\{\frac{1 - \frac{M - N\sqrt{-1}}{M - N\sqrt{-1}}}{1 + \frac{M + N\sqrt{-1}}{M - N\sqrt{-1}}}\right\}$$

$$= A.\ tang.\ \sqrt{-1} \cdot - \frac{2N\sqrt{-1}}{2M} = A.\ tang.\ \frac{N}{M}.$$ Inde haec oritur:

Regula.

Exprimatur productum $P(1 + t^{X}\sqrt{-1})$ *per* $M + N\sqrt{-1}$, *vel* $P\left(\frac{1 + t^{X}\sqrt{-1}}{1 - t^{X}\sqrt{-1}}\right)$ *per* $\frac{M + N\sqrt{-1}}{M - N\sqrt{-1}}$, *eritque summa seriei* $A.\ tang.\ t^{I} + A.\ tang.\ t^{II} + \ldots + A.\ tang.\ t^{X} = A.\ tang.\ \frac{N}{M}$.

Corollarium 2.

Si productum indefinitum $P(1 + t^{X}\sqrt{-1})$ per quantitatem exprimitur, quae ipsa ex binis pluribusue factoribus imaginariis conflata est, veluti cum fuerit $P(1 + t^{X}\sqrt{-1}) = (M^{I} + N^{I}\sqrt{-1})(M^{II} + N^{II}\sqrt{-1}) \ldots$, vel $P\left(\frac{1 + t^{X}\sqrt{-1}}{1 - t^{X}\sqrt{-1}}\right) = \frac{(M^{I} + N^{I}\sqrt{-1})(M^{II} + N^{II}\sqrt{-1}) \ldots}{(M^{I} - N^{I}\sqrt{-1})(M^{II} - N^{II}\sqrt{-1}) \ldots}$, tum ipsa summa Arcuum composita erit ex $A.\ tang.\ \frac{N^{I}}{M^{I}} + A.\ tang.\ \frac{N^{II}}{M^{II}} +$ etc. Ex theoremate 1, nimirum sponte consequitur, esse

A. tang.

$$\text{A. tang.}\ \sqrt{-1}\left\{\frac{1-\dfrac{(M^{I}+N^{I}\sqrt{-1})(M^{II}+N^{II}\sqrt{-1})\ldots}{(M^{I}-N^{I}\sqrt{-1})(M^{II}-N^{II}\sqrt{-1})\ldots}}{1+\dfrac{(M^{I}+N^{I}\sqrt{-1})(M^{II}+N^{II}\sqrt{-1})\ldots}{(M^{I}-N^{I}\sqrt{-1})(M^{II}-N^{II}\sqrt{-1})\ldots}}\right\}$$

i. e. summam arcuum, $= \text{A. tang.}\ \frac{N^{I}}{M^{I}} + \text{A. tang.}\ \frac{N^{II}}{M^{II}} + \text{etc.}$

Corollarium 3.

§. IX. Ob $(M - N\sqrt{-1})(M + N\sqrt{-1}) = M^2 + N^2$, est $\frac{M + N\sqrt{-1}}{M - N\sqrt{-1}} = \frac{(M + N\sqrt{-1})^2}{M^2 + N^2} = \frac{M^2 - N^2 + 2MN\sqrt{-1}}{M^2 + N^2}$, quod sit $= \mathfrak{M} + \mathfrak{N}\sqrt{-1}$. Quare si fuerit $P\left(\frac{1 + t^{X}\sqrt{-1}}{1 - t^{X}\sqrt{-1}}\right) = \mathfrak{M} + \mathfrak{N}\sqrt{-1}$, erit (§. VI.) S. A. tang. $a^{X} =$

$$\text{A. tang.}\ \sqrt{-1}\left(\frac{1-\mathfrak{M}-\mathfrak{N}\sqrt{-1}}{1+\mathfrak{M}+\mathfrak{N}\sqrt{-1}}\right) = \text{A. tang.}\ \frac{(1-\mathfrak{M})\left(\sqrt{-1}+\dfrac{\mathfrak{N}}{1-\mathfrak{M}}\right)}{\left(\dfrac{1+\mathfrak{M}}{\mathfrak{N}}+\sqrt{-1}\right)}$$

$= \text{A. tang.}\ \frac{1-\mathfrak{M}}{\mathfrak{N}} = \text{A. tang.}\ \sqrt{\left(\frac{1-\mathfrak{M}}{1+\mathfrak{M}}\right)}$, ob $1 - \mathfrak{M}^2 = \mathfrak{N}^2$, et $\frac{\mathfrak{N}}{1-\mathfrak{M}} = \frac{1+\mathfrak{M}}{\mathfrak{N}}$. Iam sit $\mathfrak{M} = \text{cos. m}$, erit $\sqrt{\left(\frac{1-\mathfrak{M}}{1+\mathfrak{M}}\right)} = \text{tang.}\ \frac{1}{2}\text{m}$, hinc summa $= \text{A. tang. tang.}\ \frac{1}{2}\text{m} = \frac{1}{2}\text{m}$. Quare Regula Coroll. 1. etiam sic exprimi potest: Exhibeatur productum $P\left(\frac{1 + t^{X}\sqrt{-1}}{1 - t^{X}\sqrt{-1}}\right)$ per $\mathfrak{M} + \mathfrak{N}\sqrt{-1}$, et erit A. tang. t^{I} + A. tang. t^{II} + . . . + A. tang. $t^{X} = \frac{1}{2}$ A. cos. $\mathfrak{M}$.

Scholion 1.

§. X. Ad formulam Theorematis 1. et regulam inde emanantem (§. VII.) perueni, considerando expressionem Arcus, quam Analysis infinitorum suppeditat: sc. A. tang. $q = \frac{1}{2\sqrt{-1}}\ \text{log.}\left(\frac{1 + q\sqrt{-1}}{1 - q\sqrt{-1}}\right)$ (*). Satius tamen hoc loco duxi, cuncta ex formula elementari problematis 1, quae et ipsa vsu haud caret, deriuare. Ceterum vel absque expressione ista logarithmica, demonstratio synthetica theorematis 1. ex formulis trigonometricis imaginariis peti potest. Quam breuiter indicasse sufficiat. Sit nimirum

(*) cf. Illustr. KAESTNER *Anfangsgründe der Analysis des Unendlichen*, (2te Aufl. Göttingen 1770.) pag. 254.

rum $t^{I} = \text{tang.}\ \varphi^{I}, \ldots t^{X} = \text{tang.}\ \varphi^{X}$, erit $\frac{1 + t^{X}\sqrt{-1}}{1 - t^{X}\sqrt{-1}} = \frac{\text{cos}\ \varphi^{X} + \text{s.}\ \varphi^{X}\sqrt{-1}}{\text{cos}\ \varphi^{X} - \text{s.}\ \varphi^{X}\sqrt{-1}}$.
Quare $P\left(\frac{1 + t^{X}\sqrt{-1}}{1 - t^{X}\sqrt{-1}}\right) = \frac{\text{cos.}(\varphi^{I} + \varphi^{II} \ldots + \varphi^{X}) + \text{sin.}(\varphi^{I} + \varphi^{II} \ldots + \varphi^{X})\sqrt{-1}}{\text{cos.}(\varphi^{I} + \varphi^{II} \ldots + \varphi^{X}) - \text{sin.}(\varphi^{I} + \varphi^{II} \ldots + \varphi^{X})\sqrt{-1}}$
$= \Pi$ et $\text{A. tang.}\ \sqrt{-1}\left(\frac{1 - \Pi}{1 + \Pi}\right) = \text{A. tang.}\ \frac{\text{sin.}(\varphi^{I} + \varphi^{II} \ldots + \varphi^{X})}{\text{cos.}(\varphi^{I} + \varphi^{II} \ldots + \varphi^{X})} = \varphi^{I} + \varphi^{II} \ldots$
$+ \varphi^{X} = \text{A. tang.}\ t^{I} + \text{A. tang.}\ t^{II} \ldots + \text{A. tang.}\ t^{X}$.

Scholion 2.

§. XI. Conſtat, arcus indefinite multos α, $\alpha \pm \pi$, $\alpha \pm 2\pi$, $\alpha \pm 3\pi$, . . . $\alpha \pm r\pi$, . . . communem tangentem habere. Quare expreſſiones ſummae arcuum hic traditae, quibus S. A. tang. t^{X} ad arcum certae tangentis reducitur, a formulis ſummatoriis vſitatis in eo differunt, quod per illas ſumma haud omnimode determinetur. Ad tollendam ambiguitatem in ſignificatione ipſorum ſeriei ſummandae terminorum ſub A. tang. t^{X} intelligatur arcus minimus poſitiuus, cui tangens t^{X} competit, hinc ſi tangens negatiuum valorem habeat, pro A. tang. $-$ t accipiatur $\pi -$ A. tang. $+$ t, vel complementum arcus minimi, cui eadem tangens affirmatiue ſumta competit. Exinde tamen neutiquam conſequitur, quod arcus, qui ſummam exprimit, eodem ſemper ſenſu accipiendus ſit. Varie potius pro re nata diiudicandum eſt, quodnam ſemicircumferentiae multiplum arcui minimo tangentis in ſumma expreſſae adiiciendum ſit, vt vera arcuum ſumma prodeat. Quod ſi nimirum in formula (§. VII.) A. tang. $t^{I} + \ldots +$ A. tang. $t^{X} =$ A. tang. $\frac{N}{M}$, arcus minimus cuius tangens $= \frac{N}{M}$, ſit $=$ A, erit vera arcuum ſumma $= r\pi +$ A, vbi $r\pi$ eſt quaſi Conſtans, aliunde definienda; *quaſi Conſtans* inquam, cum vaga quodammodo ſit, nec, vti in Conſtantibus ex integratione oriundis ſit, ex vno variabilis x valore certo modo determinari queat. Cuius obſeruationis vis ex ſequentibus clarius percipietur.

B. Sectio secvnda.

Inueſtigatio ſerierum algebraice ſummabilium.

Problema II.

§. XII. *Inueſtigare formam generalem ſerierum:* A. tang. $t^{I} +$ A. tang. $t^{II} + \ldots +$ A. tang. t^{X} *algebraice ſummabilium.*

Solutio.

1) Cum ſummatio harum ſerierum ad determinationem *producti indefiniti* reuocata ſit (§. VI.), caſus ſummationis ſine dubio ſimpliciſſimus eſt is, quo factores producti

ſunt

ſunt fractiones, ita ſe excipientes, vt denominatores et numeratores in producto ſe mutuo tollant, et relinquantur tantum primus numerator ac vltimus denominator. Quod quidem accidit, cum fuerit $\frac{1+t^x\sqrt{-1}}{1-t^x\sqrt{-1}}=\frac{\varphi x}{\varphi(x+1)}$, denotante φx quamuis functionem ipſius x, quippe tum erit $P\left(\frac{1+t^x\sqrt{-1}}{1-t^x\sqrt{-1}}\right)=\frac{\varphi 1}{\varphi 2}\cdot\frac{\varphi 2}{\varphi 3}\cdot\frac{\varphi 3}{\varphi 4}\cdot\cdot\frac{\varphi(x-1)}{\varphi x}\cdot\frac{\varphi x}{\varphi(x+1)}$ $=\frac{\varphi 1}{\varphi(x+1)}$.

2) Quo iam aequatio $\frac{1+t^x\sqrt{-1}}{1-t^x\sqrt{-1}}=\frac{\varphi x}{\varphi(x+1)}$ locum habere poſſit, functioni aſſumtae φx forma imaginaria tribuenda eſt. Sit igitur $\varphi x = Fx + Gx\sqrt{-1}$, denotantibus Fx et Gx functiones reales: eritque

$$(1+t^x\sqrt{-1})(F(x+1)+G(x+1)\cdot\sqrt{-1})=(1-t^x\sqrt{-1})(Fx+Gx\sqrt{-1}),$$
$$\text{vel } F(x+1)-t^xG(x+1)+(G(x+1)+t^xF(x+1))\sqrt{-1}$$
$$=Fx+t^xGx+(Gx-t^xFx)\sqrt{-1}.$$

Vnde duplex oritur aequatio:

$$F(x+1)-t^xG(x+1)=Fx+t^xGx, \text{ et } G(x+1)+t^xF(x+1)=Gx-t^xFx.$$

Hinc fit $\frac{F(x+1)-Fx}{G(x+1)+Gx}=t^x=\frac{Gx-G(x+1)}{F(x+1)+Fx}$, et multiplicando $F(x+1)^2-Fx^2$ $=Gx^2-G(x+1)^2$, vel $F(x+1)^2+G(x+1)^2=Fx^2+Gx^2$.

3) Ex qua aequatione ſponte conſequitur, functiones Fx et Gx ita eſſe accipiendas, vt Fx^2+Gx^2 ſit = quantitati conſtanti. Ponatur igitur, ſimplicitatis gratia, quae vniuerſalitati non obeſt, $Fx^2+Gx^2=1$. Iam quo Fx et Gx formam rationalem induant, conſtat ponendum eſſe $Gx=\sqrt{(1-Fx^2)}=1-Fx\,.\,fx$, denotante fx aliam functionem indicis x: vnde $Fx=\frac{2fx}{fx^2+1}$, $Gx=\frac{fx^2-1}{fx^2+1}$. Tum erit $t^x=$

$$\frac{2f(x+1)}{f(x+1)^2+1}-\frac{2fx}{fx^2+1}:\frac{f(x+1)^2-1}{f(x+1)^2+1}+\frac{fx^2-1}{fx^2+1}$$
$$=\frac{2f(x+1)fx^2+2f(x+1)-2fxf(x+1)^2-2fx}{fx^2f(x+1)^2-fx^2+f(x+1)^2-1+fx^2f(x+1)^2+fx^2-f(x+1)^2-1}$$
$$=\frac{(fx-f(x+1))(fxf(x+1)-1)}{fx^2f(x+1)^2-1}=\frac{fx-f(x+1)}{1+fxf(x+1)}.$$

4) Hinc prodit $P\left(\frac{1+t^x\sqrt{-1}}{1-t^x\sqrt{-1}}\right)=P\frac{1+fxf(x+1)+fx\sqrt{-1}-f(x+1)\sqrt{-1}}{1+fxf(x+1)-fx\sqrt{-1}+f(x+1)\sqrt{-1}}$ $=P\frac{(1+fx\sqrt{-1})(1-f(x+1)\sqrt{-1})}{(1-fx\sqrt{-1})(1+f(x+1)\sqrt{-1})}$, ſiue ob factores ſe mutuo tollentes $=\frac{(1+f1\,.\sqrt{-1})(1-f(x+1)\sqrt{-1})}{(1-f1\sqrt{-1})(1+f(x+1)\sqrt{-1})}$. Vnde erit ſumma arcuum = A. tang. $\frac{N}{M}$ (§. VII.)

 =

$= \text{A. tang.}\ \frac{f1 - f(x+1)}{1 + f1 . f(x+1)}$. Quare haec tandem obtinetur summatio:

$$\text{A. tang. } t^{I} + \text{A. tang. } t^{II} + \ldots + \text{A. tang. } t^{x} = \text{A. tang. } \frac{f1 - f(x+1)}{1 + f1 . f(x+1)},$$ posito

$$t^{x} = \frac{fx - f(x+1)}{1 + fx\, f(x+1)}.$$

5) Casum hactenus euolutum haud vnicum esse, quo productum indefinitum hincque summam arcuum inuenire liceat, facile apparet. Assumi sc. potest hypothesis generalior: $\frac{1 + t^{x}\sqrt{-1}}{1 - t^{x}\sqrt{-1}} = \frac{\varphi x}{\varphi(x+r)}$, denotante r numerum integrum; tumque erit productum indefinitum $P\,\frac{1 + t^{x}\sqrt{-1}}{1 - t^{x}\sqrt{-1}} = \frac{\varphi 1}{\varphi(1+r)} \cdot \frac{\varphi 2}{\varphi(2+r)} \ldots \frac{\varphi r}{\varphi 2r} \cdot \frac{\varphi(1+r)}{\varphi(1+2r)} \ldots\ldots \frac{\varphi x}{\varphi(x+r)} = \frac{\varphi 1 . \varphi 2 \ldots \varphi r}{\varphi(x+1)\,\varphi(x+2) \ldots \varphi(x+r)}$, i. e. reuocatum ad productum definitum. (*) Iisdem omnino calculis et ratiociniis, quae in casu praecedente adhibita sunt, quaeque repetere superfluum est, prodit iam $t^{x} = \frac{fx - f(x+r)}{1 + fx . f(x+r)}$. At vero productum

$$P\left(\frac{1 + t^{x}\sqrt{-1}}{1 - t^{x}\sqrt{-1}}\right) = P\,\frac{(1 + fx\sqrt{-1})(1 - f(x+r)\sqrt{-1})}{(1 - fx\sqrt{-1})(1 + f(x+r)\sqrt{-1})}$$ erit

$$= P\left(\frac{1 + fx\sqrt{-1}}{1 - fx\sqrt{-1}} : \frac{1 + f(x+r)\sqrt{-1}}{1 - f(x+r)\sqrt{-1}}\right)$$

$$= \frac{(1 + f1\sqrt{-1})(1 - f(x+1)\sqrt{-1})(1 + f2\sqrt{-1})(1 - f(x+2)\sqrt{-1}) \ldots}{(1 - f1\sqrt{-1})(1 - f(x+1)\sqrt{-1})(1 - f2\sqrt{-1})(1 - f(x+2)\sqrt{-1}) \ldots} \cdot \frac{(1 + fr\sqrt{-1})(1 - f(x+r)\sqrt{-1})}{(1 - fr\sqrt{-1})(1 - f(x+r)\sqrt{-1})}.$$

Hinc, combinando binos factores, summa arcuum, S. A. tang. t^{x}, $= \text{A. tang. } \frac{N^{I}}{M^{I}} + \text{A. tang. } \frac{N^{II}}{M^{II}} + \text{etc.}$ (§. VII.) $= \text{A. tang. } \frac{f1 - f(x+1)}{1 + f1 . f(x+1)} + \text{A. tang. } \frac{f2 - f(x+2)}{1 + f2\, f(x+2)} + \ldots\ldots + \text{A. tang. } \frac{fr - f(x+r)}{1 + fr\, f(x+r)}$.

Scho-

(*) Productum *indefinitum* hoc loco vocatur, cuius factorum numerus est variabilis: *definitum*, cuius factorum, quanquam variabilium, numerus tamen certus constansque est. In summatione serierum ipsa summae expressio plerumque pluribus partibus constat, quarum tamen numerus definitus est, cum partium seriei numerus (x) sit variabilis: v. c. cum Sx^{n} exprimitur per $\alpha x^{n+1} + \beta x^{n} + \gamma x^{n-1} +$ etc. Pari ratione productum indefinitum haud raro tali expressione assignatur, quae ipsa refert productum pluribus factoribus constans.

Scholion 1.

§. XIII. Formula prima praecedentis §phi (4.) conuenit cum ea, a qua EVLERVS, ceu a principio eius generis fummationum exorfus eft: quanquam paullo aliter fit expreffa. Demonftratio fynthetica huius formulae fine negotio conficitur. Eft nimirum

$$\begin{array}{ll} \text{A. tang. } t^{I} = \text{A. tang. } \frac{f1 - f2}{1 + f1.f2} = & \text{A. tang. } f1 - \text{A. tang. } f2 \\ \text{A. tang. } t^{II} = & \text{A. tang. } f2 - \text{A. tang. } f3 \\ \text{A. tang. } t^{III} = & \text{A. tang. } f3 - \text{A. tang. } f4 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \text{A. tang. } t^{x-1} = & \text{A. tang. } f(x-1) - \text{A. tang. } fx \\ \text{A. tang. } t^{x} = & \text{A. tang. } fx - \text{A. tang. } f(x+1) \end{array}$$

Quare fingulos terminos addendo, omiffis partibus, quae fe mutuo deftruunt, prodit

$$\text{S. A. tang. } t^{x} = \text{A. tang. } f1 - \text{A. tang. } f(x+1) = \text{A. tang. } \frac{f1 - f(x+1)}{1 + f1(x+1)}.$$

Satius tamen videbatur, oftendere, quo pacto haec formula analytice ex iisdem formulis generalibus euolui queat, ex quibus cafus etiam reliqui difficiliores refoluendi funt. Ceterum hanc formulam non nifi viam *indirectam* monftrare, ad fummationes perueniendi, apparet. Etenim fi ex ea fummanda effet Series Arcuum, cuius terminus generalis $=$ A. tang. X, denotante X functionem indicis x, definienda prius foret functio φx, feu refoluenda aequatio: $X = \frac{\varphi x - \varphi(x+1)}{1 + \varphi x \varphi(x+1)}$, cuius refolutio directa et vniuerfalis exhiberi nequit: indirectae et particulares folutiones obtinentur, dum affumuntur varii valores functionis φx, hincque determinatur valor functionis X. Quare fequentia problemata directe ex formulis generalibus refoluam, ita vt a termino generali ad fummam *progreffus* fiat, quin ab hac ad illum *regreffu* opus fit.

Scholion 2.

§. XIV. Formula generalis (XII. 4.) fimili ratione fynthetice comprobari poteft. Eft nimirum:

$$\begin{array}{l} \text{A, tang. } t^{I} = \text{A. tang. } f1 - \text{A. tang. } f(1+r) \\ \text{A. tang. } t^{II} = \text{A. tang. } f2 - \text{A. tang. } f(2+r) \\ \text{A. tang. } t^{III} = \text{A, tang. } f3 - \text{A, tang. } f(3+r) \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \text{A. tang. } t^{r} = \text{A. tang. } fr - \text{A. tang. } f(2r) \\ \text{A. tang. } t^{r+1} = \text{A. tang. } f(1+r) - \text{A. tang. } f(1+2r) \\ \text{A. tang. } t^{r+2} = \text{A. tang. } f(2+r) - \text{A. tang. } f(2+2r) \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \text{A. tang. } t^{x} = \text{A. tang. } fx - \text{A. tang. } f(x+r). \end{array}$$

Inde

Inde cum quilibet feriei terminus duabus partibus conftet, altera affirmatiua, altera negatiua, partes affirmatiuae ab A. tang. $f(1+r)$ vsque ad vltimam A. tang. fx deftruent partes aequales negatiuas, ac remanebit fumma

$= $ A. tang. $f1 +$ A. tang. $f2 +$ A. tang. $f3 \ldots +$ A. tang. fr

$-$ A. tang. $f(x+1) -$ A. tang. $f(x+2) \ldots -$ A. tang. $f(x+r)$,

quae expreffio confpirat cum prius inuenta (XII. 4.).

Ex quibus hactenus vniuerfe praemiffis quanquam ingens ferierum fummabilium varietas oriatur, *duae* tamen inprimis ferierum *fpecies*, fupra iam quodammodo indicatae (§. III. B. a. b.), euoluendae videntur; quarum *primam* Cap. I., *alteram* Cap. II. confiderabimus.

CAP. I.

DE IIS MAXIME SERIEBVS, QVAE CONSTANT ARCVBVS, QVORVM COTANGENTES IN SERIE ALGEBRAICA SECVNDI ORDINIS PROCEDVNT. (*)

PROBLEMA III.

§. XV. *Summare feriem Arcuum:*

$$\text{A. tang. } \frac{a}{1+b+c} + \text{A. tang. } \frac{a}{4+2b+c} + \text{A. tang. } \frac{a}{9+3b+c} + \ldots\ldots$$

$$+ \text{A. tang. } \frac{a}{x^2+bx+c}, \text{ *fuppofito* } 4c = b^2 + 4a^2 - 1.$$

Solutio.

1) Productum indefinitum $P\left(\frac{1+t^x\sqrt{-1}}{1-t^x\sqrt{-1}}\right)$ (§. VII.) eft $= P\left(\frac{x^2+bx+c+a\sqrt{-1}}{x^2+bx+c-a\sqrt{-1}}\right)$, quo reuocato ad formam $\frac{M+N\sqrt{-1}}{M-N\sqrt{-1}}$ fumma erit $=$ A. tang. $\frac{N}{M}$.

2) Refoluantur numerator et denominator quadratici in factores fimplices, eritque ille $= \left(x + \frac{b-\sqrt{(b^2-4c-4a\sqrt{-1})}}{2}\right)\left(x + \frac{b+\sqrt{(b^2-4c-4a\sqrt{-1})}}{2}\right)$, i. e. (ob $b^2 - 4c = 1 - 4a^2$, et $\sqrt{(1 - 4a^2 - 4a\sqrt{-1})} = 1 - 2a\sqrt{-1}$) $= \left(x + \frac{b-1+2a\sqrt{-1}}{2}\right)\left(x + \frac{b+1-2a\sqrt{-1}}{2}\right)$. Inde, permutando $\sqrt{-1}$ cum $-\sqrt{-1}$, prodit denominator $= \left(x+\frac{b-1-2a\sqrt{-1}}{2}\right)\left(x+\frac{b+1+2a\sqrt{-1}}{2}\right)$.

Quare producti (1) factor generalis $= \dfrac{\left(x+\frac{b-1+2a\sqrt{-1}}{2}\right)\left(x+\frac{b+1-2a\sqrt{-1}}{2}\right)}{\left(x+\frac{b-1-2a\sqrt{-1}}{2}\right)\left(x+\frac{b+1+2a\sqrt{-1}}{2}\right)}$, quem

(*) Cf. infra §. XXX.

quem esse formae $\frac{\varphi x}{\varphi(x+1)}$ (XII. 1.) facile patet. Productum ipsum reperitur, ob factores se mutuo tollentes, $= \frac{\left(b+1+2a\gamma-1\right).\left(x+\frac{b+1-2a\gamma-1}{2}\right)}{\left(b+1-2a\gamma-1\right).\left(x+\frac{b+1+2a\gamma-1}{2}\right)}$.

3) Hinc prodit summa Arcuum seriei $=$ A. tang. $\frac{N}{M}(1) =$ A. tang. $\frac{2a\left(x+\frac{b+1}{2}\right)-a(b+1)}{(b+1)\left(x+\frac{b+1}{2}\right)+2a^2}$ $=$ A. tang. $\frac{2ax}{(b+1)x+\frac{1}{2}(b+1)^2+2a^2} =$ A. tang. $\frac{2ax}{(b+1)(x+1)+2c}$, ob $4c = 4a^2 + (b+1)^2 - 2(b+1)$, vel $2c + b + 1 = 2a^2 + \frac{1}{2}(b+1)^2$.

Corollarium 1.

§. XVI. 1) Si $x = \infty$, erit summa seriei *infinitae*, (vel limes summae, dum series sine fine progreditur) $=$ A. tang. $\frac{2a}{b+1}$, vti inuenit EVLERVS (supra § III. posito $L = \frac{1}{a}$, $M = \frac{b}{a}$, $N = \frac{c}{a}$).

2) A quo arcu si arcus summam seriei finitae exhibens (XV. 3) subtrahitur, remanet A. tang. $\frac{2a}{2x+b+1}$. Hinc summa seriei finitae, termino x^{to} desinentis, etiam sic exprimi potest: S. A. tang. $\frac{a}{x^2+bx+c} =$ A. tang. $\frac{2a}{1+b}$ — A. tang. $\frac{2a}{2x+1+b}$. Eandem formulam suppeditat productum (XV. 2.) tanquam compositum, ope Coroll. IX.

Corollarium 2.

§. XVII. 1) Ponatur $\frac{2a}{b+1} = \frac{1}{\alpha}$, erit $b+1 = 2a\alpha$, $c = \frac{b^2-1+4a^2}{4}$ $= a^2(\alpha^2+1) - a\alpha$. Hinc est terminus generalis seriei hactenus consideratae $=$ A. tang. $\frac{a}{x^2+(2a\alpha-1)x+a^2(\alpha^2+1)-a\alpha} =$ A. tang. $\frac{1}{nx(x-1)+(2x-1)\alpha+m}$, posito $\frac{1}{a} = n$, et $\frac{\alpha^2+1}{n} = m$. Terminus summatorius est (ex XV. 3.) $=$ A. tang. $\frac{2ax}{2a\alpha(x+1)+2a^2(\alpha^2+1)-2a\alpha} =$ A. tang. $\frac{x}{\alpha x+m}$. Inde haec prodit summatio:

matio: $A.\ tang.\ \frac{1}{\alpha+m} + A.\ tang.\ \frac{1}{3\alpha+m+2n} + A.\ tang.\ \frac{1}{5\alpha+m+6n}$

$$+ A.\ tang.\ \frac{1}{7\alpha+m+12n} + \ldots + A.\ tang.\ \frac{1}{(2x-1)\alpha+m+x(x-1)n}$$

$= A.\ tang\ \frac{x}{\alpha x+m}$, dummodo fuerit $mn = \alpha^2 + 1$.

2) Posito $x = \infty$, prodit eiusdem seriei in infinitum continuatae summa $= A.\ tang.\ \frac{1}{\alpha}$, vti extat apud EVLERVM (l. c. §. 11.). Si $n = 1$, erit $m = \alpha^2 + 1$, et $A.\ tang.\ \frac{1}{\alpha} = A.\ tang.\ \frac{1}{\alpha^2+\alpha+1} + A.\ tang.\ \frac{1}{\alpha^2+3\alpha+3} + A.\ tang.\ \frac{1}{\alpha^2+5\alpha+7}$ $+ \ldots + A.\ tang.\ \frac{1}{\alpha^2+(2x-1)\alpha+x^2-x+1}$ + etc. (cf. l. c. §. 4.)

Corollarium 3.

§. XVIII. Aequationi $mn = \alpha^2 + 1$ (XVII. 1) innumeris modis per numeros *integros* satisfieri potest: n et α ita nimirum accipiendi sunt, vt $\alpha^2 + 1$ per n diuisibile sit. Quod cum esse nequeat, si n foret diuisor τȣ α, ponatur $\alpha = n\varkappa + r$, (denotante $\varkappa$ numerum integrum, quotientem ex diuisione τȣ α per n resultantem, r residuum), vel etiam $\alpha = n\varkappa - r$; eritque $\frac{\alpha^2+1}{n} = n\varkappa^2 \pm 2\varkappa r + \frac{r^2+1}{n} = m$. Quare si r et n ita sumantur, vt $\frac{r^2+1}{n}$ sit = numero integro, omnes numeri formae $n\varkappa \pm r = \alpha$ conditionem requisitam adimplebunt, vt scilicet eorum quadratum vnitate auctum per n diuisibile sit. Ponatur vel 1) $r^2 + 1 = n$, vel 2) $\frac{r^2+1}{2} = n$, posterius quidem, si r fuerit numerus impar, tum prodit α vel $= (r^2+1)\varkappa \pm r$, vel $= \left(\frac{r^2+1}{2}\right)\varkappa \pm r$. Hinc innumeri valores pro n, α et r obtinentur. Quorum aliquot, scilicet pro n = 5, 10, 13, 17, 25 exhibuit EVLERVS, nec tamen formula generali eos comprehendit, nec, qua ratione ad eos perueniatur, expressit.

Corollarium 4.

§. XIX. 1) Posito $n = r^2 + 1$, $\alpha = n\varkappa + r$, $m = n\varkappa^2 + 2\varkappa r + 1$, erit terminus generalis seriei (XVII. 1) $= A.\ tang.\ \frac{1}{nx^2+(2n\varkappa+2r-n)x-n\varkappa-r+n\varkappa^2+2\varkappa r+1}$

$$= A.\ tang.\ \frac{1}{n(x+\varkappa)^2+(2r-n)(x+\varkappa)-r+1} = A.\ tang.\ \frac{1}{(r^2+1)(x+\varkappa)^2-(r-1)^2(x+\varkappa)-r+1},$$

summa seriei infinitae $= A.\ tang.\ \frac{1}{\alpha}$ (XVII. 2) $= A.\ tang.\ \frac{1}{(r^2+1)\varkappa+r}$.

2) Si

2.) Si r negatiue, vel $m = nx - r$ accipiatur, erit terminus generalis $= \text{A. tang.} \frac{1}{(r^2+1)(x+x)^2 - (r+1)^2(x+x)+r+1}$ $= \text{A. tang.} \frac{1}{(r^2+1)(x+x-1)^2+(r-1)^2(x+x-1)-r+1}$; summa seriei infinitae $= \text{A. tang.} \frac{1}{(r^2+1)x-r}$.

3) Sit in (1) $x = 0$, in (2) $x = 1$, prodit S. A. tang. $\frac{1}{(r^2+1)x^2-(r-1)^2x-r+1}$ $= \text{A. tang.} \frac{1}{r}$; S. A.tang. $\frac{1}{(r^2+1)x^2+(r-1)^2x-r+1} = \text{A. tang.} \frac{1}{r^2+1-r}$. A binis seriebus ita summatis reliquae ex aliis valoribus τȣ x oriundae in eo tantum differunt, quod in *his* quidem (1 . 2) vel x, vel $x - 1$ primi *illarum* termini desint. Inde haec oritur

Summatio.

Summa seriei infinitae, cuius terminus quisque xtus $= \text{A.tang.} \frac{1}{(r^2+1)x^2 \mp (r-1)^2 x - r + 1}$ est $= \text{A. tang.} \frac{1}{(r^2+1)x \pm r}$, si series *vel* x *vel* $x - 1$ terminis initialibus (pro signis *vel* superioribus *vel* inferioribus) truncata fuerit, i. e. vel $x + 1$to vel xto termino incipiat. Hinc manifestum est, quomodo eiusdem seriei *finitae* summa determinari queat.

4) Iisdem omnino adhibitis ratiociniis, si loco $n = r^2 + 1$ ponatur $n = \frac{r^2+1}{2}$, denotante r numerum imparem, haec altera prodit

Summatio.

Seriei infinitae, cuius terminus xtus $= \text{A. tang.} \frac{1}{\frac{(r^2+1)}{2}x^2 \mp \left(\frac{r^2+1}{2} - 2r\right)x - r + 2}$ summa est $= \text{A. tang.} \frac{1}{\frac{(r^2+1)}{2}x \pm r}$, si seriei x *vel* $x - 1$ termini initiales deficiant, prouti signum superius vel inferius sumatur.

5) Ex his formulis, cum pro r in (3) quilibet numerus integer, in (4) quiuis impar, poni possit, innumerae oriuntur series summabiles. Si in (4) sumatur $r = 1$; 5; 7; in (3) $r = 1$; 2; 3; 4; proueniunt exempla EVLERI (§§. 4. 8. 10. 5. 6. 7. 9.), quorum Analysin et formulas generales easque simplices tradere haud superfluum videbatur; quare exempla in numeris addere minus necesse est.

Scholion 1.

§. XX. Summationis fundamentalis (§. XV.) *analytice inuentae* iam *demonstratio synthetica* concinnari potest. Est nimirum A. tang. $\frac{a}{x+\frac{b-1}{2}}$ — A. tang. $\frac{a}{x+\frac{b+1}{2}}$

$= \text{A. tang.}\ \frac{a}{x^2+bx+\frac{b^2-1}{4}+a^2} = \text{A. tang.}\ \frac{a}{x^2+bx+c}$, ob $4c = b^2 + 4a^2 - 1$.

Hinc sequentes o[?]iuntur aequationes:

$$\text{A. tang.}\ \frac{a}{1+b+c} = \text{A. tang.}\ \frac{2a}{b+1} - \text{A. tang.}\ \frac{2a}{b+3};$$

$$\text{A. tang.}\ \frac{a}{4+2b+c} = \text{A. tang.}\ \frac{2a}{b+3} - \text{A. tang.}\ \frac{2a}{b+5};$$

$$\text{A. tang.}\ \frac{a}{9+3b+c} = \text{A. tang.}\ \frac{2a}{b+5} - \text{A. tang.}\ \frac{2a}{b+7};$$

. . . .

$$\text{A. tang.}\ \frac{a}{x^2+bx+c} = \text{A. tang.}\ \frac{2a}{b+2x-1} - \text{A. tang.}\ \frac{2a}{b+2x+1}.$$

Quas inuicem addendo prodit summatio:

$$\text{S. A. tang.}\ \frac{a}{x^2+bx+c} = \text{A. tang.}\ \frac{2a}{b+1} - \text{A. tang.}\ \frac{2a}{2x+b+1}.$$

Scholion 2.

§. XXI. Numeratore tangentium a positiue accepto, si etiam b affirmatiuum valorem habeat, tangens summae Arcuum, $= \frac{2ax}{(b+1)x+\frac{1}{2}(b+1)^2+2a^2}$ (§. XV. 3) pro quouis valore indicis x positiua est. Proinde Arcuum summa, quousque continuetur series, quadrante semper minor prodit. Hinc sponte consequitur, pro Arcu, qui summam exhibet, Arcum minimum istius tangentis accipiendum esse (cf. §. XI.). Idem tenendum, si $b = -\beta$, et $\beta < 1$. Sin $\beta > 1 = 1 + \gamma$, erit summa

$= \text{A. tang.}\ \frac{2ax}{-\gamma x+\frac{\gamma^2}{2}+2a^2} = \text{A. tang.} \left\{ -\frac{2a}{\gamma} + a\gamma + \frac{\frac{4a^2}{\gamma}}{-\gamma x+\frac{\gamma^2}{2}+2a^2} \right\}$ aequalis Arcui, cui tangens positiua competit, si $x < \frac{\gamma}{2} + \frac{2a^2}{\gamma}$, negatiua, si $x > \frac{\gamma}{2} + \frac{2a^2}{\gamma}$.

Crescente x tangens positiua crescit, negatiuae quantitas absoluta decrescit. Hinc facile patet, vtroque casu Arcum minimum affirmatiuum tangentis in summa expressae, siue ea

positiua

positiua sit, siue negatiua, accipiendum esse (§. XI.); quippe summa Arcuum primo casu in tertium Quadrantem, altero in quartum transire nequit, cum singula seriei membra Arcus sint, semicircumferentia minores: iste nimirum transitus si in termino seriei v. c. n^{to} fieret, summae proxime praecedenti accedere deberet Arcus semicircumferentia maior.

Problema IV.

§. XXII. *Inuenire summam Arcuum:* A. tang. $\frac{a}{1+b+c}$ + A. tang. $\frac{a}{4+2b+c}$ + A. tang. $\frac{a}{9+3b+c}$ + ... + A. tang. $\frac{a}{x^2+bx+c}$, *si fuerit* $4c = b^2 + \frac{4a^2}{r^2} - r^2$, *denotante* r *numerum quemuis integrum.*

Solutio.

1) Eadem omnino ratione, ac §. XIV. 1. 2. reperitur productum indefinitum

$$\left(\frac{1+t^x\gamma-1}{1-t^x\gamma-1}\right) = P\left\{\frac{\left\{x+\frac{b-r+\frac{2a}{r}\gamma-1}{2}\right\}\left\{x+\frac{b+r-\frac{2a}{r}\gamma-1}{2}\right\}}{\left\{x+\frac{b-r-\frac{2a}{r}\gamma-1}{2}\right\}\left\{x+\frac{b+r+\frac{2a}{r}\gamma-1}{2}\right\}}\right\}, \text{ quod est}$$

formae supra (XII. 5) expositae, sumto $fx = \frac{2a}{r\left(x+\frac{b-r}{2}\right)}$.

2) Hinc prodit summa Arcuum = A. tang. f1 + A. tang. f2 ... + A. tang. fr − A. tang. f(x+1) − A. tang. f(x+2) .. − A. tang. f(x+r) = A. tang. $\frac{2a}{r(b-r+2)}$ + A. tang. $\frac{2a}{r(b-r+4)}$ + A. tang. $\frac{2a}{r(b-r+6)}$ + ... + A. tang. $\frac{2a}{r(b+r)}$ − A. tang. $\frac{2a}{r(2x+b-r+2)}$ − A. tang. $\frac{2a}{r(2x+b-r+4)}$ − .. − A. tang. $\frac{2a}{r(2x+b+r)}$.

Corollarium 1.

§. XXIII. 1) Si $x = \infty$, vel series in infinitum excurrit, eae summae partes, quae x inuoluunt, euanescunt, eritque summa seriei infinitae = A. tang. $\frac{2a}{r(b-r+2)}$ + A. tang. $\frac{2a}{r(b-r+4)}$ ... + A. tang. $\frac{2a}{r(b+r-2)}$ + A. tang. $\frac{2a}{r(b+r)}$, quae expressio r terminis constat.

 2) Ea-

2) Eadem summatio sic synthetice comprobari potest: Est A. tang. $\frac{a}{(b-r)\frac{r}{2}+x}$ $-$ A. tang. $\frac{a}{(b+r)\frac{r}{2}+x}$ $=$ A. tang. $\frac{ar^2}{(b^2-r^2)\frac{r^2}{4}+a^2+r^2bx+r^2x^2}$ $=$ A.tg. $\frac{a}{x^2+bx+c}$.

Hinc termini seriei summandae hunc in modum exprimi possunt:

A. tang. $\frac{a}{1+b+c}$ $=$ A. tang. $\frac{2a}{(b-r+2)r}$ $-$ A. tang. $\frac{2a}{(b+r+2)r}$;

A. tang. $\frac{a}{4+2b+c}$ $=$ A. tang. $\frac{2a}{(b-r+4)r}$ $-$ A. tang. $\frac{2a}{(b+r+4)r}$;

A. tang. $\frac{a}{9+3b+c}$ $=$ A. tang. $\frac{2a}{(b-r+6)r}$ $-$ A. tang. $\frac{2a}{(b+r+6)r}$;

. . . .

A. tang. $\frac{a}{r^2+rb+c}$ $=$ A. tang. $\frac{2a}{(b+r)r}$ $-$ A. tang. $\frac{2a}{(b+3r)r}$;

A. tang. $\frac{a}{(r+1)^2+(r+1)b+c}$ $=$ A. tang. $\frac{2a}{(b+r+2)r}$ $-$ A. tang. $\frac{2a}{(b+3r+2)r}$;

. . . .

Quorum additione prodit, omissis terminis se mutuo destruentibus, S. A. tang. $\frac{a}{x^2+bx+c}$ $=$ A. tang. $\frac{2a}{(b-r+2)r}$ $+$ A. tang. $\frac{2a}{(b-r+4)r}$ $+ \ldots +$ A. tang. $\frac{2a}{(b+r)r}$.

Corollarium 1.

§. XXIV. Sit $b=0$, erit, coniungendo terminum expressionis summae primum et penultimum, A. tang. $\frac{2a}{(-r+2)r}$ $+$ A. tang. $\frac{2a}{(r-2)r}$ $=\pi$; idem obtinetur, combinando quosuis binos terminos, quorum vnus a primo aeque distat, ac alter a penultimo. Iam si r fuerit numerus impar, vel $r-1$ par, habentur $\frac{r-1}{2}$ eiusmodi combinationes, quarum quaelibet summam praebet $=\pi$, quibus accedit terminus vltimus $=$ A. tang. $\frac{2a}{rr}$. Hinc erit summa seriei $=\frac{(r-1)\pi}{2}$ $+$ A. tang. $\frac{2a}{r^2}$. Si r fuerit par, vel $r-1$ impar, ab istis combinationibus excluditur terminus $\frac{r}{2}$ tus, $=$ A. tang. $\frac{2a}{(-r+r)r}$ $=$ A. tang. $\frac{r}{0}$ $=\frac{\pi}{2}$. Quare erit summa $=\frac{(r-2)\pi}{2}+\frac{\pi}{2}+$ A. tang. $\frac{2a}{rr}$. Exinde, siue r fuerit par, siue impar, haec obtinetur

Summatio.

Summatio.

Summa feriei infinitae A. tang. $\frac{a}{1+c}$ + A. tang. $\frac{a}{4+c}$ + A. tang. $\frac{a}{9+c}$ + ... + A. tang. $\frac{a}{x^2+c}$ + . . . eft $= \frac{(r-1)\pi}{2}$ + A. tang. $\frac{2a}{r^2}$, fi fuerit $c = \frac{a^2}{r^2} - \frac{r^2}{4}$, denotante r quemuis numerum integrum. Sub A. tang. $\frac{2a}{r^2}$ Arcum minimum tangentis fuae (§. XI.) intelligendum effe, ex §. XXIII. liquet.

Corollarium 3.

§. XXV. Sit $b = -1$, terminus fummae primus et vltimus A. tang. $\frac{2a}{(-r+1)r}$ + A. tang. $\frac{2a}{r(-1+r)}$ conficiunt π; idem praebent caeterorum terminorum bini a primo et vltimo aequidiftantes. Hinc fi r fuerit par, erit fumma feriei $= \frac{r\pi}{2}$; fi r impar, terminus medius expreffionis fummae combinationem haud admittens eft $=$ A. tang. $\frac{2a}{r(-1-r+r+1)} = \frac{\pi}{2}$, hinc fumma $= \frac{(r-1)\pi}{2} + \frac{\pi}{2} = \frac{r\pi}{2}$. Inde haec nafcitur

Summatio.

Summa feriei infinitae A. tang. $\frac{a}{c}$ + A. tang. $\frac{a}{1.2+c}$ + A. tang. $\frac{a}{2.3+c}$ + A. tang. $\frac{a}{3.4+c}$ . . . + A. tang. $\frac{a}{x^2-x+c}$ + . . . eft $= \frac{r\pi}{2}$, fi $4c = 1 + \frac{4a^2}{r^2} - r^2$, denotante r numerum integrum. Hinc fponte fluit haec altera fummatio:

A. tang. $\frac{a}{2+c}$ + A. tang. $\frac{a}{6+c}$ + A. tang. $\frac{a}{12+c}$ . . . + A. tang. $\frac{a}{x^2+x+c}$ + . . . $= \frac{r\pi}{2}$ — A. tang. $\frac{a}{c}$.

Corollarium 4.

§. XXVI. Ob $x^2 - x + c = \frac{(2x-1)^2 + 4c - 1}{4}$ fummatio praecedens ita exhiberi poteft: A. tang. $\frac{4a}{1+\gamma}$ + A. tang. $\frac{4a}{3^2+\gamma}$ + A. tang. $\frac{4a}{5^2+\gamma}$ + . . . + A. tg. $\frac{4a}{(2x-1)^2+\gamma}$ + . . . $= \frac{r\pi}{2}$, pofito $\gamma = \frac{4a^2}{r^2} - r^2$. Ponatur loco 4a, a; et 2r loco r; haec oritur

Summatio.

Summatio.

$$\text{A. tang.}\frac{a}{1+c}+\text{A. tang.}\frac{a}{9+c}+\text{A. tang.}\frac{a}{25+c}\ldots+\text{A. tang.}\frac{a}{(2x-1)^2+c}+\ldots$$

$=\frac{r\pi}{4}$, si fuerit $c=\frac{a^2}{r^2}-\frac{r^2}{4}$ (vti §. XXIV.) et r numerus integer par. Ceterum haec summatio ex priori (§. XXIV.) deduci potest. Ponatur nimirum illic pro a, $\frac{a}{4}$; et pro r, $\frac{r}{2}$; erit $\text{S. A. tang.}\frac{a}{4x^2+\frac{a^2}{r^2}-\frac{r^2}{4}}=\frac{(r-2)\pi}{4}+\text{A. tang.}\frac{2a}{r^2}$. Qua serie subtracta a priori, ob terminos quadrata numerorum parium inuoluentes se mutuo destruentes, remanet altera summatio modo demonstrata.

Corollarium 5.

§. XXVII. Sit $\frac{a}{r}=\frac{r}{2}$, vel $a=\frac{r^2}{2}$, erit $c=0$. Hinc summationes §. XXIV et XXVI. in has obeunt:

$$\text{A. tang.}\frac{r^2}{2.1}+\text{A. tang.}\frac{r^2}{2.4}+\text{A. tang.}\frac{r^2}{2.9}\ldots+\text{A. t.}\frac{r^2}{2.x^2}+\ldots=(2r-1)\frac{\pi}{4};$$

$$\text{A. tang.}\frac{r^2}{2.1}+\text{A. tang.}\frac{r^2}{2.9}+\text{A. tang.}\frac{r^2}{2.25}\ldots+\text{A. t.}\frac{r^2}{2(2x-1)^2}+\ldots=\frac{r\pi}{4};$$

quarum prior pro quolibet numero integro $=r$, posterior pro numero pari obtinet. Illius casum simplicissimum pro $r=1$ protulit EVLERVS (cf. supra §. II.). Altera summatio ibidem commemorata $\text{S. A. tang.}\frac{1}{x^2+x+1}=\frac{\pi}{4}$ sistit casum summationis generalioris §. XXV. demonstratae: $\text{S. A. t.}\frac{a}{x^2+x+\frac{1-r^2}{4}+\frac{a^2}{r^2}}=\frac{r\pi}{2}-\text{A. t.}\frac{4a}{1-r^2+\frac{4a^2}{r^2}}$, ex qua, posito $r=1$, fluit:

$$\text{A. tang.}\frac{a}{2+a^2}+\text{A. tang.}\frac{a}{6+a^2}+\text{A. tang}\frac{a}{12+a^2}\ldots+\text{A. tang.}\frac{a}{x^2+x+a^2}+\ldots$$
$$=\text{A. tang. } a.$$

Corollarium 6.

§. XXVIII. 1) Sequentis seriei:

$$\text{A. tang.}\frac{a}{1+b+c}+\text{A. t.}\frac{a}{9+3b+c}+\text{A. t.}\frac{a}{25+5b+c}+\ldots+\text{A. t.}\frac{a}{(2x-1)^2+(2x-1)b+c}$$

terminus generalis sic exhiberi potest: $\text{A. tang.}\frac{a:4}{x^2+\left(\frac{b}{2}-1\right)x+\frac{1-b+c}{4}}$. Quare ea ea summabilis erit (§. XXII.), si fuerit $1 - b + c = \left(\frac{b}{2} - 1\right)^2 + \frac{a^2}{4r^2} - r^2$, vel $4c = b^2 + \frac{a^2}{r^2} - 4r^2$, quae aequatio, posita $\frac{r}{2}$ loco r, in hanc abit: $4c = b^2 + \frac{4a^2}{r^2} - r^2$, vbi iam pro r numerus par sumendus est.

2) A cuius seriei duplo subtracta serie (§. XXII.), remanet series *signis alternantibus* instructa haec:

$$\text{A. tang.}\,\frac{a}{1+b+c} - \text{A. tang.}\,\frac{a}{4+2b+c} + \text{A. tang.}\,\frac{a}{9+3b+c} - \text{A. tang.}\,\frac{a}{16+4b+c} + \ldots \pm \text{A. tang.}\,\frac{a}{x^2+bx+c},$$

quae igitur summabilis erit, dummodo fuerit $4c = b^2 + \frac{4a^2}{r^2} - r^2$, denotante r numerum quemuis parem.

3) Ita posito $b = 0$, ex §. XXIV et XXVI. haec prodit

Summatio.

$$\text{A. tang.}\,\frac{a}{1+c} - \text{A. tang.}\,\frac{a}{4+c} + \text{A. tang.}\,\frac{a}{9+c} - \text{A. tang.}\,\frac{a}{16+c} + \text{etc. in inf.} = \text{A. tang.}\,\frac{r^2}{2a},$$

supposito $c = \frac{a^2}{r^2} - \frac{r^2}{4}$, et r numero pari. E. g. pro $a = \frac{r^2}{2}$ est

$$\text{A. tang.}\,\frac{r^2}{2.1} - \text{A. tang.}\,\frac{r^2}{2.4} + \text{A. tang.}\,\frac{r^2}{2.9} - \text{A. tang.}\,\frac{r^2}{2.16} + \text{etc.} = \frac{\pi}{4};$$

vel, posito $r = 2\varrho$,

$$\text{A. tang.}\,\frac{2\varrho}{1} - \text{A. tang.}\,\frac{2\varrho^2}{4} + \text{A. tang.}\,\frac{2\varrho}{9} - \text{A. tang.}\,\frac{2\varrho^2}{16} + \text{etc.} = \frac{\pi}{4}.$$

Haec itaque series Arcuum infinita pro quolibet numero integro ϱ eandem summam, Quadranti aequalem, habet.

Scholion 1.

§. XXIX. 1) Posito $a = r^2\alpha$, $b = r^2\beta$, $c = r^2\gamma$, aequatio conditionalis $4c = b^2 + \frac{4a^2}{r^2} - r^2$ (§. XXII.) in hanc abit: $4\gamma = \beta^2 + 4\alpha^2 - 1$, quae consentit cum aequatione conditionali §. XV. Hinc binae summationes Probl. 3 et 4. inuentae hoc *vno Theoremate* comprehendi possunt: summabilis est series vel finita vel infinita:

$$\text{A. tang.}\,\frac{\alpha r^2}{1+\beta r+\gamma r^2} + \text{A. tang.}\,\frac{\alpha r^2}{4+2\beta r+\gamma r^2} + \text{A. tang.}\,\frac{\alpha r^2}{9+3\beta r+\gamma r^2} + \ldots + \text{A. tang.}\,\frac{\alpha r^2}{x^2+\beta r x+\gamma r^2} + \ldots$$

denotante r quemuis numerum integrum, et supposito $4\gamma = \beta^2 + 4\alpha^2 - 1$.

2) Con-

2) Confiderentur huius feriei terminus quisque n^{tus}, et hunc infequentes $n + r^{tus}$, $n + 2r^{tus}$, $n + 3r^{tus}$ . . ., erit terminus $n + xr^{tus} = $ A. tg. $\frac{\alpha r^2}{(n + xr)^2 + \beta r(n + xr) + \gamma r^2}$ $=$ A. tang. $\frac{\alpha r^2}{x^2 r^2 + (2nr + \beta r^2)x + n^2 + \beta rn + \gamma r^2} = $ A. tang. $\frac{\alpha}{x^2 + Bx + C}$, vbi $4\left(\frac{n^2}{r^2} + \beta\frac{n}{r} + \gamma\right) = 4C = \left(\frac{2n}{\gamma} + \beta\right)^2 + 4\alpha^2 - 1 = B^2 + 4\alpha^2 - 1.$ Quare termini modo dicti a reliquis feriei (1) terminis feparati conftituunt feriem, quae conditioni Probl. 3. fatisfacit, hincque fummabilis eft. Quodfi igitur ponatur $n = 1, 2, 3, \ldots r$; feries probl. 4. difpefcitur in r feries terminorum illius feriei interuallo r inuicem diftantium, quarum quaeuis ex probl. 3. fummari poteft; ficque alia via ad folutionem prius inuentam peruenitur.

Scholion 2.

§. XXX. 1) Problemata 3 et 4. innumeras feries fummabiles praebent, quarum terminus generalis eft formae A. tang. $\frac{a}{x^2 + bx + c}$, vbi cotangens $\frac{x^2 + bx + c}{a}$ eft terminus x^{tus} feriei algebraicae *fecundi ordinis*, feu talis feriei, cuius differentiae fecundae funt inter fe aequales (cf. L. EVLERI Inft. Calc. diff. P. I. Cap. II. §. 37.). Quarum ferierum fi binae, pluresue inuicem addantur, liquet, nouas exinde oriri feries itidem fummabiles, terminum generalem habentes formae:

$$\text{A. tang.} \frac{Ax^{2m-2} + Bx^{2m-3} + Cx^{2m-4} + \ldots}{x^{2m} + bx^{2m-1} + cx^{2m-2} + \ldots}$$

vbi coëfficientes A, B, C . . .; b, c . . .; certas inter fe relationes teneant neceffe eft. Cuiusmodi arcuum refolutio in fimpliciores cum deinceps amplius exponatur, hoc loco vnum exemplum fufficiat (*). Eft nimirum A. t. $\frac{r^2}{2x^2}$ + A. t. $\frac{\varrho^2}{2x^2}$ = A. t. $\frac{2(r^2 + \varrho^2)x^2}{4x^4 - r^2\varrho^2}$, hinc S. A. t. $\frac{2(r^2 + \varrho^2)x^2}{4x^4 - r^2\varrho^2}$ (ex §. XXVII.) $= (2r - 1)\frac{\pi}{4} + (2\varrho - 1)\frac{\pi}{4} = (r + \varrho - 1)\frac{\pi}{2}$, quae fummae expreffio tantum ab $r + \varrho$, haud a fingulis r et ϱ pendet. Pro $r = 1$, $\varrho = 2$, eft v. c. S. A. tang. $\frac{5x^2}{2(x^4 - 1)} = \pi$; vel A. tang. $\frac{5 \cdot 4}{2(2^4 - 1)}$ + A. tang. $\frac{5 \cdot 9}{2(3^4 - 1)}$ + A. tang. $\frac{5 \cdot 16}{2(4^4 - 1)}$ + etc. $= \frac{\pi}{2}$.

2) Quod

(*) Nonnullae fummationes ferierum algebraice fummabilium in Sectione III. occurrent, quas quippe deinceps, ceu Corollaria ex generalioribus, facilius, quam hoc loco, demonftrare licet.

2) Quod si summabilis est series, cuius terminus generalis $=$ A. tang. X, assignari quoque poterit summa S. A. tang. $\frac{a+X}{1-aX}$ $=$ S. (A. tang. a $+$ A. tang. X) $= x$ A. tang. a $+$ S. A. tang. X. Cum sit A. tang. a $=$ $(x+1)$ A $-$ x A $=$ A. tang. tang. $(x+1)$ A $-$ A. tang. tang. x A, vel haec summatio iam sub prioribus formulis comprehenditur (§. XII.). Ita inuestigandum est, quando summari possit series, cuius terminus generalis $=$ A. tang. $\frac{Ax^2+Bx+C}{x^2+bx+c}$. Hic scilicet ponatur $=$ A. tang. α $+$ A. tang. $\frac{\beta}{x^2+\gamma x+\delta}$, et facta reductione prodit $\alpha = A$, $\gamma = b$, $\delta = \frac{AC+c}{1+A^2}$, $\beta = \frac{C-Ac}{1+A^2}$; praetereaque hae binae aequationes conditionales obtinentur: $B = Ab$, et $4\left(\frac{AC+c}{1+A^2}\right) = b^2 + \frac{4}{r^2} \cdot \left(\frac{C-Ac}{1+A^2}\right)^2 - r^2$. Ceterum cum haec omnia ex hactenus demonstratis repeti queant, et resolutio Arcuum in sequentibus vberius illustranda sit, vlterior expositio superflua videtur: indeque etiam inscriptio huius Capitis summationem earum *maxime* serierum, quas §. XV-XXIX. considerauimus, pollicebatur, quanquam eadem methodus ad longe plures series extendatur. Transeamus igitur ad *alteram speciem* serierum *algebraice summabilium*, supra §. III. commemoratam.

CAP. II.

DE IIS MAXIME SERIEBVS, QVAE CONSTANT ARCVBVS, QVORVM COTANGENTES PROCEDVNT IN SERIE RECVRRENTE SECVNDI ORDINIS, VEL PVRA VEL AFFECTA. (*)

Problema V.

§. XXXI. *Summare seriem Arcuum:* A. t. $\frac{a}{E+bE^{-1}+c}$ $+$ A. t. $\frac{a}{E^2+bE^{-2}+c}$ $+$ A. tang. $\frac{a}{E^3+bE^{-3}+c}$ $+ \ldots +$ A. tang. $\frac{a}{E^x+bE^{-x}+c}$, *posito* $\frac{b}{E} = \frac{c^2}{(E+1)^2} + \frac{a^2}{(E-1)^2}$.

Solutio.

1) Productum indefinitum $P\left(\frac{1+\iota^x\Upsilon-1}{1-\iota^x\Upsilon-1}\right)$ (§. VII.) est

$$= P\left(\frac{E^x+bE^{-x}+c+a\Upsilon-1}{E^x+bE^{-x}+c-a\Upsilon-1}\right) = P\left(\frac{E^{2x}+(c+a\Upsilon-1)E^x+b}{E^{2x}+(c-a\Upsilon-1)E^x+b}\right).$$ Facta resolu-

(*) Cf. infra §. XXXIII.

solutione numeratoris et denominatoris in factores simplices, prodit ille $=$ $\left(E^x + \frac{c+a\sqrt{-1}-\sqrt{((c+a\sqrt{-1})^2-4b)}}{2}\right) \cdot \left(E^x + \frac{c+a\sqrt{-1}+\sqrt{((c+a\sqrt{-1})^2-4b)}}{2}\right)$.
Est autem $(c+a\sqrt{-1})^2 - 4b = c^2 + 2ac\sqrt{-1} - a^2 - 4b = c^2 - a^2 - \frac{4c^2E}{(E+1)^2} - \frac{4a^2E}{(E-1)^2} + 2ac\sqrt{-1} = c^2\left(\frac{E-1}{E+1}\right)^2 - a^2\left(\frac{E+1}{E-1}\right)^2 + 2ac\sqrt{-1}$, $=$ quadrato, cuius radix est $= c\left(\frac{E-1}{E+1}\right) + a\left(\frac{E+1}{E-1}\right)\cdot\sqrt{-1}$. Hinc fit numerator $= \left(E^x + \frac{c}{E+1} - \frac{a}{E-1}\cdot\sqrt{-1}\right)\left(E^x + \frac{cE}{E+1} + \frac{aE}{E-1}\cdot\sqrt{-1}\right)$, et, permutando $\sqrt{-1}$ cum $-\sqrt{-1}$, denominator $=$
$\left(E^x + \frac{c}{E+1} + \frac{a}{E-1}\cdot\sqrt{-1}\right)\left(E^x + \frac{cE}{E+1} - \frac{aE}{E-1}\cdot\sqrt{-1}\right)$, indeque prodit factor generalis $= \frac{\left(E^x + \frac{c}{E+1} - \frac{a}{E-1}\cdot\sqrt{-1}\right)\left(E^{x-1} + \frac{c}{E+1} + \frac{a}{E-1}\cdot\sqrt{-1}\right)}{\left(E^x + \frac{c}{E+1} + \frac{a}{E-1}\cdot\sqrt{-1}\right)\left(E^{x-1} + \frac{c}{E+1} - \frac{a}{E-1}\cdot\sqrt{-1}\right)}$,
quem esse formae $\frac{\varphi x}{\varphi(x+1)}$ (§. XII.) sponte liquet. Quare erit productum, ob factores se mutuo destruentes, $=$

$$\frac{\left(1 + \frac{c}{E+1} + \frac{a}{E-1}\cdot\sqrt{-1}\right)\left(E^x + \frac{c}{E+1} - \frac{a}{E-1}\cdot\sqrt{-1}\right)}{\left(1 + \frac{c}{E+1} - \frac{a}{E-1}\cdot\sqrt{-1}\right)\left(E^x + \frac{c}{E+1} + \frac{a}{E-1}\cdot\sqrt{-1}\right)}.$$

2) Hinc prouenit summa Arcuum $=$ A. tang. $\frac{N}{M}$ (§. VII.) $=$

$$\text{A. tang.} \frac{\frac{a}{E-1}\left(E^x + \frac{c}{E+1}\right) - \frac{a}{E-1}\left(1 + \frac{c}{E+1}\right)}{\left(1 + \frac{c}{E+1}\right)\left(E^x + \frac{c}{E+1}\right) + \frac{a^2}{(E+1)^2}}$$

$$= \text{A. tang.} \frac{\frac{a}{E-1}\left(E^x - 1\right)}{E^x\left(1 + \frac{c}{E+1}\right) + \frac{c}{E+1} + \frac{b}{E}}.$$

Corollarium 1.

§. XXXII. 1) Si series in infinitum excurrit, vel x ponitur $= \infty$, tum duo casus sunt discernendi, prouti fuerit $E > 1$ vel $E < 1$. Priori casu, ob $E^x = \infty$, erit summa ma $=$ A. tang. $\dfrac{\frac{aE^x}{E-1}}{E^x\left(1+\frac{c}{E+1}\right)}$ $=$ A. tang. $\dfrac{a(E+1)}{(E-1)(E+1+c)}$. Altero casu, ob

$E^x = o$, prodit summa $=$ A. tang. $\dfrac{\frac{a}{1-E}}{\frac{c}{E+1}+\frac{b}{E}}$ $=$ A. tang. $\dfrac{aE(E+1)}{(1-E)((c+b)E+b)}$.

Ceterum posterior casus ad priorem reduci potest, ponendo $E = \frac{1}{e}$, vt sit $e > 1$, vnde erit terminus generalis $=$ A. tang. $\dfrac{a}{e^{-x}+be^x+c}$ $=$ A. tang. $\dfrac{a:b}{e^x+e^{-x}:b+c:b}$ $=$ A. tang. $\dfrac{\alpha}{e^x+\beta e^{-x}+\gamma}$, vbi est $\dfrac{E}{b} = \dfrac{c^2}{b^2\left(\frac{1}{E}+1\right)^2} + \dfrac{a^2}{b^2\left(\frac{1}{E}-1\right)^2}$, vel $\dfrac{\beta}{e} = \dfrac{\gamma^2}{(e+1)^2} + \dfrac{\alpha^2}{(e-1)^2}$. Hinc ex summa pro casu priori summa pro altero casu prodit, indeque semper $E > 1$ sumere licet.

2) Summa seriei finitae, simili ratione ac §. XVI. tanquam differentia binorum Arcuum exhiberi potest; sc. S. A. tang. $\dfrac{a}{E^x+bE^{-x}+c}$ $=$ A. tang. $\dfrac{a:(E-1)}{1+c:(E+1)}$ $-$ A. tang. $\dfrac{a:(E-1)}{E^x+c:(E+1)}$, quorum alter pro $x = \infty$ euanescit.

Corollarium 2.

§. XXXIII. 1) Quodsi accuratius considerentur tangentes inuersae vel cotangentes arcuum seriei, apparebit, eas formare seriem recurrentem secundi ordinis *cum appendice*. (*) Sit nimirum $\dfrac{E^x+bE^{-x}+c}{a} = z$, et cotangentes Arcuum seriei proxime infe-

(*) Denominationem serierum recurrentium cum *appendice* ab Analystis Italis passim vsurpatam (cf. Gli Elementi teorico-pratici delle Matematiche pure del Padre Odoardo *Gherli*, Tom. II. Modena 1771. 4. pag. 408. sq.) primus adhibuit *Vinc. Riccati*. Cf. de Bononiensi Scientiarum et Artium Instituto atque Academia Commentarii. Tomi V. Pars I. Bonon. 1767. 4. p. 87-108. „De termino generali serierum recurrentium cum appendice." Tomi V. Pars II. p. 415-420. „Additamentum ad opusculum de termino generali serierum etc." Illic p. 87. haec verba extant: „. . . Aliud serierum recurrentium genus, quarum termini singuli determinantur, si aliquot antecedentibus per constantes multiplicatis addas vel demas quantitatem item constantem. Has autem propter terminum constantem qui additur, licet nominare series recurrentes cum appendice." Cum igitur insequentium, siue $x+1^{ti}$ et $x+2^{ti}$, sint z^I, z^{II}, tum erit $z^{II} = \frac{E^2+1}{E}.z^I - z - \frac{(E-1)^2 c}{aE}$. Ponatur enim $\frac{1}{a}E^x + \frac{b}{a}E^{-x} = \zeta$, vel $\zeta = z - \frac{c}{a}$, erit ζ terminus generalis seriei recurrentis vulgaris, cuius scalam relationis constituunt $\frac{E+1}{E}$, $-\frac{E \cdot 1}{E} = -1$, vt sit $\zeta^{II} = \frac{E^2+1}{E}.\zeta^I - \zeta$. Hinc sponte fluit altera aequatio inter z^{II}, z^I, z.

2) Proinde summatio §. XXXI. XXXII. ita quoque exhiberi potest: S. A. tang. $\frac{1}{z}$

$$= \text{A.t.} \frac{a\left\{\frac{E^x-1}{E-1}\right\}}{E^x\left\{1+\frac{c}{E+1}\right\}+\frac{c}{E+1}+\frac{b}{E}} = \text{A.t.} \frac{a:(E-1)}{1+c:(E+1)} - \text{A.t.} \frac{a:(E-1)}{E^x+c:(E+1)},$$ si

fuerit z terminus generalis seriei recurrentis, cuius lex hac aequatione continetur: $z^{II} = \frac{(E^2+1)}{E}.z^I - z - \frac{(E-1)^2 c}{aE}$. Requiritur autem insuper, vt sint bini priores huius seriei termini $= \frac{E+bE^{-1}+c}{a}$, $\frac{E^2+bE^{-2}+c}{a}$, posito $\frac{b}{E} = \frac{c^2}{(E+1)^2} + \frac{a^2}{(E-1)^2}$. Inter quos igitur terminos et coëfficientes aequationis legem seriei τῶν z exprimentis certa relatio locum habeat necesse est. Quae relatio sequenti problemate accuratius definietur.

Problema 6.

§. XXXIV. *Summare seriem Arcuum:* A. cotang. A + A. cotang. B + A. cot. C + . . . + A. cotang. z, *progredientibus* A, B, C . . . z, z^I, z^{II} *in serie recurrente tali, vt sit* $z^{II} = mz^I - z - n$; *ac assumta insuper inter terminos binos priores huius seriei* A, B *et coëfficientes* m, n *hac aequatione:* $(m+2)(AB-1) = (A+B)(A+B+n)$.

Solutio.

1) Comparatio huius seriei cum serie prius summata (§. XXXIII. 2.) sequentes quatuor offert aequationes:

1) m =

igitur discrimen inter series recurrentes vulgares et series alterius generis, seu recurrentes cum appendice, in eo positum sit, quod aequationi, quae *illarum* legem exprimit, aeque ac earundem termino generali adiiciendus sit pro *his* nouus terminus, e. g. — n, aequationi §. XXXIV, et c : a termino generali §. XXXIIII; ad similitudinem aequationum quadraticarum, quae in puras et *affectas* diuiduntur, series recurrentes alterius generis haud incommode *affectae* vocari videntur. Ceterum series recurrentes affectae n^{ti} ordinis reducuntur ad series recurrentes puras $n+1^{ti}$ ordinis.

1) $m = \frac{E^2+1}{E}$

2) $n = \frac{(E-1)^2 c}{aE}$

3) $A = \frac{E}{a} + \frac{c^2}{a(E+1)^2} + \frac{a}{(E-1)^2} + \frac{c}{a}$

4) $B = \frac{E^2}{a} + \frac{c^2}{aE(E+1)^2} + \frac{a}{E(E-1)^2} + \frac{c}{a}$

Ex quibus non tantum determinari poſſunt quantitates c, a, E; verum etiam infertur aequatio inter ipſas quantitates A, B, m, n. Quo ex 3 et 4, adhibendo 2, eliminentur c et a, habetur $BE - A = E\left[\frac{E^2-1}{a}\right] + \frac{c}{a}(E-1)$, $AE - B = \frac{c^2}{a(E+1)^2} \cdot \left[\frac{E^2-1}{E}\right] + \frac{a}{(E-1)^2} \cdot \left[\frac{E^2-1}{E}\right] + \frac{c}{a}(E-1)$; vel, ob $\frac{c}{a} = \frac{nE}{(E-1)^2}$, $\frac{c^2}{a} = \frac{a \cdot c^2}{a^2} = \frac{an^2E^2}{(E-1)^4}$; erit: $BE - A = E\frac{(E^2-1)}{a} + \frac{nE}{E-1}$, $AE - B = \frac{an^2E}{(E^2-1)(E-1)^2} + \frac{a(E+1)}{(E-1)E} + \frac{En}{E-1}$. Quae aequationes combinatae praebent: $\left[BE - A - \frac{En}{E-1}\right]\left[AE - B - \frac{En}{E-1}\right] = \frac{E^2n^2}{(E-1)^2} + (E+1)^2$, vel euoluendo, $(BE - A)(AE - B) - En(B+A) = (E+1)^2$. Hinc porro fit $BA(E^2+1) - (A^2+B^2)E - En(B+A) = E^2 + 1 + 2E$, ſeu ob $E^2 + 1 = mE$, (ex 1), $Bam - A^2 - B^2 - n(B+A) = m+2$; vnde prodit $m = \frac{A^2+B^2+2+n(A+B)}{BA-1}$, et $m + 2 = \frac{(A+B)^2+n(A+B)}{AB-1} = \frac{(A+B)(A+B+n)}{AB-1}$. Sic igitur inuenta eſt *aequatio conditionalis* definiens relationem neceſſariam inter quantitates A, B, m, n, eandem in problemate iam enuntiatam.

2) Quod iam ad quantitates c et a ac ſummam inde determinandam attinet, eſt, ex prius (1) demonſtratis, $BE - A - \frac{En}{E-1} = \frac{E(E^2-1)}{a}$, hinc $\frac{a}{E-1} = \frac{E(E^2-1)}{(BE-A)(E-1)-En}$; et $\frac{c}{E+1} = \frac{anE}{(E-1)^2(E+1)} = \frac{nE^2}{(BE-A)(E-1)-En}$. Quare ſumma ſeriei infinitae eſt (§. XXXIII. 2.) $=$ A. tang. $\frac{E(E^2-1)}{(BE-A)(E-1)-En+nE^2}$ $=$ A. tang. $\frac{E(E+1)}{BE-A+En}$.

Summa ſeriei finitae reperitur $=$ A. tang. $\frac{E(E+1)}{BE-A+nE}$

$-$ A. tang. $\frac{E(E+1)}{E^x(BE-A)-nE^2(E^{x-1}-1):(E-1)}$. Quantitas E ex aequatione E^2+1

$E^2 + 1 = mE$ determinatur, ita quidem, vt fit $E > 1$, nullo ad fignum refpectu habito. Hinc eft $E = \frac{m + \sqrt{(m^2 - 4)}}{2}$, vel $= -\frac{\mu - \sqrt{(\mu^2 - 4)}}{2}$, fi m negatiuum valorem habeat $= -\mu$.

Corollarium 1.

§. XXXV. Expreffio fummae feriei infinitae fequenti ratione transformatur. Ob $(BE - A)(EA - B) = nE(B + A) + (E + 1)^2$ (§. XXXIV. 1.), eft $B(BE - A) + nE(B + A) = -(E + 1)^2 + EA(BE - A)$, ac vtrinque addito $A(BE - A)$, $(B + A)(BE - A + En) = (E + 1)(A(BE - A) - E - 1)$, hinc $\frac{E(E+1)}{BE - A + En} = \frac{B + A}{AB - 1 - (A^2 + 1):E}$. Quare erit, ferie in infinitum producta, S. A. cotang. $z =$ A. tang. $\frac{A + B}{AB - 1 - (A^2 + 1):E}$, vel etiam = A. tang. $\frac{1}{A}$ + A. tang. $\frac{E + 1}{BE - A}$.

Corollarium 2.

§. XXXVI. 1) Ob $C = mB - A - n$, habetur $B^2 + C^2 + n(B + C) + 2 = B^2 + m^2 B^2 - 2mBA + A^2 - 2mnB + 2nA + n^2 + nB + nmB - nA - n^2 + 2 = B^2 + A^2 + nA + nB + 2 + m^2 B^2 - 2mBA - mnB = m(BA - 1) + m^2 B^2 - 2mBA - mnB = m(mB^2 - AB - nB - 1) = m(BC - 1)$. Quare aequatio conditionalis inter m, n et terminos feriei cotangentium primum et fecundum fuppofita locum etiam habet de fecundo et tertio, hinc de quibusuis terminis fibi inuicem proximis: vt fit $z^I z^I + z^{II} z^{II} + n(z^I + z^{II}) + 2 = m(z^I z^{II} - 1)$, feu $(m + 2)(z^I z^{II} - 1) = (z^I + z^{II})(z^I + z^{II} + n)$.

2) Hinc feriei ab inde termino $x + 1$to vel A. cotang. z^I in infinitum excurrentis erit fumma A. cotang. z^I + A. cotang. z^{II} + ... in inf. = A. tang. $\frac{z^I + z^{II}}{z^I z^{II} - 1 - (z^I z^I + 1):E}$. Qua ferie a priori (§. XXXV.) fubducta remanet fumma feriei finitae ad vsque terminum xtum = A. cotang. z productae, A. cotang. A + A. cotang. B + ... + A. cotang. z = A. tang. $\frac{A + B}{AB - 1 - (AA + 1):E}$ — A. tang. $\frac{z^I + z^{II}}{z^I z^{II} - 1 - (z^I z^I + 1):E}$, dum fint z^I et z^{II} cotangentes vltimam z proxime infequentes. Erit igitur, pofita vltimam praecedente $= {}^Iz$, $z^I = mz - {}^Iz - n$; $z^{II} = mz^I - z - n = (m^2 - 1)z - m\,{}^Iz - (m + 1)n$. Quare iam folutionem problematis 6. fequens complectitur:

THEO-

THEOREMA GENERALE.

Summa seriei infinitae

$$A.\,\text{cotang.}\,A + A.\,\text{cot.}\,B + \ldots + A.\,\text{cot.}\,z + A.\,\text{cot.}\,z^{I} + A.\,\text{cot.}\,z^{II} + \ldots$$

est $= A.\,\text{tang.}\,\frac{A+B}{AB - 1 - (A^2+1):E}$, *posito* $z^{II} = m z^{I} - z - n$, *et* $(m+2)(AB-1) = (A+B)(A+B+n)$. *A qua summa, si series termino* A. cotang. z *finitur, subtrahendus est Arcus* A. tang. $\frac{z^{I}+z^{II}}{z^{I}z^{II} - 1 - (z^{I}z^{I}+1):E}$.

Est autem $\frac{1}{E} = \frac{m - \sqrt{(m^2-4)}}{2} < 1$. (§. XXXIV. 2.)

Scholion 1.

§. XXXVII. Operae pretium esse videtur, ostendere, quo pacto solutio praecedentis problematis inuoluat simul solutionem problematis 3. (§. XV.), instar casus particularis. Series nimirum algebraicae secundi ordinis considerari possunt tanquam recurrentes affectae eiusdem ordinis, ob differentias secundas constantes. Posito igitur §. XV. $\frac{x^2+bx+c}{a} = z$, erit $z^{II} - 2z^{I} + z$ differentia secunda $= \frac{2}{a}$; hinc est §. XXXIV. $m = 2$, $n = -\frac{2}{a}$, $E = 1$. Aequatio conditionalis $(B-A)^2 + n(B+A) + 4 = 0$ in hanc abit: $\left[\frac{3+b}{a}\right]^2 - \frac{2}{a}\left[\frac{5+3b+2c}{a}\right] + 4 = 0$. vnde fit $4c = 4a^2 + b^2 - 1$. Summa seriei infinitae est $= A.\,\text{tang.}\,\frac{E(E+1)}{BE - A + nE} = A.\,\text{tang.}\,\frac{2}{\frac{3+b-2}{a}}$ $= A.\,\text{tang.}\,\frac{2a}{b+1}$. Quae omnia cum supra inuentis apprime conspirant.

Scholion 2.

§. XXXVIII. Summatio problematis 5. §. XXXI. simili ratione ac summatio §. XX. demonstrari potest, resoluendo terminum generalem in differentiam binorum Arcuum.

1) Sit nimirum $A.\,\text{tang.}\,\frac{a}{E^{x} + bE^{-x} + c} = A.\,\text{tang.}\,\frac{\alpha}{E^{x}+\beta} - A.\,t.\,\frac{\alpha}{E^{x+1}+\beta}$,

erit $\frac{a}{E^{x} + bE^{-x} + c} = \frac{\alpha E^{x}(E-1)}{E^{2x+1} + \beta E^{x}(E+1) + \beta^2 + \alpha^2} = \frac{\alpha(E-1):E}{E^{x} + \frac{(\beta^2+\alpha^2)}{E}E^{-x} + \frac{\beta(E+1)}{E}}$.

Hinc tres oriuntur aequationes: $a = \alpha(E-1):E$; $b = \frac{\beta^2+\alpha^2}{E}$; $c = \frac{\beta(E+1)}{E}$. Ex prima

prima et tertia habetur $\alpha = \frac{aE}{E-1}$, $\beta = \frac{cE}{E+1}$, secunda praebet aequationem conditionalem: $\frac{b}{E} = \frac{c^2}{(E+1)^2} + \frac{a^2}{(E-1)^2}$. Est igitur A. tang. $\frac{a}{E^x + bE^{-x} + c} =$ A. tang. $\frac{a:(E-1)}{E^{x-1} + c:(E+1)}$ — A. tang. $\frac{a:(E-1)}{E^x + c:(E+1)}$

2) Hinc termini seriei summandae sequentem in modum exprimi possunt:

$$\text{A. tang.} \frac{a}{E + bE^{-1} + c} = \text{A. tang.} \frac{a:(E-1)}{1 + c:(E+1)} - \text{A. tang.} \frac{a:(E-1)}{E + c:(E+1)}$$

$$\text{A. tang.} \frac{a}{E^2 + bE^{-2} + c} = \text{A. tang.} \frac{a:(E-1)}{E + c:(E+1)} - \text{A. tang.} \frac{a:(E-1)}{E^2 + c:(E+1)}$$

$$\text{A. tang.} \frac{a}{E^3 + bE^{-3} + c} = \text{A. tang.} \frac{a:(E-1)}{E^2 + c:(E+1)} - \text{A. tang.} \frac{a:(E-1)}{E^3 + c:(E+1)}$$

. . . . . .

. . . . . .

$$\text{A. tang.} \frac{a}{E^x + bE^{-x} + c} = \text{A. tang.} \frac{a:(E-1)}{E^{x-1} + c:(E+1)} - \text{A. tang.} \frac{a:(E-1)}{E^x + c:(E+1)}.$$

Quos inuicem addendo prodit summa

$$\text{S. A. tang.} \frac{a}{E^x + bE^{-x} + c} = \text{A. tang.} \frac{a:(E-1)}{1 + c:(E+1)} - \text{A. tang.} \frac{a:(E-1)}{E^x + c:(E+1)}.$$

(cf. §. XXXII. 2.)

Scholion 3.

§. XXXIX. Ex hactenus demonstratis, cum aequationi $(m+2)(AB-1) = (A+B)(A+B+n)$, quatuor quantitates indeterminatas inuoluenti, infinitis modis satisfieri possit, innumerae obtinentur series summabiles Arcuum, quorum cotangentes ad legem seriei recurrentis latiori sensu acceptae procedunt. In sequentibus series inprimis *infinitas* consideremus, ad quas finitarum summatio sine negotio reuocatur (§. XXXVI.). Deinde in eos casus maxime inquirendum videtur, quibus singulae Cotangentes *numeris integris* exprimuntur: quod fit, cum A, B, m, n, istiusmodi numeris aequantur. Quanquam resolutio aequationis praedictae in numeris integris ad *Analysin Diophanteam* pertineat, nec sit huius loci, primarios tamen casus euoluamus, et quidem 1) quando cotangentes A, B, . . . z . . sunt termini seriei recurrentis *strictius* sic dictae (§. XXXIX-XLVI.); 2) quando eaedem aequantur quadratis eiusmodi terminorum, vel horum quadratorum aequemultiplis (§. XLVII-LVI.); 3) cum A, B, . . . z . . sint numeri integri in serie recurrente quacunque *affecta* progredientes (§. LVII-LXXXV.). Theoremata huc spectantia *particularia* vocantur, quoniam ea subsunt theoremati *generali* §. XXXVI. (*)

THEO-

(*) Ad hanc disquisitionem, a difficultatibus haud liberam, ac, fateor, satis longam, exinde maxime perductus sum, quod in quatuor exemplis ab EVLERO prolatis (§. III.) cotangentes numeris *integris*

Theorema particvlare I.

§. XXXIX. b. *Numeris* A, B, . . . z, z^I, z^{II} . . . *progredientibus in ſerie recurrente ſcalae* m, — 1, *vt ſit* $z^{II} = m z^I - z$; *erit ſumma ſeriei infinitae*

$$A.\,\text{cotang.}\,A + A.\,\text{cot.}\,B + \ldots + A.\,\text{cotang.}\,z + \ldots = \tfrac{1}{2}\,A.\,\text{cot.}\,\frac{A(1-m)+B}{2},$$

dum fuerit $m + 2 = \dfrac{(A+B)^2}{AB-1}$.

Demonſtratio.

1) Ponatur §. XXXVI. n = 0, tum binae aequationes inter z^{II}, z^I, z; ac inter A, B, m, n ſponte in eas ipſas abeunt, quas theorema enuntiat.

2) Iam ſumma ſeriei infinitae eſt $= A.\,\text{tang.}\,\dfrac{B+A}{AB-1-(A^2+1):E} =$

$$A.\,t.\,\frac{B+A}{AB-1-\frac{(A^2+1)(m-\surd(m^2-4))}{2}} = A.\,t.\,\frac{1}{\frac{AB-1-(A^2+1)m:2}{A+B} + \frac{(A^2+1)}{A+B}\cdot\surd\left(\frac{m^2}{4}-1\right)}$$

Eſt autem $\left(\dfrac{AB-1-(A^2+1)\frac{m}{2}}{A+B}\right)^2 + 1$

$$= \frac{A^2B^2 - 2AB + 1 - (AB-1)(A^2+1)m + (A^2+1)^2\frac{m^2}{4} + A^2 + 2AB + B^2}{(A+B)^2}$$

$$= \frac{A^2B^2 - 2AB + 1 - (A^2+B^2+2)(A^2+1) + (A^2+1)^2\cdot\frac{m^2}{4} + A^2 + 2AB + B^2}{(A+B)^2}$$

$$= \frac{A^4 - 2A^2 - 1 + (A^2+1)^2\frac{m^2}{4}}{(A+B)^2} = \frac{(A^2+1)^2\left(\frac{m^2}{4}-1\right)}{(A+B)^2}.$$ Quare poſito

$\dfrac{AB-1-(A^2+1)m:2}{A+B} = T$, erit ſumma $= A.\,\text{tang.}\,\dfrac{1}{T+\surd(T^2+1)} = A.\,t.\,\surd(T^2+1) - T$

$= \frac{1}{2}$

integris exprimantur. Quanquam haec exempla ex theoremate noſtro generali (§. XXXVI.) haud difficulter deduci queant: hoc tamen ipſum theorema innumeras ſeries praebet, quarum cotangentes numeris *fractis* exprimuntur. Inde ſtrictius oſtendendum videtur, ex eodem innumeras etiam ſeries fluere, quarum cotangentes numeris *integris* aequantur. Quomodo tales ſeries directe ac a priori ſint inueſtigandae, in ſequentibus ita declaratum eſt, vt vix quicquam reliquum eſſe videatur. Sic etiam quatuor iſta exempla ad formas generales reuocantur (§§. XLIII, LIV, LVI, LXXII.). Pro binis ex illis ab EVLERO legem progreſſus cotangentium haud expreſſam eſſe, ſupra iam monui (§. III.); et quartum quidem a ceteris ita diſcrepat, vt lex progreſſus alia omnino ratione eaque magis abſcondita inueſtiganda ſit.

$= \frac{1}{2}$ A. cotang. T. Eſt porro $(A+B)^2 = 2(AB-1) + m(BA-1)$ vel $(A+B)^2 - m(A^2+AB) = 2(AB-1) - m(1+A^2)$, hinc $T = \frac{A+B-mA}{2}$; vnde conſequitur ipſa ſummatio demonſtranda.

Corollarium 1.

§. XL. 1) Summa ſeriei finitae A. cotang. A + A. cot. B + A. cot. C + ... + A. cot. z ex §. XXXVI. eſt $= \frac{1}{2}$ A. tang. $\frac{2}{A+B-Am} - \frac{1}{2}$ A. tang. $\frac{2}{z^I+z^{II}-mz^I}$

$= \frac{1}{2}$ A. tang. $\frac{2}{A+B-Am} - \frac{1}{2}$ A. tang. $\frac{2}{z^I-z}$

2) Ob A. cotang. B + A. cotang. C + ... + A. cotang. z + ... $= \frac{1}{2}$ A. tang. $\frac{2}{B+C-Bm} = \frac{1}{2}$ A. tang. $\frac{2}{B-A}$, apparet ſummam ſeriei infinitae etiam ſic exprimi poſſe: A. cotang. A + A. cotang. B + ... + A. cotang. z + ... $=$ A. cotang. A $+ \frac{1}{2}$ A. cot. $\frac{B-A}{2}$.

Corollarium 2.

§. XLI. Si quantitatum B et m vtraque negatiue accipiatur, tum ex praecedente ſummatione obtinetur ſeriei ſignis alternantibus inſtructae:

A. cotang. A — A. cotang. B + ... ± A. cotang. z ∓ in inf.

ſumma $= \frac{1}{2}$ A. cotang. $\frac{A(1+m)-B}{2}$, poſito $z^{II} = mz^I - z$, et $m - 2 = \frac{(A-B)^2}{AB+1}$.

Quae ſumma etiam ſic exprimi poteſt: S ± A. cotang. z = A. tang. $\frac{1}{A} - \frac{1}{2}$ A. t. $\frac{2}{A+B}$.

Cum ſumma ſeriei neceſſario minor eſſe debeat, quam A. cotang. A, ob ſeriem A, B, ... z ... creſcentem, facile apparet, in ſummae expreſſione Arcus minimos intelligendos eſſe, quales etiam in ipſis ſeriei terminis ſupponuntur (cf. §. XL.).

Corollarium 3.

§. XLII. Ex aequatione conditionali §. XXXIX. $m = \frac{A^2+B^2+2}{AB-1}$ conſequitur

$B = \frac{mA \pm \sqrt{(m^2A^2 - 4(A^2+m+2))}}{2}$. Hinc apparet, proditurum eſſe valorem rationalem τȣ B, ponendo $A^2 + m + 2 = 0$, ſeu $m = -A^2 - 2$, vnde fit $B = mA$. Exinde naſcitur ſequens

Summa-

Summatio.

$\text{A. cot. } A - \text{A. cot. } A(A^2+2) + \text{A. cot. } (A(A^2+2)^2 - A) - \text{etc.} \pm \text{A. cot. } z \pm \text{in inf.}$
$= \frac{1}{2} \text{ A. cotang. } \frac{A}{2}$, cotangentibus $A,\ A(A^2+2),\ A(A^2+2)^2 - A, \ldots z, z^{I}, z^{II} \ldots$
hac lege progredientibus, vt fit $z^{II} = (A^2+2)z^{I} - z$.

Exempla in numeris.

§. XLIII. 1) Pofito $A = 1$, erit $\text{A. t. } 1 - \text{A. t. } \frac{1}{3} + \text{A.t. } \frac{1}{8} - \text{A.t. } \frac{1}{21} + \text{A.t. } \frac{1}{55} - \text{etc.}$ $= \frac{1}{2} \text{ A. tang. } 2$; vbi eft $8 = 3 \cdot 3 - 1$; $21 = 3 \cdot 8 - 3$; $55 = 3 \cdot 21 - 8$; ... $z^{II} = 3z^{I} - z$.

2) Pro $A = 2$, habetur $\text{A.tang. } \frac{1}{2} - \text{A.t. } \frac{1}{12} + \text{A.t. } \frac{1}{70} - \text{A.t. } \frac{1}{408} + \text{A.t. } \frac{1}{2378} - \text{etc.}$ $= \frac{1}{2} \text{ A. tang. } 1 = \frac{\pi}{8}$; exiftentibus numeris $70 = 6 \cdot 12 - 2$; $408 = 6 \cdot 70 - 12$; $2378 = 6 \cdot 408 - 70$; ... $z^{II} = 6z^{I} - z$.

3) Pofito $A = 3$, eft $\text{A. tang. } \frac{1}{3} - \text{A. t. } \frac{1}{33} + \text{A. t. } \frac{1}{360} - \text{A. t. } \frac{1}{3927} + \text{etc.}$ $= \frac{1}{2} \text{ A. tang. } \frac{2}{3}$; vbi lex numerorum his aequationibus exprimitur: $360 = 11 \cdot 33 - 3$; $3927 = 11 \cdot 360 - 33$; ... $z^{II} = 11z^{I} - z$.

4) Pro $A = 4$ habetur $\text{A. tang. } \frac{1}{4} - \text{A. t. } \frac{1}{72} + \text{A. t. } \frac{1}{1292} - \text{A. t. } \frac{1}{23184} + \text{etc.}$ $= \frac{1}{2} \text{ A. t. } \frac{2}{4} = \frac{1}{2} \text{ A. t. } \frac{1}{2}$; vbi eft $1292 = 18 \cdot 72 - 4$; $23184 = 18 \cdot 1292 - 72$; ... $z^{II} = 18z^{I} - z$.

5) Pro $A = 5$ habetur $\text{A. tang. } \frac{1}{5} - \text{A. t. } \frac{1}{135} + \text{A. t. } \frac{1}{3640} - \text{A. t. } \frac{1}{98145} + \text{etc.}$ $= \frac{1}{2} \text{ A. t. } \frac{2}{5}$; vbi eft $135 = 27 \cdot 5$; $3640 = 27 \cdot 135 - 5$; $98145 = 27 \cdot 3640 - 135$; ... $z^{II} = 27z^{I} - z$.
Quae feries quomodo pro lubitu continuandae fint, manifeftum eft. Exemplum (2) fupra iam commemoratum (§. II.) extat apud EVLERVM (l. c.), quod igitur iam ad formulam generalem reuocatum eft, quae tamen ipfa alio refpectu particularis eft.

Alia exempla numerica.

§. XLIV. Quo pateat, praeter feries fub formula §. XLII. comprehenfas alias infuper exhiberi poffe feries, ad cafum 1. fupra (§. XXXVIII.) expofitum pertinentes, fequentia adiungam exempla.

1) Sit $A = 1$, erit $m + 2 = \frac{(1+B)^2}{B-1} = \frac{(\beta+2)^2}{\beta} = \beta + 4 + \frac{4}{\beta}$, fumendo $B - 1 = \beta$. Hinc poni debet $\beta = 1$; vel $= 2$; vel $= 4$; vnde B obtinet valores 2; 3; 5; et m valores 7; 6; 7. Quare fequentes tres oriuntur fummationes:

a) Ex $B = 2$, $\text{A. tang. } 1 + \text{A. t. } \frac{1}{2} + \text{A. t. } \frac{1}{13} + \text{A. t. } \frac{1}{89} + \text{A. t. } \frac{1}{610} + \text{etc.}$
$= \frac{\pi}{4} + \frac{1}{2} \text{ A. tang. } 2$; (§. XL. 2.)

 feu

seu A. tang. $\frac{1}{2}$ + A. t. $\frac{1}{13}$ + A. t. $\frac{1}{89}$ + A. t. $\frac{1}{610}$ + etc. = $\frac{1}{2}$ A. tang. 2; vbi est $13 = 7 \cdot 2 - 1$; $89 = 7 \cdot 13 - 2$; . . . $z^{II} = 7z^{I} - z$.

b) Ex $B = 5$ prodit A. tang. 1 + A. tang. $\frac{1}{5}$ + A. tang. $\frac{1}{34}$ + A. tang. $\frac{1}{233}$ + etc. $= \frac{\pi}{4} + \frac{1}{2}$ A. tang. $\frac{1}{2}$, seu A. tang. $\frac{1}{5}$ + A. tang. $\frac{1}{34}$ + A. tang. $\frac{1}{233}$ + etc. = $\frac{1}{2}$ A. t. $\frac{1}{2}$; vbi est $34 = 7 \cdot 5 - 1$; $233 = 7 \cdot 34 - 5$; . . . $z^{II} = 7z^{I} - z$.

c) Ex $B = 3$, fit A. tang. 1 + A. t. $\frac{1}{3}$ + A. t. $\frac{1}{17}$ + A. t. $\frac{1}{99}$ + A. t. $\frac{1}{577}$ + etc. = A. tang. 1 + $\frac{1}{2}$ A. t. $\frac{2}{2} = \frac{3\pi}{8}$; vbi est $17 = 6 \cdot 3 - 1$; $99 = 6 \cdot 17 - 3$; $577 = 6 \cdot 99 - 17$; . . . $z^{II} = 6z^{I} - z$.

2) Sit $A = 2$, erit $m + 2 = \frac{(2+B)^2}{2B-1}$, seu, posito $2B - 1 = \beta$, $4(m+2) = \frac{(\beta+5)^2}{\beta} = \beta + 10 + \frac{25}{\beta}$. Vnde ponendum $\beta = 1; 5; 25$; $B = 1; 3; 13$; $m = 7; 3; 7$. Extremi valores haud nouas praebent summationes; ex mediis autem, seu pro $B = 3$, haec oritur summatio:

A. tang. $\frac{1}{2}$ + A. t. $\frac{1}{3}$ + A. t. $\frac{1}{7}$ + A. t. $\frac{1}{18}$ + A. t. $\frac{1}{47}$ + A. t. $\frac{1}{123}$ + etc. $= \frac{1}{2}$ A. tang. 2 + A. tang. $\frac{1}{2} = \frac{\pi}{2} - \frac{1}{2}$ A. tang. 2; seu

A. tang. $\frac{1}{3}$ + A. tang. $\frac{1}{7}$ + A. tang. $\frac{1}{18}$ + A. tang. $\frac{1}{47}$ + etc. = $\frac{1}{2}$ A. tang. 2; vbi est $7 = 3 \cdot 3 - 2$; $18 = 3 \cdot 7 - 3$; $47 = 3 \cdot 18 - 7$; $123 = 3 \cdot 47 - 18$; . . . $z^{II} = 3z^{I} - z$.

Scholion 1.

§. XLV. Termini generales serierum hactenus inuestigatarum facile exhiberi possunt, cum cotangentes in serie recurrente procedant. Ita terminus generalis seriei §. XLII. summatae est $= \pm$ A. cotang. $z = \pm$ A. cot. $\frac{(\sqrt{(g^2+1)}+g)^{2x} - (\sqrt{(g^2+1)}-g)^{2x}}{2\sqrt{(g^2+1)}}$, posito $\frac{A}{2} = g$. Exinde nimirum prodit pro $x = 1$, terminus primus = A. cot. $2g$ = A. cot. A; pro $x = 2$, terminus alter $= -$ A. cot. $\frac{8g \cdot (2g^2+1) \cdot \sqrt{(g^2+1)}}{2\sqrt{(g^2+1)}}$ $= -$ A. cotang. $A(A^2+2)$; et lex terminorum hac aequatione exprimitur: $z^{II} = 2(2g^2+1)z^{I} - z = (A^2+2)z^{I} - z$. Pro exemplo EVLERI (§. XLIII. 2.) est $g = 1$, et terminus generalis $= \pm$ A. cotang. $\frac{(\sqrt{2}+1)^{2x} - (\sqrt{2}-1)^{2x}}{2\sqrt{2}}$ $= \pm$ A. cot. $\frac{(3+2\sqrt{2})^{x} - (3-2\sqrt{2})^{x}}{2\sqrt{2}}$.

Scho-

Scholion 2.

§. XLVI. 1) Generatim casus hactenus expositus $n = o$, idem est ac si in formulis (§. XXXI. XXXII.) ponatur $c = o$. Tum est $b = \frac{E a^2}{(E-1)^2}$, et haec prodit summatio:

$$A.\,tang. \frac{a}{E + bE^{-1}} + A.\,t. \frac{a}{E^2 + bE^{-2}} + A.\,t. \frac{a}{E^3 + bE^{-3}} \ldots + A.\,t. \frac{a}{E^x + bE^{-x}}$$

$= A.\,t. \frac{a}{E-1} - A.\,t. \frac{a}{E^x(E-1)} = A.\,t. \frac{a(E^x - 1):(E-1)}{E^x + b:E}$. Summa seriei infinitae est $= A.\,t. \frac{a}{E-1}$, si $E > 1$; eadem $= A.\,t. \frac{1-E}{a}$, si $E < 1$. Sit $\frac{a}{E-1} = \frac{1}{\alpha}$, erit

$$A.\,tang. \frac{\alpha(E-1)}{\alpha^2 E + 1} + A.\,t. \frac{\alpha(E-1)}{\alpha^2 E^2 + E^{-1}} + A.\,t. \frac{\alpha(E-1)}{\alpha^2 E^3 + E^{-2}} \ldots + A.\,t. \frac{\alpha(E-1)}{\alpha^2 E^x + E^{1-x}}$$

$= A.\,cotang.\, \alpha - A.\,cotang.\, \alpha E^x$.

2) Posito $E = -\varepsilon$, summatio modo inuenta in hanc abit:

$$A.\,tang. \frac{\alpha(\varepsilon+1)}{\alpha^2\varepsilon - 1} - A.\,t. \frac{\alpha(\varepsilon+1)}{\alpha^2\varepsilon^2 - \varepsilon^{-1}} + A.\,t. \frac{\alpha(\varepsilon+1)}{\alpha^3\varepsilon^3 - \varepsilon^{-2}} \ldots \pm A.\,t. \frac{\alpha(\varepsilon+1)}{\alpha^2\varepsilon^x - \varepsilon^{1-x}}$$

$= A.\,cotang.\, \alpha \pm A.\,cot.\, \alpha\varepsilon^x$; signo superiori pro x pari, inferiori pro impari sumto.

3) Sit $\varepsilon = \alpha^2$, prodibit:

$$A.\,t. \frac{\alpha^2+1}{\alpha(\alpha^2 - \alpha^{-2})} - A.\,t. \frac{\alpha^2+1}{\alpha(\alpha^4 - \alpha^{-4})} + A.\,t. \frac{\alpha^2+1}{\alpha(\alpha^6 - \alpha^{-6})} \ldots \pm A.\,t. \frac{\alpha^2+1}{\alpha(\alpha^{2x} - \alpha^{-2x})}$$

$= A.\,cotang.\, \alpha \pm A.\,cotang.\, \alpha^{2x+1}$. Posito $\alpha = \sqrt{(g^2+1)} + g$, hinc emanat summatio §. XLV, scilicet $S. \pm A.\,tang. \frac{2\sqrt{(g^2+1)}}{(\sqrt{(g^2+1)}+g)^{2x} - (\sqrt{(g^2+1)}-g)^{2x}} =$
$A.\,tang.\, (\sqrt{(g^2+1)} - g) \pm A.\,tang.\, (\sqrt{(g^2+1)} - g)^{2x+1}$, vbi est
$A.\,tang.\, (\sqrt{(g^2+1)} - g) = \frac{1}{2} A.\,cotang.\, g$.

Exposito iam *primo* casu, quo cotangentes in serie recurrente procedunt, transeamus ad *alterum* casum (§. XXXIX.), cum eae aequentur quadratis, eorumue aequemultiplis, quorum radices istiusmodi seriem constituunt.

Huc spectat sequens

THEOREMA PARTICVLARE 2.

§. XLVII. *Summa seriei*

$$A.\,tang. \frac{a}{(e+\gamma e^{-1})^2} + A.\,t. \frac{a}{(e^2+\gamma e^{-2})^2} + A.\,t. \frac{a}{(e^3+\gamma e^{-3})^2} + \ldots$$

$$+ A.\,t. \frac{a}{(e^x+\gamma e^{-x})^2} \textit{ est } = A.\,t. \frac{a:(e^2-1)}{1+2\gamma:(e^2+1)} - A.\,t. \frac{a:(e^2-1)}{e^{2x}+2\gamma:(e^2+1)},$$

posito $a = \frac{\gamma(e^2-1)^2}{e(e^2+1)}$.

Demon-

Demonstratio.

In formulis §. XXXI. XXXII. ponatur $c = 2\sqrt{b}$, vel $b = \frac{c^2}{4}$, tum aequatio conditionalis in hanc abit: $\frac{c^2}{4E} = \frac{c^2}{(E+1)^2} + \frac{a^2}{(E-1)^2}$, vel $\frac{c^2(E-1)^2}{4E(E+1)^2} = \frac{a^2}{(E-1)^2}$, et, posito $E = e^2$, $c = 2\gamma$, $\frac{\gamma(e^2-1)}{e(e^2+1)} = \frac{a}{e^2-1}$. Terminus generalis fit $=$ A. tang. $\frac{a}{e^{2x} + \frac{c^2}{4} e^{-2x} + c} =$ A. tang. $\frac{a}{(e^{x} + \gamma e^{-x})^2}$. Summa ex §. XXXII. 2. sponte innotescit.

Corollarium 1.

§. XLVIII. Summa seriei infinitae est $=$ A. tang. $\frac{\gamma(e^2-1)}{e(e^2+1)\left[1+\frac{2\gamma}{e^2+1}\right]}$ $=$ A. tang. $\frac{\gamma(e^2-1)}{e(e^2+1+2\gamma)}$. Summa seriei finitae $=$ A. tang. $\frac{\gamma(e^2-1)}{e(e^2+1+2\gamma)}$ $-$ A. tang. $\frac{\gamma(e^2-1)}{e(e^{2x}(e^2+1)+2\gamma)} =$ A. t. $\frac{\gamma(e^2-1)(e^{2x}-1)}{e^{2x+2}(e^2+1+2\gamma)+\gamma^2(e^2+1)+2\gamma e^2}$.

Corollarium 2.

§. XLIX. 1) Posito $\gamma = -1$, erit A. tang. $\frac{1}{e(e^2+1)} \cdot \left[\frac{e^2-1}{e-e^{-1}}\right]^2$ $+$ A. t. $\frac{1}{e(e^2+1)} \cdot \left[\frac{e^2-1}{e^2-e^{-2}}\right]^2$ $+$ A. t. $\frac{1}{e(e^2+1)} \cdot \left[\frac{e^2-1}{e^3-e^{-3}}\right]^3$ . . . $+$ A. t. $\frac{1}{e(e^2+1)} \cdot \left[\frac{e^2-1}{e^{x}-e^{-x}}\right]^2 =$ A. tang. $\frac{1}{e}$ $-$ A. tang. $\frac{e^2-1}{e(e^{2x}(e^2+1)-2)}$ $=$ A. t. $\frac{e(e^{2x}(e^2+1)-e^2-1)}{e^2(e^{2x}(e^2+1)-2)+e^2-1} =$ A. t. $\frac{e(e^{2x}-1)}{e^{2x+2}-1}$. Summa seriei infinitae est $=$ A. tang. $\frac{1}{e}$, si $e > 1$; $=$ A. tang. e, si $e < 1$.

2) Ponatur in hac summatione $e = g + \sqrt{(g^2-1)}$, erit $\frac{(e^2-1)^2}{e(e^2+1)} = \frac{2(g^2-1)}{g}$, hinc:

A. tang. $\frac{2(g^2-1):g}{((g+\sqrt{(g^2-1)})-(g-\sqrt{(g^2-1)}))^2}$ $+$ A. t. $\frac{2(g^2-1):g}{((g+\sqrt{(g^2-1)})^2-(g-\sqrt{(g^2-1)})^2)^2}$ $+$ etc. $+$ A. t. $\frac{2(g^2-1):g}{((g+\sqrt{(g^2-1)})^{x}-(g-\sqrt{(g^2-1)})^{x})^2} =$ A. t. $\frac{1}{g+\sqrt{(g^2-1)}}$ $=$ A. t. $g - \sqrt{(g^2-1)} = \frac{1}{2}$ A. sin. $\frac{1}{g}$.

Corol-

Corollarium 3.

§. L. Sit $\gamma = -2$, erit $a = -\frac{(e^2-1)^2}{e^2+1}$, hinc: A. tang. $\frac{1}{e^2+1} \cdot \left[\frac{e^2-1}{e-1}\right]^2$.
$+$ A. tang. $\frac{1}{e^2+1} \cdot \left[\frac{e^2-1}{e^2-e-1}\right]^2 +$ A. tang. $\frac{1}{e^2+1} \cdot \left[\frac{e^2-1}{e^3-e-2}\right]^2 + \ldots$
$+$ A. tang. $\frac{1}{e^2+1} \cdot \left[\frac{e^2-1}{e^X-e^1-X}\right]^2 =$ A. tang. $\left[\frac{e+1}{e-1}\right]\left[\frac{e^2X-1}{e^2X+1}\right]$. Summa feriei infinitae eft $=$ A. t. $\frac{e+1}{e-1}$ pro $e > 1$; $=$ A. tang. $\frac{e+1}{1-e}$ pro $e < 1$.

THEOREMA PARTICVLARE 3.

§. LI. Summa feriei infinitae A. cotang. A $+$ A. cot. B $+ \ldots +$ A. cot. z $+ \ldots$ eft $=$ A. tang. $\frac{A+B}{AB-1-\frac{1}{2}(A^2+1)(\nu^2-2-\nu\gamma(\nu^2-4))}$, quodfi radices cotangentium in ferie recurrente hac lege procedant, vt, pofito $z = \zeta\zeta$, fit $\zeta^{II} = \nu\zeta^I - \zeta$; ac infuper fuerit $\nu = \frac{A+B}{\gamma AB \pm 1}$.

Demonftratio.

1) Comparata hac ferie cum prius fummata (§. XLVII.) ponatur $\zeta = \frac{e^X + \gamma e^{-X}}{\gamma^2}$, erit $\zeta^{II} = (e + \frac{1}{e})\zeta^I - \zeta$, hinc $\nu = e + \frac{1}{e}$. Eft porro $z^{II} = (e^2 + \frac{1}{e^2})z^I - z \mp 2(e + \frac{1}{e})$, (§. XXXIII.), ob $E = e^2$, $\frac{c}{aE}(E-1)^2 = \pm 2\frac{(e^2+1)}{e}$; hinc (§. XXXIV.) $m = e^2 + \frac{1}{e^2}$, $n = \pm 2\frac{(e^2+1)}{e} = \pm 2\nu$; $m + 2 = e^2 + \frac{1}{e^2} + 2 = \frac{n^2}{4} = \nu^2$. Quare aequatio conditionalis $(m+2)(AB-1) = (A+B)(A+B+n)$ in hanc abit: $\nu^2(AB-1) = (A+B)(A+B \pm 2\nu)$. Exinde pro $n = +2\nu$, fit $\nu = \frac{A+B \pm \gamma((A+B)^2 + (A+B)^2(AB-1))}{AB-1} = \frac{(A+B)(1 \pm \gamma AB)}{AB-1} = \frac{A+B}{\gamma AB - 1}$, quippe radicibus cotangentium i. e. ζ, ζ^I, ζ^{II} pofitiue fumtis etiam quantitati v valor pofitiuus tribuendus eft. Pro $n = -2\nu$ prodit eodem modo $\nu = -\frac{(A+B)(1 \pm \gamma AB)}{AB-1} = \frac{A+B}{\gamma AB + 1}$.

2) Summa

2) Summa seriei infinitae est ex §. XXXV. $= \text{A. tang.}\ \frac{A+B}{AB-1-\frac{(A^2+1)}{E}}$, vbi $\frac{E^2+1}{E} = m = \nu^2 - 2$, hinc $E = \frac{1}{2}(\nu^2 - 2 + \nu\sqrt{(\nu^2-4)})$, et $\frac{1}{E} = \frac{\nu^2 - 2 - \nu.\sqrt{(\nu^2-4)}}{2}$. Quomodo exinde summa seriei finitae determinanda sit, ex §. XXXVI. manifestum est.

Scholion.

§. LII. Praecedentis theorematis alia quoque demonstratio, absque theoremate §. XLVII. exhiberi potest. Posito $\zeta^{II} = \nu\zeta^{I} - \zeta$, et $\nu = \frac{A+B}{\sqrt{AB}\pm 1}$, erit C, tertia cotangens, $= (\nu\sqrt{B} - \sqrt{A})^2$, et $\frac{B+C}{\sqrt{BC}\pm 1} = \frac{B+(\nu\sqrt{B}-\sqrt{A})^2}{\sqrt{B}.(\nu\sqrt{B}-\sqrt{A})\pm 1} = \frac{B+\nu^2 B - 2\nu\sqrt{BA} + A}{\nu B - \sqrt{AB}\pm 1} = \nu$, ob $B + A - \nu\sqrt{BA} = \pm\nu$. Hinc pro quibusuis terminis seriei cotangentium, z, z^{I}, erit ob $\zeta\zeta = z$, $\nu = \frac{z^{I}+z}{\zeta^{I}\zeta\pm 1}$, vel $\nu\zeta^{I}\zeta = z^{I} + z \mp \nu$.
Iam ex aequatione $\zeta^{II} = \nu\zeta^{I} - \zeta$, prodit quadrando: $\zeta^{II}\zeta^{II} = \nu^2\zeta^{I}\zeta^{I} - 2\nu\zeta^{I}\zeta + \zeta\zeta$, siue $z^{II} = \nu^2 z^{I} - 2z^{I} - 2z \pm 2\nu + z = (\nu^2 - 2)z^{I} - z \pm 2\nu$. Posito iam $m = \nu^2 - 2$, $n = \mp 2\nu$, aequationi conditionali $(m+2)(AB-1) = (A+B)(A+B+n)$ adsumta aequatione $\nu = \frac{A+B}{\sqrt{AB}\pm 1}$ satisfieri sponte liquet. Quare iam praecedens theorema redit ad formulam §. XXXV.

Corollarium 1.

§. LIII. Ex $\nu = \frac{A+B}{\sqrt{AB}\pm 1}$ fit $B - \nu\sqrt{A}.\sqrt{B} = -A \pm \nu$, indeque $\sqrt{B} = \nu\sqrt{A} \pm \sqrt{(\nu^2 A - 4A \pm 4\nu)}$; $B = \frac{(\nu^2-2)A \pm 2\nu \pm \nu\sqrt{((\nu^2-4)A^2 \pm 4\nu A)}}{2}$.
Hinc vt B euadat numerus integer, quantitas $(\nu^2 - 4)A^2 \pm 4\nu A$ quadrato aequanda est. Id quidem fit, signo $+$ adhibito, si ponatur $A = \nu$, vnde est $B = \frac{(\nu^2-2)\nu + 2\nu + \nu^3}{2} = \nu^3$.
Summa est $= \text{A. tang.}\ \frac{\nu + \nu^3}{\nu^4 - 1 - \frac{1}{2}(\nu^2+1)(\nu^2 - 2 - \nu\sqrt{(\nu^2-4)})} =$
$\text{A. t.}\ \frac{\nu}{\nu^2 - 1 - \frac{1}{2}(\nu^2 - 2 - \nu\sqrt{(\nu^2-4)})} = \text{A. t.}\ \frac{2}{\nu + \sqrt{(\nu^2-4)}} = \text{A. t.}\ \frac{\nu}{2} - \sqrt{\left[\frac{\nu^2}{4} - 1\right]}$
$= \frac{1}{2}$

$= \frac{1}{2}$ Arc. sin. $\frac{2}{\nu}$; quippe posito $\frac{2}{\nu} =$ sin. λ, habetur tang. $\frac{1}{2}\lambda = \frac{1 - \text{cos.}\,\lambda}{\text{sin.}\,\lambda} =$
$\frac{\frac{\nu}{2} - \sqrt{\left(1 - \frac{4}{\nu^2}\right)}}{2:\nu} = \frac{\nu}{2} - \sqrt{\left(\frac{\nu^2}{4} - 1\right)}$. Exinde, posito $A = \nu = \mathfrak{A}^2$, haec prodit

Summatio.

$\text{A.cot.}\,\mathfrak{A}^2 + \text{A.cot.}\,\mathfrak{A}^6 + \text{A.cot.}\,\mathfrak{A}^2(\mathfrak{A}^4 - 1)^2 \ldots + \text{A.cot.}\,\zeta\zeta + \ldots = \frac{1}{2}\,\text{A. sin.}\,\frac{2}{\mathfrak{A}\mathfrak{A}}$;
vbi cotangentes sunt quadrata, quorum radices formant seriem recurrentem talem, vt sit $\zeta^{II} = \mathfrak{A}\mathfrak{A}.\zeta^{I} - \zeta$. Cotangentes ipsae hac lege procedunt, vt, existente $z = \zeta\zeta$, habeatur $z^{II} = (\mathfrak{A}^4 - 2)\,z^{I} - z + 2\,\mathfrak{A}^2$.

Exemplum.

§. LIV. Posito $\mathfrak{A} = 2$, erit ob A. sin. $\frac{1}{2} = \frac{\pi}{6}$, A. tang. $\frac{1}{4}$ + A. tang. $\frac{1}{64}$ + A. tang. $\frac{1}{900}$ + A. tang. $\frac{1}{12544}$ + etc. $= \frac{\pi}{4}$; vbi denominatores sunt quadrata, quorum radices hanc progressionem constituunt: 2, 8, 30, 112, 418, 1560, ... ζ ... Est autem $30 = 4.8 - 2$; $112 = 4.30 - 8$; $418 = 4.112 - 30$; ... $\zeta^{II} = 4\zeta^{I} - \zeta$; et terminus x^{tus} $\zeta = \frac{(2+\sqrt{3})^x - (2-\sqrt{3})^x}{\sqrt{3}}$; indeque haec summatio etiam sub Coroll. 1. §. XLIX. continetur, posito $e = 2 + \sqrt{3}$, siue $g = 2$. Hoc exemplum, cuius supra iam mentio est iniecta (§. II.) extat apud EVLERVM l. c. §. 20. Apparet ex forma generaliore, cui illud subest, innumera similia deriuari posse.

Corollarium 2.

§. LV. 1) Sit $A = m\mathfrak{A}\mathfrak{A} = \nu$ (§. LIII.), erit $B = m^3\mathfrak{A}^6$. Hinc prodeunt cotangentes aeque multiplae secundum numerum m quadratorum $(\mathfrak{A})^2$, $(m\mathfrak{A}^3)^2$, $(m^2\mathfrak{A}^5 - \mathfrak{A})^2 \ldots u^2$, seu secundum $m\mathfrak{A}^2$ quadratorum 1, $(m\mathfrak{A}^2)^2$, $(m^2\mathfrak{A}^4 - 1)^2 \ldots v^2$; eritque A. cotang. $m(\mathfrak{A})^2$ + A. cot. $m(m\mathfrak{A}^3)^2$ + A. cot. $m(m^2\mathfrak{A}^5 - \mathfrak{A})^2 \ldots$ + A. cot. $m(u)^2$ + etc. siue A. cotang. $m\mathfrak{A}^2.(1)^2$ + A. cot. $m\mathfrak{A}^2(m\mathfrak{A}^2)^2$ + A. cot. $m\mathfrak{A}^2(m^2\mathfrak{A}^4 - 1)^2 \ldots$ + A. cot. $m\mathfrak{A}^2(v)^2$ + etc. $= \frac{1}{2}$ A. sin. $\frac{2}{m\mathfrak{A}^2}$, posito $u^{II} = m\mathfrak{A}^2 u^{I} - u$ seu $v^{II} = m\mathfrak{A}^2 v^{I} - v$. Exempli gratia posito $\mathfrak{A} = 1$, $m = 2$, habetur A. cotang. 2.1 + A. cot. 2.4 + A. cot. $2.9 \ldots$ + A. cot. $2.x^2$ etc. $= \frac{\pi}{4}$, vti supra iam inuentum est (§. XXVII.).

2) Generatim cum ob expressionem quantitatis ν (§. LII.), $\sqrt{AB}$ rationalis esse debeat, fit $AB = K^2$, vel $\frac{AB}{A^2} = \frac{B}{A} = \frac{K^2}{A^2}$. Fractione $\frac{K}{A}$ ad minimam denomina-

tionem

tionem reducta $= \frac{\mathfrak{B}}{\mathfrak{A}}$, erit $\frac{B}{A} = \frac{\mathfrak{B}^2}{\mathfrak{A}^2}$, et $B = r\mathfrak{B}^2$, $A = r\mathfrak{A}^2$, existentibus $\mathfrak{B}$, $\mathfrak{A}$ numeris inter se primis, et r numero integro. Hinc et reliquae cotangentes aequemultiplae secundum r numerorum quadratorum erunt, ob $z^{II} = \nu z^I - z$. Porro est $\nu = \frac{r(\mathfrak{A}^2 + \mathfrak{B})^2}{r\mathfrak{A}\mathfrak{B} \pm 1}$, $\nu r \mathfrak{A}\mathfrak{B} \pm \nu = r(\mathfrak{A}^2 + \mathfrak{B}^2)$, hinc ν multiplum esse debet numeri r, $= rN$; eritque $2\mathfrak{B} = rN\mathfrak{A} \pm \sqrt{((r^2N^2 - 4)\mathfrak{A}^2 \pm 4N)}$.

Alia exempla numerica.

§. LVI. Formula §. LIII. inuenta est, posito $\nu = \frac{A+B}{\sqrt{AB+1}}$. Iam si ponatur $\nu = \frac{A+B}{\sqrt{AB-1}}$, nouae prodeunt summationes, vti sequentia exempla declarant:

1) Sit $A = 1$, $B = \mathfrak{B}^2$, erit $\nu = \frac{1+\mathfrak{B}^2}{\mathfrak{B}-1} = \mathfrak{B} + 1 + \frac{2}{\mathfrak{B}-1}$, quare ponendum est vel $\mathfrak{B} = 2$, vel $\mathfrak{B} = 3$. Inde duplex oritur

Summatio.

a) $A.\,\text{tang.}\,1 + A.\,t.\,\frac{1}{(2)^2} + A.\,t.\,\frac{1}{(9)^2} + A.\,t.\,\frac{1}{(43)^2} + \text{etc.} + A.\,t.\,\frac{1}{\zeta\zeta} + \text{etc.}$

$= A.\,\text{tang.}\,\frac{1}{\sqrt{21}-4}$.

b) $A.\,\text{tang.}\,1 + A.\,t.\,\frac{1}{(3)^2} + A.\,t.\,\frac{1}{(14)^2} + A.\,t.\,\frac{1}{(67)^2} + \text{etc.} + A.\,t.\,\frac{1}{\zeta\zeta} + \text{etc.}$

$= A.\,\text{tang.}\,\frac{2}{\sqrt{21}-3}$.

Pro vtraque serie est $\zeta^{II} = 5\zeta^I - \zeta$; e. g. $9 = 5 \cdot 2 - 1$; $14 = 5 \cdot 3 - 1$. Prior etiam sic exhiberi potest: $A.\,\text{tang.}\,\frac{1}{(2)^2} + A.\,t.\,\frac{1}{(9)^2} + A.\,t.\,\frac{1}{(43)^2} + \text{etc.}$

$= A.\,\text{tang.}\,\frac{2}{\sqrt{21}+3}$.

2) Sit $A = 2$, erit $\nu = \frac{2+B}{\sqrt{2B-1}}$, quare ponendum est $B = 2\varrho^2$, hinc $\nu = \frac{2\varrho^2+2}{2\varrho-1} = \varrho + \frac{\varrho+2}{2\varrho-1}$, vnde $\varrho = 1$, vel $= 3$. Ex $\varrho = 3$ sequens oritur

Summatio.

$A.\,\text{tang.}\,\frac{1}{2(1)^2} + A.\,t.\,\frac{1}{2(3)^2} + A.\,t.\,\frac{1}{2(11)^2} + A.\,t.\,\frac{1}{2(41)^2} + A.\,t.\,\frac{1}{2(153)^2} + \text{etc.}$
$+ A.\,\text{tang.}\,\frac{1}{2\zeta\zeta}\,\text{etc.} = A.\,\text{tang.}\,\frac{1}{\sqrt{3}} = \frac{\pi}{6}$.

Hoc

Hoc exemplum extat apud EVLERVM l. c. pag. 51. (cf. §. II.). Lex ibi non expressa, quam sequuntur numeri 1, 3, 11, ... ζ.., his aequationibus continetur: $11 = 4 \cdot 3 - 1$; $41 = 4 \cdot 11 - 3$; ... $\zeta^{II} = 4\zeta^{I} - \zeta$. Cotangentes ipsae, seu numeri 2, 18, 242, 3362, ... $z = 2\zeta\zeta$, ... hanc legem obseruant, vt sit $242 = 14 \cdot 18 - 2 - 8$; $3362 = 14 \cdot 42 - 18 - 8$; ... $z^{II} = 14 z^{I} - z - 8$. Posito $\varrho = 1$ eadem summatio recurrit.

3) Sit $A = 3$, erit $\nu = \frac{3+B}{\sqrt{3B-1}}$, quare ponendum est $B = 3\varrho^2$, vnde $\nu = \frac{3+3\varrho^2}{3\varrho - 1} = \varrho + \frac{\varrho + 3}{3\varrho - 1}$, vbi ϱ vel $= 1$, vel $= 2$ esse debet. Posito $\varrho = 2$, haec prodit

Summatio.

$$A.\,\text{tang.}\,\frac{1}{3(1)^2} + A.\,t.\,\frac{1}{3(2)^2} + A.\,t.\,\frac{1}{3(5)^2} + A.\,t.\,\frac{1}{3(13)^2} + \ldots + A.\,t.\,\frac{1}{3\zeta\zeta} + \text{etc.}$$

$= A.\,\text{tang.}\,\frac{1}{\sqrt{5}}$, vbi est $\zeta^{II} = 3\zeta^{I} - \zeta$. Positio $\varrho = 1$ haud nouam seriem praebet.

Scholion.

§. LVII. Transeundum iam est ad *tertium* casum supra §. XXXIX. commemoratum, cum cotangentes Arcuum, seriei summandae terminorum, aequantur numeris integris, qui in serie recurrente quacunque *affecta* secundi ordinis procedunt. Aequatio conditionalis $m + 2 = \frac{(A+B)(A+B+n)}{AB-1}$ duplici ratione considerari potest, 1) dum A et B pro cognitis habentur, indeque m et n debito modo definiuntur; vel 2) dum datis quantitatibus m et n, vnaue earum, determinandae sunt A et B, seu earum alterutra. Prior consideratio sequens suppeditat

Theorema particulare 4.

§. LVIII. Summa seriei infinitae $A.\,\text{tang.}\,\frac{1}{A} + A.\,t.\,\frac{1}{B} + A.\,t.\,\frac{1}{C} + \text{etc.} + A.\,t.\,\frac{1}{z} + \ldots$ est $= A.\,\text{tang.}\,\frac{2}{2A - r(A^2+1) + (A^2+1)\sqrt{r\left(r - \frac{4}{A+B}\right)}}$, si cotangentes A, B, ... z, z^{I}, z^{II} ... hac lege procedant, vt sit z^{II}

$$= ((A+B)r - 2)z^{I} - z - r(AB-1) + A + B,$$

denotante r numerum quemuis integrum.

Demonstratio.

1) Simplicissimus modus, aequationi conditionali pro datis A et B (§. LVII. 1.) satisfaciendi, est is, vt ponatur $\frac{A+B+n}{AB-1}$ = numero integro $= r$; vnde fit

$n = (AB-1)r - A - B$; $m + 2 = (A+B)r$, vel $m = (A+B)r - 2$.

2) Sum-

2) Summa feriei eft ex §. XXXVI. $\frac{2(A+B)}{2AB-2-(A^2+1)m+(A^2+1)\surd(m^2-4)}$

$= A.\ \text{tang.}\ \frac{2(A+B)}{2AB-2-(A^2+1)(A+B)r+2(A^2+1)+(A^2+1)\surd(m^2-4)}$

$= A.\ \text{tang.}\ \frac{2}{2A-r(A^2+1)+(A^2+1)\surd\frac{(m+2)(m-2)}{(A+B)(A+B)}}$, vnde expreffio theorematis fponte fequitur.

Corollarium 1.

§. LIX. In ferie praecedente reperitur cotangens tertii membri $C = r(B^2+1) - B$, ex lege τῶν z. Si a fumma primum membrum A. tang. $\frac{1}{A}$ fubtrahitur, remanet

A. tang. $\frac{2}{B-A+\surd\left[(B+A)(B+A-\frac{4}{r})\right]}$. Quod fi igitur terminus fecundus tanquam primus confideretur, et ponatur A pro B, a pro A, haec obtinetur

Summatio.

$$A.\ \text{tang.}\ \frac{1}{A} + A.\ \text{tang.}\ \frac{1}{r(A^2+1)-A} + \text{etc.} + A.\ \text{tang.}\ \frac{1}{z} + \text{etc.}$$

$$= A.\ \text{tang.}\ \frac{2}{A-a+\surd\left[(A+a)(A+a-\frac{4}{r})\right]},$$

fi fuerit $z^{II} = (r(A+a)-2)z^{I} - z - r(Aa-1) + A + a.$

Corollarium 2.

§. LX. Cum in binis fummationibus modo demonftratis tres quantitates indeterminatae occurrant, A, B, r; A, a, r; quarum quaeuis numero cuilibet integro aequari poteft, innumerae oriuntur feries fummabiles Arcuum, quorum cotangentes numeris integris expreffae in ferie recurrente affecta fecundi ordinis procedunt. Ceterum aequationi conditionali pro datis A et B plerumque aliis infuper modis fatisfieri poffe, quam pofitione §. LVIII. affumta, manifeftum eft. Quodfi enim diuidendo $A+B$ et $AB-1$ per factorem communem maximum prodeant quotientes f, g, ponendum eft $n \pm gr - A - B$. Hinc maior adhuc varietas ferierum fummabilium oritur.

Corollarium 3.

§. LXI. Aequatio conditionalis ita transformari poteft, vt m et n exprimantur per A et B et nouam quantitatem ϱ. Eft nimirum $(m+2)A = \frac{(A^2+1)(A+B+n)}{AB-1} + A + B + n.$

Hinc

Hinc $\frac{(A^2+1)(A+B+n)}{AB-1}$ numero integro ϱ aequari debet. Inde fit $(m+2)A = \varrho+A+B+n$; $n=(m+1)A-B-\varrho$; porro $(m+2)A=\varrho+\frac{\varrho(AB-1)}{A^2+1}$, seu $m+2=\frac{\varrho(A+B)}{A^2+1}$. Sumto igitur pro ϱ numero integro tali, vt $\varrho(A+B)$ per A^2+1 diuidi queat, erit $m=\frac{\varrho(A+B)}{A^2+1}-2$; $n=(m+1)A-B-\varrho$. Summa seriei infinitae reperitur $=$ A. tang. $\frac{2}{2A-\varrho+\sqrt{(\varrho(\varrho-4(\frac{A^2+1}{A+B})))}}$. Prouti iam vel $\frac{\varrho}{A^2+1}$ vel $\frac{A+B}{A^2+1}=$ numero integro $=r$ ponitur, peruenitur ad summationem §. LVIII, vel ad alteram §. LIX.

Corollarium 4.

§. LXII. 1) Quod si aequatio conditionalis *altero* respectu (§. LVII. 2.) consideretur, et quidem primo m et A pro cognitis habeantur, tum ob $\varrho=\frac{(m+2)(A^2+1)}{A+B}$ numerator $(m+2)(A^2+1)$ quocunque modo in binos factores f, F, resoluendus est, et alteri eorum f aequandus denominator; quo facto habetur $B=f-A$, $\varrho=F$; hinc $n=(m+2)A-F-f$; et summa $=$ A. tang. $\frac{2}{2A-F+\sqrt{F(F-4\frac{(A^2+1)}{f})}}$. Hinc apparet, pro datis m et A certum tantum numerum serierum summabilium oriri, haud innumeras, vt priori casu, assumtis A et B (§. LX.).

2) Exempli gratia, posito $f=A^2+1$ prodit: A. tang. $\frac{1}{A}+$ A. t. $\frac{1}{A^2-A+1}$ $+$ etc. $+$ A. tang. $\frac{1}{z}+$ etc. $=$ A. tang. $\frac{2}{2A-m-2+\sqrt{(m^2-4)}}$, vbi est $z^{II}= mz^{I}-z-(m+2)(A-1)+A^2+1$. Pro $f=m+2$, habetur A. tang. $\frac{1}{A}$ $+$ A. tang. $\frac{1}{m+2-A}+$ etc. $+$ A. t. $\frac{1}{z}+$ etc. $=$ A. t. $\frac{2}{-(A-1)^2+(A^2+1)\sqrt{\frac{m-2}{m+2}}}$, supposito iterum $z^{II}=mz^{I}-z-(m+2)(A-1)+A^2+1$. Quarum summationum prior etiam ex §. LIX, altera ex §. LVIII. consequitur, sumendo $r=1$.

Scholion.

§. LXIII. Si vel n et A, vel m et n pro cognitis fumantur, vt aequationem conditionalem altero refpectu (§. LVII. 2.) confiderare pergamus, tum eam aequationem alio modo tractare conuenit. Eius nimirum refolutio praebet:

$$B = \frac{mA - n + \sqrt{(m^2A^2 - 2mAn + n^2 - 4(A^2 + nA + m + 2))}}{2}$$

Quam igitur formulam ad rationalitatem perducere oportet. Quod negotium modo generali abfolui vix poteft: fi quidem poftuletur folutio vniuerfalis omnes valores idoneos trium quantitatum m, A, n, complectens. Etenim fi 1) A fpectetur tanquam quantitas indeterminata, tum methodi notae (cf. infra §. LXXVII. LXXVIII.) fupponunt, vnum illius valorem cognitum effe, qui formulam quadraticam rationalem reddat, ex quo deinceps innumeros alios valores deriuare licet. At is ipfe valor vnde eliciendus fit, haud liquet: deinde ex vno tali valore haud plures easque reuera diuerfas feries fummabiles Arcuum prodire, infra docebitur (§. LXXXVI.). Quodfi 2) quantitates m vel n determinandae fint pro datis A, n vel A, m, tum, quoniam eae coëfficientes habent A^2 et 1, i. e. quadratos, methodi vfitatae hoc cafu ne adhiberi quidem poffunt (cf. §. LXXVII.). Quare in eo acquiefcendum videtur, vt inter quantitates A, m, n, tales relationes supponantur, pro quibus formula quadratica fponte rationalis prodeat. Quem in finem duae fe offerunt pofitiones, quas iam euoluamus, cum fcilicet fuerit vel 1) $A^2 + nA + m + 2 = 0$; vel 2) $-2mAn + n^2 - 4(A^2 + nA + m + 2) = 0$. Prior hypothefis ad eum cafum fpectat, quo dantur A et n; alterius ope pro dato m innumeri valores quantitatum A, B, n, certo ordine procedentes reperiuntur. Ex *illa* fequens petitur

THEOREMA PARTICVLARE 5.

§. LXIV. Summa feriei infinitae A. cotang. A + A. cot. B + etc. + A. cot. z + etc. eft $=$ A. cotang. $\frac{A}{2} + \sqrt{\left(\frac{A^2}{4} - \frac{A}{N-A}\right)}$, pofito $B = ((N-A)A - 2).A + N$, et $z^{II} = ((N-A)A - 2)z^{I} - z + N$.

Demonftratio.

1) Pofito $A^2 + nA + m + 2 = 0$, et $n = -N$, erit $m = (N - A)A - 2$, hinc (§. LVII.) $B = mA - n = ((N-A) - 2)A + N = N(A^2 + 1) - A(A^2 + 2)$; porro lex cotangentium hac aequatione exprimitur: $z^{II} = mz^{I} - z - n = ((N - A)A - 2)z^{I} - z + N$.

2) Summa feriei ex §. XXXV. eft $=$ A. tang. $\frac{A+B}{AB - 1 - (A^2+1)\left(\frac{m - \sqrt{(m^2 - 4)}}{2}\right)}$.

Eft autem $\sqrt{(m^2 - 4)} = \sqrt{((N - A)^2 A^2 - 4(N - A)A)}$;

A + B

$A+B=(N-A)(A^2+1)$; $2AB-2-(A^2+1)m=2AB-2-(A^2+1)(N-A)A+2(A^2+1)=2A(B+A)-(A^2+1)(N-A)A=A(N-A)(A^2+1)$. Inde fit summa $=$ A. t. $\frac{2(N-A)}{A(N-A)+\sqrt{((N-A)^2A^2-4(N-A)A)}}$

$=$ A. tang. $\frac{1}{\frac{A}{2}+\sqrt{\left(\frac{A^2}{4}-\frac{A}{N-A}\right)}}$.

Corollarium 1.

§. LXV. Si N negatiuum valorem habeat, vel n positiuum, tum B negatiuum valorem obtinet: nec non reliquarum cotangentium signa alternantur. Inde haec oritur

Summatio.

A. cotang. A $-$ A. cotang. B $+$ A. cotang. C . . . $\pm$ A. cotang. z $\mp$ etc.

est $=$ A. cotang. $\frac{A}{2}+\sqrt{\left(\frac{A^2}{4}+\frac{A}{n+A}\right)}$, si fuerit $B=((n+A)A+2)A-n$, et $z^{II}=((n+A)A+2)z^I-z\mp n$, in qua posteriori aequatione signum superius pro affirmatiuis z vel z^{II}, inferius pro negatiuis obtinet.

Corollarium 2.

§. LXVI. 1) Posito $N=2A$, ex summatione §. LXIV. prodit summatio §. LIII. Posito $n=0$, ex altera summatione §. LXV. sequitur summatio §. XLII.

2) Posito $A=1$ (§. LXIV.), erit A. tang. 1 $+$ A. t. $\frac{1}{2N-3}$ $+$ A. tang. $\frac{1}{2N^2-8N+8}$ $+$ etc. $+$ A. tang. $\frac{1}{z}$ $+$ etc. $=$ A. tang. $\frac{2}{1+\sqrt{\left(\frac{N-5}{N-1}\right)}}$, vbi est $z^{II}=(N-3)z^I-z+N$.

3) Posito $A=2$, prodit A. tang. $\frac{1}{2}$ $+$ A. t. $\frac{1}{5N-12}$ $+$ A. t. $\frac{1}{10N^2-53N+70}$ $+$ etc. $+$ A. tang. $\frac{1}{z}$ $+$ etc. $=$ A. tang. $\frac{1}{1+\sqrt{\left(\frac{N-4}{N-2}\right)}}$, vbi est $z^{II}=(2N-6)z^I-z+N$.

4) Sit $A=4$, erit A. tang. $\frac{1}{4}$ $+$ A. t. $\frac{1}{17N-72}$ $+$ etc. $+$ A. tang. $\frac{1}{z}$ $+$ etc. $=$ A. tang. $\frac{1}{2+2\sqrt{\left(\frac{N-5}{N-4}\right)}}$, vbi est $z^{II}=(4N-18)z^I-z+N$.

Scho-

Scholion.

§. LXVII. Summationes §§. LXIV. LXV. etiam ex ſummatione §. LIX. deduci poſſunt, ponendo $a = 0$, $r = N - A$ vel $= -n - A$. Quare cum r cuiuis numero integro, ſiue affirmatiuo, ſiue negatiuo aequari poſſit, Theorema §. LXIV. haberi poteſt pro Corollario theorematis §. LVIII, et ſeries ſummabiles, quas illud ſuppeditat, iam comprehenduntur ſub ſummatione §. LIX. Aequatio conditionalis $m + 2 = \frac{(A+B)(A+B+n)}{AB-1}$ pro datis A et n ita etiam reſolui poteſt, vt ponatur $B = 0$; tum eſt eſt $m + 2 = -A(A+n)$. Hinc prodit ſeries $A.t.\frac{1}{A} + A.t.\frac{1}{0} + A.t.\frac{1}{-A-n}$ . . . . quae omiſſis binis prioribus terminis, et poſito N loco $-n$, A loco $N - A$, in ſeriem §. LXIV. ſummatam abit.

THEOREMA PARTICVLARE 6.

§. LXVIII. Summa ſeriei infinitae A. cotang. A + A. cot. B + etc. + A. cot. z + etc. eſt $= A.\ \text{tang.}\ \frac{A+B}{AB - 1 - (A^2+1)\left(\frac{m - \sqrt{(m^2-4)}}{2}\right)}$, ſi fuerit $z^{II} = mz^{I} - z + 2\nu$; $m + 2 = \frac{A^2 - \nu^2}{A\nu - 1}$; et B vel $= mA + \nu$, vel $= \nu$.

Demonſtratio.

Hoc theorema ſponte conſequitur ex altera poſitione §. LXIII. commemorata: $-2mAn + n^2 - 4(A^2 + nA + m + 2) = 0$. Hinc n numero pari aequandus eſt. Sit igitur $n = -2\nu$, et erit $m = \frac{A^2 - \nu^2 - 2\nu A + 2}{A\nu - 1}$, vel $m + 2 = \frac{A^2 - \nu^2}{A\nu - 1}$. Ob $B = \frac{mA - n \pm mA}{2}$ duplex valor prodit: 1) $B = mA + \nu$; 2) $B = \nu$. Summae expreſſio ex §. XXXV. petitur.

Corollarium 1.

§. LXIX. 1) Si m negatiuum valorem habeat, tum in ſummae expreſſione pro $-\sqrt{(m^2-4)}$ ponendum eſt $+\sqrt{(m^2-4)}$, quoniam §. XXXIV. $E > 1$, vel $\frac{1}{E} < 1$ ſupponitur.

2) Poſito $B = mA + \nu$, ſummae formula, calculis rite peractis in hanc transformatur: S. A. cotang. $z = A.\ \text{tang.}\ \frac{2}{A - \nu + \frac{(A\nu - 1)}{A - \nu}.\sqrt{(m^2-4)}}$

$= A.$

$= \text{A. t.} \dfrac{2}{A - \nu + (A + \nu)\sqrt{\left(\dfrac{m-2}{m+2}\right)}}$. Similiter si fuerit B $= \nu$, prodit summa

$$= \text{A. t.} \frac{2}{3A + \dfrac{\nu - A^3}{A\nu - 1} + \dfrac{(A^2+1)}{A+\nu} \cdot \sqrt{(m^2-4)}} = \text{A. t.} \frac{2}{2A + \dfrac{(A^2+1)(\nu - A)}{A\nu - 1}\left(1 - \sqrt{\left(\dfrac{m-2}{m+2}\right)}\right)}.$$

Corollarium 2.

§. LXX. 1) Sit $\nu = 1$, erit $m + 2 = \dfrac{A^2 - 1}{A - 1} = A + 1$, vel $m = A - 1$. Posito iam $B = mA + \nu = A^2 - A + 1$, est summa (§. LXI. 2.) $=$

$$\text{A. t.} \frac{2}{A - 1 + \sqrt{((A-1)^2 - 4)}} = \text{A. t.} \frac{A-1}{2} - \sqrt{\left(\frac{(A-1)^2}{4} - 1\right)} = \tfrac{1}{2}\,\text{A. sin.} \frac{2}{A-1}.$$

Inde haec prodit

Summatio.

Summa seriei infinitae $\text{A. tang.} \dfrac{1}{A} + \text{A. tang.} \dfrac{1}{(A-1)A+1} + \text{etc.} + \text{A. t.} \dfrac{1}{z} + \text{etc.}$ est $= \tfrac{1}{2}\,\text{A. sin.} \dfrac{2}{A-1}$; vbi denominatores ea lege procedunt, vt sit $z^{II} = (A-1)z^{I} - z + 2$.

2) Addito seriei (1) eiusque summae, $\text{A. tang.}\ 1 = \dfrac{\pi}{4}$, ob $\dfrac{\pi}{2} + \text{A. sin.} \dfrac{2}{A-1} = \pi - \text{A. cos.} \dfrac{2}{A-1}$, consequitur inde haec summatio:

$$\text{A. tang.} \frac{1}{1} + \text{A. tang.} \frac{1}{A} + \text{A. tang.} \frac{1}{(A-1)A+1} + \text{etc.} + \text{A. t.} \frac{1}{z} + \text{etc.}$$

$= \dfrac{\pi}{2} - \tfrac{1}{2}\,\text{A. cos.} \dfrac{2}{A-1}$, vbi itidem est $z^{II} = (A-1)z^{I} - z + 2$, quam legem iam tertius denominator $(A-1)A+1$ obseruat. Eadem summatio ex theoremate §. LXVIII. immediate prodit, ponendo $A = 1$, $\nu = 1$; vnde aequatio conditionalis in identicam abit, et quantitas m indeterminata manet. Positio $B = \nu$, loco $B = mA + \nu$ (1.), haud nouam summationem praebet.

Corollarium 3.

§. LXXI. Sumatur A negatiue, ex §. LXX. 1. haec oritur summatio:

$$\text{A. tang.} \frac{1}{A} - \text{A. tang.} \frac{1}{A(A+1)+1} + \text{etc.} \pm \text{A. tang.} \frac{1}{z} \mp \text{etc.} = \tfrac{1}{2}\,\text{A. sin.} \frac{2}{A+1};$$

quae addito vtrinque $-$ A. tang. 1, ob $\dfrac{\pi}{2} - \text{A. sin.} \dfrac{2}{A+1} = \text{A. cos.} \dfrac{2}{A+1}$, in hanc abit:

A. tang.

$\text{A. tang.}\,\frac{1}{1} - \text{A. tang.}\,\frac{1}{A} + \text{A. tang.}\,\frac{1}{A^2+A+1} + \text{etc.} \pm \text{A. tang.}\,\frac{1}{z} \mp \text{etc.}$

$= \frac{1}{2}\,\text{A. cos.}\,\frac{2}{A+1}$, vbi denominatores constituunt seriem recurrentem affectam, pro qua est $z^{II} = (A+1)z^{I} - z \pm 2$; signo superiori obtinente pro terminis seriei, z^{II}, z, signo $+$ instructis, inferiori contra.

Exempla.

§. LXXII. 1) Sit $A = 5$, erit ex (§. LXX. 1.) A. tang. $\frac{1}{5}$ + A. tang. $\frac{1}{21}$ + A. tang. $\frac{1}{81}$ + A. tang. $\frac{1}{305}$ + etc. $= \frac{\pi}{12}$; siue A. tang. $\frac{1}{1}$ + A. t. $\frac{1}{5}$ + A. t. $\frac{1}{21}$ + A. tang. $\frac{1}{81}$ + A. tang. $\frac{1}{305}$ + etc. $= \frac{\pi}{3}$; vbi est $21 = 4 \cdot 5 - 1 + 2$; $81 = 4 \cdot 21 - 5 + 2$; $305 = 4 \cdot 81 - 21 + 2$; . . . $z^{II} = 4z^{I} - z + 2$.

2) Sit $A = 3$, ex §. LXXI. prodit: A. tang. 1 — A. t. $\frac{1}{3}$ + A. t. $\frac{1}{13}$ — A. t. $\frac{1}{47}$ + A. t. $\frac{1}{177}$ — A. t. $\frac{1}{659}$ + etc. $= \frac{\pi}{8}$. Hoc exemplum protulit EVLERVS (l. c. nr. 20. cf. supra §. IL); nec tamen legem exhibuit, secundum quam denominatores progrediuntur. Quae lex ex §. LXXI. hisce aequationibus continetur: $13 = 4 \cdot 3 - 1 + 2$; $47 = 4 \cdot 13 - 3 - 2$; $177 = 4 \cdot 47 - 13 + 2$; $659 = 4 \cdot 177 - 47 - 2$; . . . $z^{II} = 4z^{I} - z \pm 2$; vel etiam numeri alternatim affirmatiui et negatiui 1, — 3, 13, — 47, 177, — 659, etc. progrediuntur in serie recurrente scalae — 4, — 1, cum *appendice* + 2. Eadem summatio sic quoque exhiberi potest:

$\text{A. tang}\,\frac{3}{2+1} - \text{A. tang.}\,\frac{3}{10-1} + \text{A. tang.}\,\frac{3}{38+1} - \text{A. tang.}\,\frac{3}{142-1} + \text{etc.}$

$\pm \text{A. tang.}\,\frac{3}{u \pm 1} \mp \text{etc.} = \frac{\pi}{8}$; vbi numeri 2, 10, 38, 142 . . . constituunt seriem recurrentem vulgarem, scalae 4, — 1, vt sit $u^{II} = 4u^{I} - u$.

Scholion.

§. LXXIII. 1) Summationes §. LXX. 1. 2. ex summationibus §§. LIX et LVIII. inuentis deducuntur, pro $r = 1$, $a = 1$; $r = 1$, $A = 1$. Ex iisdem corollaria adhuc generaliora peti possunt.

2) Posito nimirum $A = 1$, ex §. LVIII. habetur A. tang. 1 + A. tang. $\frac{1}{B}$ + etc.

$+ \text{A. tang.}\,\frac{1}{z} + \text{etc.} = \text{A. tang.}\,\frac{1}{1 - r + \gamma r\left(r - \frac{4}{B+1}\right)}$, vbi est $z^{II} =$

$((B+1)r - 2)z^{I} - z - r(B-1) + B + 1$. Si B est numerus impar, tum pro r posito $\frac{o}{2}$, $(B+1)r$ et $(B-1)r$ manent numeri integri; vel etiam ex §. LIX. pro $A = 1$, haec obtinetur summatio:

A. t.

$A.\,t.\,1 + A.\,t.\,\frac{1}{2r-1} + \text{etc.} + A.\,t.\,\frac{1}{z} + \text{etc.} = A.\,t.\,\frac{2}{1-a+\sqrt{\left((1+a)(1+a-\frac{4}{r})\right)}}$,

vbi eſt $z^{II} = (r(1+a)-2)z^{I} - r(a-1) + a + 1$. Exempli gratia pro $r = 2$, ex priori ſerie fit: $A.\,\text{tang.}\,1 + A.\,\text{tang.}\,\frac{1}{B} + \text{etc.} + A.\,\text{tang.}\,\frac{1}{z} + \text{etc.}$

$= A.\,\text{tang.}\,\frac{1}{-1+2\sqrt{\left(\frac{B-1}{B+1}\right)}}$, poſito $z^{II} = 2Bz^{I} - z - B + 3$; ex altera:

$A.\,\text{tang.}\,1 + A.\,\text{tang.}\,\frac{1}{3} + A.\,\text{tang.}\,\frac{1}{5a+2} + \text{etc.} + A.\,\text{tang.}\,\frac{1}{z} + \text{etc.}$

$= A.\,\text{tang.}\,\frac{2}{1-a+\sqrt{(a^2-1)}}$, vbi eſt $z^{II} = 2az^{I} - a + 3$.

3) Pro $A = 2$ prodit ex §. LVIII. $A.\,\text{tang.}\,\frac{1}{2} + A.\,\text{tang.}\,\frac{r}{B} + \text{etc.} + A.\,t.\,\frac{1}{z} + \text{etc.}$

$= A.\,\text{tang.}\,\frac{2}{4-5r+5\sqrt{r\left(r-\frac{4}{B+2}\right)}}$, vbi eſt $z^{II} = ((2+B)r-2)z^{I} - z - r(2B-1) + B + 2$; item ex §. LIX. $A.\,t.\,\frac{1}{2} + A.\,t.\,\frac{1}{5r-2} + \text{etc.} + A.\,t.\,\frac{1}{z} + \text{etc.} = A.\,\text{tang.}\,\frac{2}{2-a+\sqrt{(2+a)\left(2+a-\frac{4}{r}\right)}}$; poſito $z^{II} = (r(a+2)-2)z^{I} - z - r(2a-1) + a + 2$. Exempli gratia pro $r = 1$ ex priori ſummatione fit:

$A.\,\text{tang.}\,\frac{1}{2} + A.\,t.\,\frac{1}{B} + \text{etc.} + A.\,\text{tang.}\,\frac{1}{z} + \ldots = A.\,\text{tang.}\,\frac{2}{5\sqrt{\left(\frac{B-2}{B+2}\right)}-1}$,

ex altera: $A.\,\text{tang.}\,\frac{1}{2} + A.\,\text{tang.}\,\frac{1}{3} + A.\,t.\,\frac{1}{2a+1} + \text{etc.} + A.\,\text{tang.}\,\frac{1}{z} + \text{etc.}$

$= A.\,\text{tang.}\,\frac{2}{2-a+\sqrt{(a^2-4)}}$, pro quarum ſerierum illa eſt $z^{II} = Bz^{I} - z - B + 3$, hac: $z^{II} = az^{I} - z - a + 3$.

Corollarium 4.

§. LXXIV. 1) Poſito §. LXVIII. $A = 1$, erit $m + 2 = -\nu - 1$, $\nu = -m - 3$. Hinc, ſi fuerit $B = mA + \nu$, haec obtinetur ſummatio:

$A.\,\text{tang.}\,1 - A.\,\text{tang.}\,\frac{1}{3} - A.\,t.\,\frac{1}{5m+7} - \text{etc.} - A.\,\text{tang.}\,\frac{1}{z} - \text{etc.}$

 $= A.$

$= A. tang. \frac{2}{m+4-\sqrt{(m^2-4)}}$. Inde fit $A. tang. \frac{1}{3} + A. t. \frac{1}{5m+7} + etc. + A. t. \frac{1}{z}$ $+ etc. = A. tang. \frac{1}{2+\sqrt{\left(\frac{m-2}{m+2}\right)}}$, vbi est $z^{II} = m z^{I} - z + 2(m+3)$.

2) Si m negatiuum valorem induit $= -\mu$, haec ex priore oritur summatio: $A. tang. 1 - A. tang. \frac{1}{2} + A. tang. \frac{1}{5\mu - 7} + etc. \pm A. tang. \frac{1}{z} \mp etc.$ $= A. tang. \frac{2}{4-\mu+\sqrt{(\mu^2-4)}}$, vbi est $z^{II} = \mu z^{I} - z \pm 2(\mu - 3)$; quae pro $\mu = 4$ exhibet EVLERI seriem (§. LXXII. 2.). Ex altera serie (1) prodit: $A. tang. \frac{1}{3}$ $- A. tang. \frac{1}{5\mu-7} + etc. \pm A. tang. \frac{1}{z} \mp etc. = A. tang. \frac{1}{2+\sqrt{\left(\frac{\mu+2}{\mu-2}\right)}}$, vbi iam est $z^{II} = \mu z^{I} - z \mp 2(\mu - 3)$.

3) Si loco $B = m A + \nu$ (1) ponatur $B = \nu$ (§. LXVIII.), erit $A. tang. 1 + A. tang. \frac{1}{\nu} - A. tang. \frac{1}{\nu^2+\nu+1} + etc. \pm A. tang. \frac{1}{z} \mp etc.$ $= A. tang. \frac{1}{2-\sqrt{\left(\frac{\nu+5}{\nu+1}\right)}}$. Inde fit $A. tang. \frac{1}{\nu} - A. tang. \frac{1}{\nu^2+\nu+1} + etc.$ $\pm A. tang. \frac{1}{z} \mp etc. = A. tang. \frac{2}{\nu-1+\sqrt{((\nu+3)^2-4)}}$, vbi est $z^{II} = (\nu+3) z^{I}$ $- z \pm 2\nu$. Introducta loco ν quantitate $m = -\nu - 3$, hae summationes aliam formam induunt. Est nimirum $A. tang. 1 - A. t. \frac{1}{m+3} - A. t. \frac{1}{(m+3)(m+2)+1} - etc.$ $- A. tang. \frac{1}{z} - etc. = A. tang. \frac{1}{2-\sqrt{\left(\frac{m-2}{m+2}\right)}}$, et $A. tang. \frac{1}{m+3} +$ $A. tang. \frac{1}{(m+3)(m+2)+1} + etc. + A. t. \frac{1}{z} + etc. = A. t. \frac{2}{m+4+\sqrt{(m^2-4)}}$, vbi est $z^{II} = m z^{I} - z + 2(m+3)$.

Scholion 1.

§. LXXV. Summationum §. LXXIV. 1. prima fluit ex §. LIX. pro $A = 1$, $r = -1$, $a = -m - 3$; altera pro $A = 3$, $a = -1$, $2r = m + 2$. Nec non summationum §. LXXIV. 2. prima ex §. LVIII. pro $A = 1$, $r = -1$; vltima ex §. LIX. pro $A =$

$A = m + 3$, $r = 1$, $a = -1$ deriuatur. Eadem ratione ac §. LXXIII. summationes generaliores elici possunt. Ita, posito §. LIX. $A = 3$, $2r = m + 2$, et $a = -2\alpha - 1$, haec obtinetur summatio: $A.\ tang.\ \frac{1}{3} + A.\ t.\ \frac{1}{5m+7} + etc. + A.\ t.\ \frac{1}{z^{I}} + etc.$

$= A.\ tang.\ \frac{1}{2 + \alpha + \surd\left((1-\alpha)\left(\frac{m-2}{m+2} - \alpha\right)\right)}$, posito $z^{II} = (m(1-\alpha) - 2\alpha)z^{I} - z + 2(m+3) + (3m+4)\alpha$; (cf. §. LXXIV. 1.). Perinde habetur pro $A = m + 3$, $r = 1$, $a = \alpha - 1$: $A.\ tang.\ \frac{1}{m+3} + A.\ t.\ \frac{1}{(m+3)(m+2)+1} + etc. + A.t.\ \frac{1}{z} + etc.$

$= A.\ t.\ \frac{2}{m + 4 - \alpha + \surd((m+2+\alpha)(m-2+\alpha))}$, posito $z^{II} = (m+\alpha)z^{I} - z + 2(m+3) - (m+2)\alpha$, (cf. §. LXXIV. 3.).

Scholion 2.

§. LXXVI. Aequationi §. LXVIII: $m + 2 = \frac{A^2 - \nu^2}{A\nu - 1}$ aliis adhuc modis, quam ponendo $\nu = 1$ vel $A = 1$, vti §§. LXX. LXXIV, per *numeros integros* satisfieri potest. Resolutio huius aequationis quadraticae praebet:

$A = \frac{(m+2)\nu \pm \surd(((m+2)^2+4)\nu^2 - 4(m+2))}{2}$. Quare ii valores quantitatis ν eligendi sunt, qui expressionem $((m+2)^2+4)\nu^2 - 4(m+2)$ *quadrato* aequalem reddunt. Huc facit sequens

Lemma.

§. LXXVII. Inuenire valores quantitatis indeterminatae x, qui formulam $\surd(\alpha x^2 + \gamma) = y$ rationalem reddant (*).

Solutio.

1) Vnus valor problemati satisfaciens cognitus esse debet. Qui sit $= a$, et $\surd(\alpha a^2 + \gamma) = b$.

2) Iam

(*) De hoc problemate cf. L. EVLERI Algebra P. II. Sect. II. Cap. IV. V. VI; Eiusdem binae Commentationes: de Solutione problematum diophanteorum per numeros integros, Comment. Acad. Petrop. T. VI. p. 175; de resolutione formularum quadraticarum indeterminatarum per numeros integros, Nou. Comment. Acad. Petrop. T. IX. pag. 3. Post EVLERVM inprimis Ill. LA GRANGE hoc problema, eique affinia, pertractauit: vid. Miscellan. Taurin. Tom. IV.; Memoires de l'Academie de Berlin 1767. 1768.; Eiusdem additiones ad versionem Gallicam Algebrae EVLERI.

2) Iam quaerendi sunt pro p et q tales numeri, vt sit $\sqrt{(\alpha q^2 + 1)} = p$. (*)

3) Quo facto innumeri valores τῶν x et y, quaesito satisfacientes prodibunt ex his seriebus: $x = a,\ a^{I},\ a^{II},\ a^{III},\ a^{IV} \ldots P,\ Q,\ R \ldots$

$y = b,\ b^{I},\ b^{II},\ b^{III},\ b^{IV} \ldots S,\ T,\ V \ldots$

quarum vtraque est recurrens hac lege, vt sit $R = 2pQ - P$, $V = 2pT - S$. Secundi earum termini ex his aequationibus definiuntur: $a^{I} = pa + qb$, $b^{I} = \alpha qa + pb$. Quare cum b affirmatiue et negatiue accipi queat, tam pro x quam pro y duae series reperiuntur.

Corollarium 1.

§. LXXVIII. Formula $\sqrt{(\alpha x^2 + \beta x + \gamma)} = y$ simili modo rationalis redditur. Sit $\sqrt{(\alpha a^2 + \beta a + \gamma)} = b$, $\sqrt{(\alpha q^2 + 1)} = p$; eritque pro seriebus τῶν x et y, $a^{I} = pa + qb + \frac{\beta}{2\alpha}(p - 1)$, $b^{I} = \alpha qa + pb + \frac{1}{2}\beta q$; earumque serierum lex progressus his aequationibus continetur: $x^{II} = 2px^{I} - x + \frac{\beta}{\alpha}(p - 1)$; $y^{II} = 2py^{I} - y$.

Corollarium 2.

§. LXXIX. 1) Quae praemissa vt ad formulam (§. LXXVI.) $\sqrt{(((m+2)^2+4)\nu^2 - 4(m+2))} = y$ traducantur, est (§. LXXVII. 1.) $a = 1$, primus valor quantitatis ν, et $b = m$; porro aequationi $\sqrt{(((m+2)^2+4)q^2+1)} = p$ (2) satisfit, sumendo $q = \frac{m+2}{2}$, $p = 1 + \frac{1}{2}(m+2)^2$. Hinc prodit (3) $a^{I} = pa \pm qb$, vel $= 1 + (m+1)(m+2) = m^2 + 3m + 3$, vel $= m + 3$; $b^{I} = \alpha qa \pm pb$, vel $= (m+1)(m+2)^2 + 3m + 4 = (m+2)^3 - m(m+1)$, vel $= (m+2)^2 + m + 4 = (m+2)(m+3) + 2$. Exinde duplex oritur series valorum quantitatis ν: I) 1; $1 + (m+1)(m+2)$; $\ldots \nu^{r} \ldots$

II) 1; $m+3$; $\ldots \nu^{r} \ldots$

Vtraque est recurrens, et obseruat hanc legem communem, vt sit $\nu^{r+2} = ((m+2)^2 + 2)\nu^{r+1} - \nu^{r}$. His seriebus respondent sequentes binae series valorum formulae radicalis y: I) m;

(*) De quo Cf. EVLERI Algebra l. c. Cap. VI. *Idem* de vsu noui Algorithmi in Problemate *Pelliano* soluendo, Nou. Comment. Acad. Petrop. Tom. XI. pag. 28.; vbi tabula numerorum p et q pro omnibus valoribus numeri α vsque ad 100 traditur; porro EVLERI opuscula analytica T. I. Petropoli 1783. 4.: Noua subsidia pro resolutione formulae $axx + 1 = yy$, pag. 310. Problema rectius *Fermatianum* vocari videtur, quippe a Fermatio primum Anglis propositum; cf. IOH. WALLISII de Algebra Tractatus, Opp. Math. Vol. II. Oxonii 1693. fol. pag. 418 sq. Commerc. Epistol. e. l. pag. 767. 789. 862. Ceterum numerus α *non quadratus* supponitur. Quare etiam solutio problematis, Lemmatis instar hic propositi, ad hunc casum restringitur (cf. supra §. LXIII.).

I) $m;\ (m+2)^3 - m(m+1);\ \ldots\ y^r\ \ldots$

II) $-m;\ \ (m+2)^2 + m + 4;\ \ \ldots\ y^r\ \ldots$

quae eandem legem progreſſus tenent.

2) Cuilibet ν reſpondet duplex valor quantitatis $A = \frac{(m+2)\nu \pm y}{2}$. Hinc quatuor naſcuntur ſeries $\tau\tilde{\omega}\nu$ A, duplex pro vtraque ſerie $\tau\tilde{\omega}\nu$ ν, ſcilicet

I. pro prima ſerie $\tau\tilde{\omega}\nu$ ν.

I. 1) $m+1;\ (m+2)^3 - (m+1)^2;\ \ldots\ A^r\ \ldots$

I. 2) $1;\ \ \ -(m+1);\ \ \ldots\ A^r\ \ldots$

II. pro altera ſerie $\tau\tilde{\omega}\nu$ ν

II. 1) $1;\ \ 1 + (m+2)(m+3);\ \ldots\ A^r\ \ldots$

II. 2) $m+1;\ \ \ -1;\ \ \ \ldots\ A^r\ \ldots$

Cum conſtet A productis conſtantium in quantitatis ν et y, quae in ſerie recurrente eiusdem ſcalae progrediuntur, manifeſtum eſt, ſeries $\tau\tilde{\omega}\nu$ A ſimiles eſſe ſeries, iisque legem communem hanc, vt ſit $A^{r+2} = ((m+2)^2+2)A^{r+1} - A^r$. Series I. 2) a ſerie I. 1) quoad ſigna tantum differt, eſtque illius terminus rtus aequalis termino huius r-1to negatiue ſumto. Terminus quippe tertius ſeriei I. 2) eſt $= -(m+1)((m+2)^2+2) - 1$ $= -(m+1)(m+2)^2 - 2m - 3 = -((m+2)^3 - (m+1)^2)$. Idem valet de ſeriebus II. 1) et II. 2).

3) Cum poni poſſit B vel a) $= mA + \nu$, vel b) $= \nu$ (§. LXVIII.), pro quolibet A et ν, B duplicem valorem habet. Hinc ex binis ſeriebus $\tau\tilde{\omega}\nu$ ν (1) et quatuor $\tau\tilde{\omega}\nu$ A (2) octo prodeunt ſeries $\tau\tilde{\omega}\nu$ B.

I. pro prima ſerie $\tau\tilde{\omega}\nu$ ν

1. pro prima ſerie $\tau\tilde{\omega}\nu$ A

I. 1. a) $m(m+1)+1;\ (m+1)^3(m+2) + (m+1)(2m+1);\ \ldots\ B^r\ \ldots$

I. 1. b) $1;\ \ \ 1 + (m+1)(m+2);\ \ \ldots\ B^r\ \ldots$

2. pro altera ſerie $\tau\tilde{\omega}\nu$ A

I. 2. a) $m+1;\ \ \ 2m+3;\ \ \ldots\ B^r\ \ldots$

I. 2. b) $1;\ 1 + (m+1)(m+2);\ \ldots\ B^r\ \ldots$

II. pro altera ſerie $\tau\tilde{\omega}\nu$ ν

1. pro tertia ſerie $\tau\tilde{\omega}\nu$ A

II. 1. a) $m+1;\ m + (m+1)^2(m+3);\ \ldots\ B^r\ \ldots$

II. 1. b) $1;\ \ \ m+3;\ \ \ \ldots\ B^r\ \ldots$

2. pro

2. pro quarta ferie τῶν A

II. 2. a) $m(m+1)+1$; 3; . . . B^r . . .

II. 2. b) 1; $m+3$; . . . B^r . . .

Hisce feriebus τῶν B legem progreſſus communem eſſe cum feriebus τῶν A et ν, vel $B^{r+2} = ((m+2)^2+2)B^{r+1} - B^r$, ex (2) manifeſtum eſt.

Corollarium 3.

§. LXXX. 1) Quibus feriebus τῶν B ſi feries τῶν A et ν debito modo iungantur, prouti deſignatio hic adhibita ſatis clare indicat, octo obtinentur combinationes trium ſerierum pro ν, A et B, quae ita ſunt comparatae, vt ſi earum termini quilibet ſibi inuicem reſpondentes pro ν, A et B ſumantur, prodeant ſeries Arcuum, quorum cotangentes in ſerie recurrente *affecta* ſecundi ordinis progredientes numeris *integris* exprimuntur; quae quidem ſeries hac forma comprehenſae: A. cotang. A + A. cot. B + A. cot. C + ... + A. cotang. z + . . ., vbi eſt $z^{II} = mrz^{I} - z + 2\nu$, ſummam habent $= \text{A. tang} \dfrac{A+B}{AB - 1 - \frac{1}{2}(A^2+1)(m - \sqrt{(m^2-4)})}$. Innumeras inde eiusmodi ſeries oriri manifeſtum eſt, quarum quaelibet ob quantitatem m indeteminatam aeque innumera exempla complectitur.

2) Accuratior tamen conſideratio docet, *octo* iſtas combinationes ad *quatuor* redire. Series nimirum Arcuum ex ſerie I. 1. a) oriundae a feriebus, quas feries I. 2. b) ſuppeditat, non differunt, niſi quod illae duobus primis harum terminis truncatae ſint, ſeu ſerius incipiant. Deſignentur enim illarum termini initiales per A. cotang. A^r, A. cot. B^r, A. cot. C^r; harum per A. cot. $\mathfrak{A}^r$, A. cot. $\mathfrak{B}^r$, A. cot. $\mathfrak{C}^r$, A. cot. $\mathfrak{D}^r$: erit $A^r = \dfrac{(m+2)^r + y^r}{2}$, $\mathfrak{A}^r = \dfrac{(m+2)^r - y^r}{2}$, huc $A^r + A^r = (m+2)\nu^r = m\mathfrak{B}^r + 2\nu^r$, ob $\mathfrak{B}^r = \nu^r$; quare $A^r = m\mathfrak{B}^r - \mathfrak{A}^r + 2\nu^r = \mathfrak{C}^r$; porro $B^r = mA^r + \nu^r = m\mathfrak{C}^r - \mathfrak{B}^r + 2\nu^r = \mathfrak{D}^r$; i. e. illarum ſerierum termini primi et ſecundi aequantur harum tertiis et quartis, et ſic porro. Simili omnino ratione perſpicitur, ſeries Arcuum ex ſerie I. 1. b) oriundas, easque quae ex I. 2. a) prodeunt, pro identicis habendas eſſe, quippe hae duobus tantum prioribus illarum terminis truncatae ſunt. Nec minus manifeſtum eſt, idem ratiocinium ad alteram ſeriem τῶν ν patere, et ex feriebus II. 1. a, II. 2. b; atque II. 1. b, II. 2. a. haud diuerſas ſummationes reſultare.

3) Quod ſi igitur feries tantum I. 2. a, I. 2. b, II. 2. a, II. 2. b, retineantur, eaeque cum feriebus τῶν ν et A rite conjungantur, ſequentes oriuntur ſeries, ex quibus valores trium quantitatum ν, A et B peti poſſunt: qui in ſerie A. cotang. A + A. cot. B + . . . + A. cot. z + etc. ſubſtituti praebent progreſſiones Arcuum ſummabiles:

1) 1;

1) 1; $1+(m+1)(m+2)$; . . . ν^r . . . (I. 2. a)
1; $-(m+1)$; . . . A^r . . .
$m+1$; $2m+3$; . . . B^r . . .

2) 1; $1+(m+1)(m+2)$; . . . ν^r . . . (I. 2. b)
1; $-(m+1)$; . . . A^r . . .
1; $1+(m+1)(m+2)$; . . . B^r . . .

3) 1; $m+3$; . . . ν^r . . . (II. 2. a)
$m+1$; -1; . . . A^r . . .
$1+(m+1)(m+2)$; 3; . . . B^r . . .

4) 1; $m+3$; . . . ν^r . . . (II. 2. b)
$m+1$; -1; . . . A^r . . .
1; $m+3$; . . . B^r . . .

Corollarium 4.

§. LXXXI. 1) Harum ſerierum termini primi pro ν, A et B ſubſtituti haud alias progreſſiones Arcuum ſuppeditant, ac eas, quae iam §§. LXX. LXXI. exhibitae ſunt. Ex terminis ſecundis ſerierum 3. 4. oriuntur progreſſiones §. LXXIV ſummatae.

2) Conſideremus igitur terminos ſecundos ſerierum 1. 2, ex quibus primo, adhibitis ſeriebus 1, haec oritur ſummatio:

$$\text{A. tang.} -\frac{1}{m+1} + \text{A. t.} \frac{1}{2m+3} + \text{A. t.} \frac{1}{2(2m+3)(m+1)+1} + \text{etc.} + \text{A. t.} \frac{1}{z} + \ldots$$

$$= \text{A. tang.} \frac{2}{-(m+2)^2 + (1+(m+1)^2)\sqrt{\frac{m-2}{m+2}}}.$$ Demto termino primo haec inde obtinetur ſummatio:

$$\text{A. tang.} \frac{1}{2m+3} + \text{A. tang.} \frac{1}{2(2m+3)(m+1)+1} + \text{etc.} + \text{A. tang.} \frac{1}{z} + \ldots$$

$$= \text{A. tang.} \frac{2}{3m+4+\sqrt{(m^2-4)}},$$ vbi eſt $z^{II} = m z^{I} - z + 2 + 2(m+1)(m+2)$.

Poſito $m = -\mu$, prior ſummatio in hanc abit:

$$\text{A. tang.} \frac{1}{\mu-1} - \text{A. t.} \frac{1}{2\mu-3} + \text{A. t.} \frac{1}{2(\mu-1)(2\mu-3)+1} + \text{etc.} \pm \text{A. t.} \frac{1}{z} \mp \text{etc.}$$

$= A.\ tang.\ \frac{2}{-(\mu-2)^2+(1+(\mu+1)^2)\sqrt{\frac{\mu+2}{\mu-2}}}$, vbi est $z^{II} =$
$\mu z^I - z \pm 2(1+(\mu-1)(\mu-2))$; altera in hanc:
$A.\ tang.\ \frac{1}{2\mu-3} - A.t.\ \frac{1}{2(\mu-1)(2\mu-3)+1} + A.t.\ \frac{1}{4(\mu-1)^3-\mu+1}$ — etc.
$\pm$ A. tang. $\frac{1}{z} \mp$ etc. $= A.\ tang.\ \frac{2}{3\mu-4+\sqrt{(\mu^2-4)}}$, vbi iam est $z^{II} =$
$\mu z^I - z \mp 2(1+(\mu-1)(\mu-2))$.

3) Simili ratione serierum (2) (§. LXXX.) termini secundi hanc praebent summationem: $A.\ tang.\ \frac{1}{-(m+1)} + A.t.\ \frac{1}{1+(m+1)(m+2)} + A.t.\ \frac{1}{(m+2)^3-(m+1)^2}$ + etc. + A. tang. $\frac{1}{z}$ + etc. $= A.\ tang.\ \frac{-2}{3m+4-\sqrt{(m^2-4)}}$, vnde fit
$A.\ tang.\ \frac{1}{1+(m+1)(m+2)} + A.\ tang.\ \frac{1}{(m+2)^3-(m+1)^2}$ + etc. + A. tang. $\frac{1}{z}$ + etc.
$= A.\ tang.\ \frac{2}{(m+2)^2+(1+(m+1)^2)\sqrt{\frac{m-2}{m+2}}}$, vbi est $z^{II} = mz^I - z +$
$2(1+(m+1)(m+2))$. Si m negatiue accipitur $= -\mu$, sequitur inde:
$A.\ tang.\ \frac{1}{1+(\mu-1)(\mu-2)} - A.\ tang.\ \frac{1}{(\mu-2)^3+(\mu-1)^2}$ + etc. $\pm$ A. tang. $\frac{1}{z} \mp$ etc.
$= A.\ tang.\ \frac{2}{(\mu-2)^2+(1+(\mu-1)^2)\sqrt{\frac{\mu+2}{\mu-2}}}$, vbi est $z^{II} =$
$\mu z^I - z \pm 2(1+(\mu-1)(\mu-2))$.

Exempla.

§. LXXXII. Pro $m = 3$ ex §. LXXXI. 2. sequitur: A. tang. $\frac{1}{9}$ + A. tang. $\frac{1}{73}$ + A. tang. $\frac{1}{252}$ + etc. + A. t. $\frac{1}{z}$ + etc. $=$ A. t. $\frac{2}{13+\sqrt{5}}$; vbi est $252 = 3.73 - 9 + 42$; . . . $z^{II} = 3z^I - z + 42$. Ex §. LXXXI. 3. prodit: A. tang. $\frac{1}{21}$ + A. tang. $\frac{1}{109}$ + A. tang. $\frac{1}{286}$ + etc. + A. tang. $\frac{1}{z}$ + etc. $=$ A. t. $\frac{2}{25+17\sqrt{\frac{1}{5}}}$, vbi itidem est $z^{II} = 3z^I - z + 42$. Pro $\mu = 3$ ex §. LXXXI. 2. habetur: A. tang. $\frac{1}{3}$ — A. tang. $\frac{1}{5}$ + A. t. $\frac{1}{13}$ — A. t. $\frac{1}{30}$ + etc. $\pm$ A. t $\frac{1}{z} \mp$ etc. $=$ A. tang. $\frac{2}{5\sqrt{5}-1}$; vbi est $13 = 3.3 - 2 + 6$; $30 = 3.13 - 3 - 6$; . . . $z^{II} = 3z^I - z \pm 6$;

fine

siue $A.t.\frac{1}{3} - A.t.\frac{1}{13} + A.t.\frac{1}{38} - \text{etc.} \pm A.t.\frac{1}{z} \mp \text{etc.} = A.\text{tang}.\frac{2}{5+\sqrt{5}}$. Pro $\mu = 4$ prodit: $A.\text{tang}.\frac{1}{2} - A.t.\frac{1}{3} + A.t.\frac{1}{31} - A.t.\frac{1}{161} + \text{etc.} \pm A.t.\frac{1}{z} \mp \text{etc.}$ $= A.\text{tang}.\frac{2}{5\sqrt{3}-2}$; vbi est $z^{II} = 4z^{I} - z \pm 14$; siue $A.\text{tang}.\frac{1}{3} - A.\text{tang}.\frac{1}{31}$ $+ A.t.\frac{1}{161} + \text{etc.} \pm A.\text{tang}.\frac{1}{z} \mp \text{etc.} = A.t.\frac{1}{4+\sqrt{3}}$. Pro $\mu = 4$ ex §. LXXXI. 3. prodit: $A.t.\frac{1}{7} - A.t.\frac{1}{17} + A.t.\frac{1}{71} - A.t.\frac{1}{269} + \text{etc.} \pm A.t.\frac{1}{z} \mp \text{etc.}$ $= A.\text{tang}.\frac{1}{5\sqrt{3}+2}$; vbi etiam est $z^{II} = 4z^{I} - z \pm 14$.

Scholion.

§. LXXXIII. 1) Summationum §. LXXXI. 2. prima ex §. LVIII. pro $A = -m-1$, $B = 2m+3$, $r = 1$, fluit; altera ex §. LIX. pro $A = 2m+3$, $r = 1$. Summationum §. LXXXI. 3. prima ex §. LIX. deriuatur pro $A = -m-1$, $r = 1$.

2) Quibus collatis cum §§. LXXIII. LXXV, apparet, generatim progressiones Arcuum, quae ex terminis primis et secundis serierum pro ν, A et B, (§. LXXX.) oriuntur, iam sub summationibus §§. LVIII. LIX. comprehensas esse; cuius consensus ratio in eo sita est, quod pro illis progressionibus vel $\frac{A+B+n}{AB-1}$ vel $\frac{A+B}{A^2+1}$ numero integro aequetur (cf. §. LXI.). At si termini (§. LXXX.) post secundos adhibeantur, tum nouae prodeunt summationes, a prioribus (§§. LVIII. LIX.) diuersae: quas tamen amplius euoluere superfluum videtur.

Corollarium 5.

§. LXXXIV. 1) Quanquam valores quantitatis ν §§. LXXX. LXXXI. inuenti etiam negatiue accipi queant, hinc tamen haud nouae series Arcuum obtinentur: Quippe tum etiam $A = \frac{-(m+2)\nu \pm y}{2} = \frac{-((m+2) \mp y)}{2}$, et $B = mA + \nu$ vel $= \nu$ signa tantum mutant. Porro aequatio $m+2 = \frac{A^2 - \nu^2}{A\nu - 1}$ (§. LXXVI.) ita quoque resolui potest, vt ν per A exprimatur: $\nu = \frac{-(m+2)A \pm \sqrt{(((m+2)^2+4)A^2+4(m+2))}}{2}$. Quod si vero super hac formula ad rationalitatem perducenda, ratiocinia prioribus similia instituantur, series inde ab iam inuentis haud diuersae prodeunt.

2) Cum in aequatione conditionali $m+2 = \frac{(A+B)(A+B+n)}{AB-1}$ (§. XXXIV.), quae in vniuersa hactenus inuestigatione fundamenti loco posita fuit, quantitates A et B per-

 mutari

mutari inuicem queant, manifestum est, ex seriebus Arcuum ad regulas §. LXXX. inuentis nouas formari posse, dum illarum termini primi et secundi inuicem permutantur, seu qui erant primi, secundo loco ponantur, et vice versa: vnde ex lege $z^{II} = mz^{I} - z + 2v$ etiam reliqui termini nouarum serierum alios valores recipiant necesse est. Itaque numerus serierum summabilium duplo maior quam ex §. LXXX. prodire videtur. Rem tamen secus se habere, accuratior consideratio ostendit. Designentur enim seriei cuiuspiam Arcuum ex (1) I. 2. a. (§. LXXX.) deriuatae termini initiales per A. cot. A^r, A. cot. B^r, A. cot. C^r; eius contra seriei, quae ex (2) I. 2. b. oritur, per A. cot. $\mathfrak{A}^r$, A. cot. $\mathfrak{B}^r$, . . . ; erit $A^r = \mathfrak{A}^r = \frac{(m+2)v^r - y^r}{2}$; $B^r = mA^r + v^r$; $\mathfrak{B}^r = v^r$.

Quod si iam in serie priori A et B seu termini primus et secundus inuicem permutentur, erit pro noua serie inde oriunda cotangens tertia $= mA^r - B^r + 2v^r = v^r = \mathfrak{B}^r$ = secundae cotangenti alterius seriei ex (2) deriuatae, vti secunda illius $= A^r = \mathfrak{A}^r$ = primae huius. Hinc euidens est, seriem Arcuum ex permutatione τῶν A et B ortam a serie ex (2) I. 2. b petita haud differre, nisi quod haec primo illius termino truncata sit. Eadem omnino ratione series Arcuum ex permutatione τῶν A et B in progressionibus, ex I. 2. b ortis, prodeuntes haud diuersae sunt a seriebus ex I. 2. a deriuatis: quippe illarum cuiuspiam cotangens tertia $= m\mathfrak{A}^r - \mathfrak{B}^r + 2v^r = mA^r + v^r = B^r$ = secundae cotangenti seriei ex I. 2. a ortae. Idem prorsus obtinet de seriebus Arcuum ex (3) et (4) vel II. 2. a. II. 2. b deriuatis. Proinde iam, quo ea, quae §§. LXXIX. LXXX. fusius inuestigata sunt, quam licet breuissime complectamur, sequens ex istis condi potest:

THEOREMA PARTICVLARE 7.

§. LXXXV. Quod si sequentes binae formentur conternationes trium serierum:

1)	1;	$1+(m+1)(m+2)$;	. . . v^r . . .
	-1;	$-(m+1)$;	. . . A^r vel B^r . .
	$m+1$;	$2m+3$;	. . . B^r vel A^r . .
2)	1;	$m+3$;	. . . v^r . . .
	$m+1$;	-1;	. . . A^r vel B^r . .
	$1+m(m+1)$;	3;	. . . B^r vel A^r . .

quarum quaeuis est recurrens, scala relationis communi existente $(m+2)^2+2$; -1; si porro quicunque terminus primae seriei alterutrius conternationis pro v, et termini illi respondentes secundae et tertiae seriei eiusdem conternationis pro A et B, vel vice versa pro B et A substituantur; tum orientur progressiones summabiles Arcuum, scilicet A. cot. A + A. cot.

A. cot. B + etc. + A. cot. z + etc. quorum cotangentes numeris *integris* exprimuntur, et in ferie recurretne affecta procedunt, hac lege, vt fit $z^{II} = m z^{I} - z - 2 v$. Summa erit $= A.\ tang. \frac{A + B}{AB - 1 - \frac{1}{2}(A^2 + 1)(m - \surd(m^2 - 4))}$. Demonstratio huius theorematis ex iis ipsis consequitur, quae hactenus exposita sunt. Quanquam ea breuius concinnari possit, sumendo tantum §. LXXIX. (2) $A = \frac{(m+2)v - y}{2}$, (3) $B = mA + v$, praestare tamen mihi videbatur, omnium casuum, qui ex hypothesi altera §. LXIII. commemorata, resultare posse videntur, enumerationem completam exhibere, simulque ostendere, eorum apparentem multitudinem et varietatem ad eam simplicitatem reduci, quam Theorema indicat. Ceterum ex §. XXXV. apparet, summam progressionis, si ea termino A. cot. z finiatur, esse $= A.\ tang. \frac{A + B}{AB - 1 - \frac{1}{2}(A^2 + 1)(m - \surd(m^2 - 4))}$ $- A.\ tang. \frac{z^I + z^{II}}{z^I z^{II} - 1 - \frac{1}{2}(z^I z^I + 1)(m - \surd(m^2 - 4))}$.

Scholion.

§. LXXXVI. 1) Cum sit $B = \frac{mA - n \pm \surd((m^2 - 4)A^2 - 2An(m+2) + n^2 - 4m - 2)}{2}$ $= \frac{mA - n \pm u}{2}$, (§. LXIII.) ex vnico valore τῶ A, qui pro certis m et n formulam radicalem rationalem reddat, innumeri alii valores obtinentur idem praestantes (§. LXXVII. LXXVIII). Aequationi nimirum $((\surd m^2 - 4)(q^2 + 1) = p$ satisfaciunt $q = \frac{1}{2}$, $p = \frac{m}{2}$. Hinc ex valoribus $A = a$, $u = b$ noui prodeunt valores: $A^I = \frac{m}{2} a + \frac{1}{2} b \frac{-n}{m-2}\left(\frac{m}{2} - 1\right) = \frac{ma + b - n}{2}$; $b^I = (m^2 - 4)\frac{a}{2} + \frac{m}{2} b - n\frac{(m+2)}{2}$; ex quibus simili ratione noui obtinentur, et sic porro. Proinde ex seriebus Arcuum, quas theorema §. LXXXV. suppeditat, innumerae aliae deriuari posse videntur, dum pro certo valore τῶ v ex vno valore τῶ A vi theorematis cognito alii valores reperiuntur. Nihilominus tamen haud nouas exinde oriri summationes, a prioribus diuersas, sequenti ratione apparet. Namque cum sit $b = u = 2B - mA + n$, erit $A^I = \frac{mA + b - n}{2} = B$; inde fit $B^I =$

$$\frac{mA^I - n \pm b^I}{2} = \frac{mB - n + (m^2 - 4)\frac{A}{2} + \frac{m}{2} b - n\frac{(m+2)}{2}}{2}$$

$= \frac{mB - n + mB - 2A - n}{2} = C$. Quare series Arcuum ex valore A^I oriunda ab ea,

quam

quam valor A ex theoremate praebet, haud reuera diuersa est: sicque et reliqui valores haud nouas summationes praebent. Formulam u, si pro certo valore τȣ A rationalis reddatur, etiam rationalem fieri, si pro A substituantur B, C, D, et ceterae cotangentes illam sequentes, exinde iam apparet, quod aequatio conditionalis non tantum de primo et secundo termino seriei cotangentium, A et B, verum etiam de quibusuis sibi proximis, z^{I}, z^{II} locum habeat (§. XXXVI.): sin autem A et B rationaliter exprimantur, etiam reliquas cotangentes rationales fore, euidens est.

2) Ex quibus apparet, resolutionem aequationis conditionalis (§. LXXVIII.), quatenus ea spectetur tanquam aequatio inter indeterminatas A et B, haud quicquam emolumenti afferre, ad detegendas plures progressiones Arcuum summabiles (cf. §. LXIII.). Ex vna enim cognita haud nouae prodirent ac reuera diuersae series. Quibus haud obstantibus innumeras eiusmodi series exhiberi posse, hactenus declaratum est: aequatio scilicet conditionalis considerari potest tanquam aequatio inter tres indeterminatas A, B et n.

PROBLEMA VII.

§. LXXXVII. *Summare seriem Arcuum*

$$\text{A. tang.}\frac{a}{E+bE^{-1}+c}+\text{A. t.}\frac{a}{E^2+bE^{-2}+c}+\text{etc.}+\text{A. t.}\frac{a}{E^x+bE^{-x}+c},$$

posito $\frac{b}{E^r}=\frac{c^2}{(E^r+1)^2}+\frac{a^2}{(E^r-1)^2}$, *denotante* r *numerum quemuis integrum.*

Solutio.

Iisdem omnino ratiociniis, quae §. XXX. (cf. §. XXII.) adhibita sunt, quaeque repetere superfluum est, productum indefinitum $P\left(\frac{1+t^x\gamma-1}{1-t^x\gamma-1}\right)$ reducitur ad formam §. XII. 5. expositam; hincque prodit summa Arcuum

$$=\text{A. tang.}\frac{aE^r}{(E^r-1)\left\{E+\frac{cE^r}{E^r+1}\right\}}+\text{A. tang.}\frac{aE^r}{(E^r-1)\left\{E^2+\frac{cE^r}{E^r+1}\right\}}+\ldots$$

$$+\text{A. tang.}\frac{aE^r}{(E^r-1)\left\{E^r+\frac{cE^r}{E^r+1}\right\}}$$

— A.

$$- \text{A. tang.} \frac{aE^r}{(E^r - 1)\left\{E^1 + x + \frac{cE^r}{E^r+1}\right\}} - \text{A. tang.} \frac{aE^r}{(E^r - 1)\left\{E^2 + x + \frac{cE^r}{E^r+1}\right\}} - \ldots$$

$$- \text{A. tang.} \frac{aE^r}{(E^r - 1)\left\{E^{r+x} + \frac{cE^r}{E^r+1}\right\}}.$$

Si ſeries in infinitum excurrit, eae ſummae partes, quae x inuoluunt, omittuntur, ſuppoſito $E > 1$.

Corollarium 1.

§. LXXXVIII. Sit $c = 0$, et $\frac{aE^r}{(E^r - 1)E} \cdot \frac{aE^r}{(E^r - 1)E^r} = 1$, vel $a^2 = \frac{(E^r - 1)^2}{E^{r-1}}$, $b = \frac{a^2 E^r}{(E^r - 1)^2} = E$; tum in ſummae expreſſione primus et vltimus arcus conficiunt ſummam $\frac{\pi}{2}$, quippe tangens vnius = cotangenti alterius; eandem ſummam praebent quicunque bini termini a primo et vltimo aequidiſtantes. Hinc erit ſumma, ſi r impar fuerit, $= \frac{\pi}{2} \frac{(r-1)}{2}$

$+ \text{A. tang.} \frac{aE^r}{(E^r - 1)\, E^{\frac{r+1}{2}}} = \frac{\pi r}{4}$; pro r pari ſumma etiam eſt $= \frac{\pi}{2} \cdot \frac{r}{2} = \frac{\pi r}{4}$. Exinde haec oritur ſummatio, poſito $E = e^2$; $S.\ \text{A. tang.} \frac{(e^{2r} - 1)}{e^{r-1}(e^{2x} + e^{2-2x})} = \frac{\pi r}{4}$;

vel $\text{A. tang.} \frac{e(e^r - e^{-r})}{e^2 + 1} + \text{A. tang.} \frac{e(e^r - e^{-r})}{e^4 + e^{-2}} + \text{etc.} + \text{A. tang.} \frac{e(e^r - e^{-r})}{e^{2x} + e^{2-2x}}$

$+ \text{etc.} = \frac{\pi r}{4}$.

Corollarium 2.

§. LXXXIX. Sit $c = 0$, et $\frac{aE^r}{(E^r - 1)E} \cdot \frac{aE^r}{(E^r - 1)E^{r-1}} = 1$, ſeu $a^2 = \frac{(E^r - 1)^2}{E^r}$, $b = 1$; tum primus et penultimus terminus ſummae (§. LXXXVII.) coniunctim adaequant quadrantem, aeque ac ſecundus et antepenultimus, et ſic porro. Inde erit ſumma $= \frac{\pi}{2} \frac{(r-1)}{}$

$(r-1)\frac{\pi}{4}+\text{A. tang.}\frac{a}{E^r-1}$. Posito igitur $E=e^2$, haec obtinetur summatio:

$$\text{A. tang.}\frac{e^r-e^{-r}}{e^2+e^{-2}}+\text{A. tang.}\frac{e^r-e^{-r}}{e^4+e^{-4}}+\text{etc.}+\text{A. t.}\frac{e^r-e^{-r}}{e^{2x}+e^{-2x}}+\text{etc.}$$

$$=(r-1)\frac{\pi}{4}+\text{A. tang.}\frac{1}{e^r}=(r+1)\frac{\pi}{4}-\text{A. tang. } e^r.$$

Scholion.

§. XC. 1) Summationis §. LXXXVII. demonstratio synthetica condi potest ex resolutione termini generalis in binorum Arcuum differentiam. Est nimirum

$$\text{A. tang.}\frac{a}{E^x+bE^{-x}+c}=\text{A. t.}\frac{aE^r}{(E^r-1)\left\{E^x+\frac{cE^r}{E^r+1}\right\}}-\text{A. t.}\frac{aE^r}{(E^r-1)\left\{E^{x+r}+\frac{cE^r}{E^r+1}\right\}}.$$

Qua ratione si singuli Arcus seriei exprimantur, omissis terminis se mutuo destruentibus, prouenit summae expressio prius exhibita (cf. §§. XXXVIII. XXIII.) Summationes §. LXXXVII et XXXI. sub hoc theoremate comprehendi possunt: summabilem esse seriem Arcuum, cuius terminus generalis est A. tang. $\frac{a}{e^{\frac{x}{r}}+be^{\frac{-x}{r}}+c}$, denotante r numerum quemuis integrum, dum fuerit $\frac{b}{e}=\frac{c^2}{(e+1)^2}+\frac{a^2}{(c-1)^2}$ (cf. §. XXIX. 1.)

2) Quodsi serierum innumerarum, quae ex hactenus demonstratis summari possunt, duae pluresque inuicem addantur, nouae prodeunt series summabiles Arcuum, quorum cotangentes haud amplius in serie recurrente affecta secundi ordinis, verum ad aliam legem magis compositam procedunt (cf. §. XXX.). Cum tamen istiusmodi serierum contemplatio vix ad theoremata generalia et simplicia perducere videatur, cumque superior iam expositio nimis forte longa fuerit, ad *Sectionem tertiam* transeundum est.

C. Se-

C. SECTIO TERTIA.

Inueftigatio ferierum tranfcendenter fummabilium.

CAP. I.

PROBLEMATA FVNDAMENTALIA ET SIMPLICIORA.

Lemma. (*)

§. XCII. 1) Producti infinite multis factoribus conftantis:

$$\frac{(1+z^2)(4+z^2)(9+z^2)\ldots(x^2+z^2)\ldots}{1\,.\,4\,.\,9\ldots\ldots x^2\ldots\ldots} \text{ valor eft} = \frac{e^{\pi z}-e^{-\pi z}}{2\pi z}.$$

2) Productum infinitum

$$\frac{(1+z^2)(9+z^2)(25+z^2)\ldots((2x-1)^2+z^2)\ldots}{1\,.\,9\,.\,25\ldots\ldots(2x-1)^2\ldots\ldots} \text{ eft} = \frac{e^{\frac{\pi z}{2}}+e^{-\frac{\pi z}{2}}}{2}.$$

3) Productum infinitum

$$\left(1+\frac{z^2}{1}\right)\left(1+\frac{z^2}{(2m-1)^2}\right)\left(1+\frac{z^2}{(2m+1)^2}\right)\left(1+\frac{z^2}{(4m-1)^2}\right)\left(1+\frac{z^2}{(4m+1)^2}\right)\ldots$$

$$\text{eft} = \frac{e^{\frac{\pi z}{m}}-2\operatorname{cof.}\frac{\pi}{m}+e^{-\frac{\pi z}{m}}}{2\left(1-\operatorname{cof.}\frac{\pi}{m}\right)}.$$

PROBLEMA VIII.

§. XCIII. Summare feriem infinitam:

$$\text{A. tang. }\frac{a}{1+b}+\text{A. t. }\frac{a}{4+b}+\text{A. t. }\frac{a}{9+b}+\text{A. t. }\frac{a}{16+b}+\text{etc.}+\text{A. t. }\frac{a}{xx+b}+\text{etc.}$$

Solu-

(*) Expreffiones (1) et (2) fponte fluunt ex formulis pro finu et cofinu, quas inuenit IOH. BERNOULLIVS (Opera omnia T. IV. Laufannae 1742. 4. Nr. CLII.), quarumque demonftrationem a dubiis liberam tradidit Ill. KAESTNERVS (*Analyfis des Unendlichen* §. 338-342.). Aliam demonftrationem exhibuit EVLERVS (Introduct. T. I. Cap. IX.), cui nuper ex principiis methodi limitum maiorem rigorem conciliauit Cel. L'HUILIER (Memoires de l'Academie Royale des Sciences et Belles Lettres 1788. 1789. Berlin 1793. pp. 326-368.). Formula (3), hic paullo aliter ac ab EVLERO (l. c. §. 159.) expreffa, ex prioribus fine negotio deriuatur (cf. L'HUILIER l. c. p. 356.).

Solutio.

1) Productum indefinitum $P\left(\frac{1+t^x\sqrt{-1}}{1-t^x\sqrt{-1}}\right)$ (§. VII.) est pro hac serie $= P\left(\frac{x^2+b+a\sqrt{-1}}{x^2+b-a\sqrt{-1}}\right) = P\left(\frac{x^2+(\beta+\alpha\sqrt{-1})^2}{x^2+(\beta-\alpha\sqrt{-1})^2}\right)$, posito $\sqrt{(b\pm a\sqrt{-1})} = \beta\pm\alpha\sqrt{-1}$. Cuius iam producti valor, factorum numero in infinitum excurrente, ex Lemmate praemisso assignari potest. Est nimirum $P\left(\frac{x^2+z^2}{z^2}\right)=\frac{e^{\pi z}-e^{-\pi z}}{2\pi z}$, $P\left(\frac{x^2+\zeta^2}{\zeta^2}\right)=\frac{e^{\pi\zeta}-e^{-\pi\zeta}}{2\pi\zeta}$, hinc diuidendo $P\left(\frac{x^2+z^2}{x^2+\zeta^2}\right)=\frac{\zeta}{z}\cdot\left\{\frac{e^{\pi z}-e^{-\pi z}}{e^{\pi\zeta}-e^{-\pi\zeta}}\right\}$. Sumendo igitur $z=\beta+\alpha\sqrt{-1}$, $\zeta=\beta-\alpha\sqrt{-1}$, productum illud infinitum abit in

$$\left(\frac{\beta-\alpha\sqrt{-1}}{\beta+\alpha\sqrt{-1}}\right)\cdot\left\{\frac{e^{\pi(\beta+\alpha\sqrt{-1})}-e^{-\pi(\beta+\alpha\sqrt{-1})}}{e^{\pi(\beta-\alpha\sqrt{-1})}-e^{-\pi(\beta+\alpha\sqrt{-1})}}\right\}$$

$$=\left(\frac{\beta-\alpha\sqrt{-1}}{\beta+\alpha\sqrt{-1}}\right)\cdot\left\{\frac{(e^{\pi\beta}-e^{-\pi\beta})\,\text{cos}.\,\pi\alpha+(e^{\pi\beta}+e^{-\pi\beta})\,\text{sin}.\,\pi\alpha.\sqrt{-1}}{(e^{\pi\beta}-e^{-\pi\beta})\,\text{cos}.\,\pi\alpha-(e^{\pi\beta}+e^{-\pi\beta})\,\text{sin}.\,\pi\alpha.\sqrt{-1}}\right\},$$

ob $e^{\pm\alpha\pi\sqrt{-1}} = \text{cos}.\,\alpha\pi\pm\sqrt{-1}.\,\text{sin}.\,\alpha\pi$.

2) Qua igitur ratione productum reuocatum est ad formam Coroll. 2. §. VIII, posito $N^I=-\alpha$, $M^I=\beta$; $N^{II}=(e^{\pi\beta}+e^{-\pi\beta})\,\text{sin}.\,\pi\alpha$, $M^{II}=(e^{\pi\beta}-e^{-\pi\beta})\,\text{cos}.\,\pi\alpha$. Hinc prouenit summa Arcuum $=$ A. tang. $\frac{N^I}{M^I}$ $+$ A. tang. $\frac{N^{II}}{M^{II}} = -$ A. tang. $\frac{\alpha}{\beta}+$ A. t. $\left\{\frac{e^{\pi\beta}+e^{-\pi\beta}}{e^{\pi\beta}-e^{-\pi\beta}}\right\}$ tang. $\pi\alpha$. Huius summae pars posterior aliter exprimi potest. Habetur nimirum, quicquid sint f, g et h, A. tang. $\left(\frac{f+g}{f-g}\cdot\text{tang}.\,h\right)$ — A. tang. (tang. h) $=$ A. t. $\frac{2g\,\text{tang}.\,h}{f-g-(f+g)(\text{tang}.\,h)^2}$ $=$ A. tang. $\frac{2g\,\text{tang}.\,h}{f(1+\text{tang}.\,h^2)-g(1-\text{tang}.\,h^2)}$ $=$ A. tang. $\frac{2g\,\text{sin}.\,h\,.\,\text{cos}.\,h}{f-g(\text{cos}.\,h^2-\text{sin}.\,h^2)}$, hinc A. tang. $\left(\frac{f+g}{f-g}\cdot\text{tang}.\,h\right) = h+$ A. tang. $\left(\frac{g\,\text{sin}.\,2h}{1-g\,\text{cos}.\,2h}\right)$. Quare fit summa Arcuum

$$=\pi\alpha - \text{A. tang.}\,\frac{\alpha}{\beta}+\text{A. tang.}\left\{\frac{\text{sin}.\,2\pi\alpha}{e^{2\pi\beta}-\text{cos}.\,2\pi\alpha}\right\}.$$

3) Quod

3) Quod ad determinationem quantitatum α et β attinet, eſt (1) $b + a\sqrt{-1} = \beta^2 + 2\alpha\beta\sqrt{-1} - \alpha^2$, hinc $b = \beta^2 - \alpha^2$, $a = 2\alpha\beta$; vnde $b = \frac{a^2}{4\alpha^2} - \alpha^2$, ſeu $\alpha^4 + b\alpha^2 = \frac{a^2}{4}$; et $\alpha^2 = \frac{\sqrt{(b^2 + a^2)} - b}{2}$, quia valor poſitiuus ſumi debet; porro $\beta^2 = \alpha^2 + b = \frac{\sqrt{(b^2 + a^2)} + b}{2}$. Eſt tandem $A.\,tang.\,\frac{\alpha}{\beta} = \frac{1}{2}\,A.\,t.\,\frac{2\alpha:\beta}{1 - \frac{\alpha^2}{\beta^2}}$ $= \frac{1}{2}\,A.\,tang.\,\frac{2\alpha\beta}{\beta^2 - \alpha^2} = \frac{1}{2}\,A.\,tang.\,\frac{a}{b}$.

4) Quibus combinatis plenam iam problematis ſolutionem ſequens complectitur

Summatio.

$$A.\,tang.\,\frac{a}{1+b} + A.\,t.\,\frac{a}{4+b} + A.\,t.\,\frac{a}{9+b} + \text{etc.} + A.\,t.\,\frac{a}{xx+b} + \text{in inf.} = \pi\alpha - \frac{1}{2}\,A.\,tang.\,\frac{a}{b} + A.\,tang.\left\{\frac{\text{ſin.}\,2\pi\alpha}{e^{2\pi\beta} - \text{coſ.}\,2\pi\alpha}\right\},$$

exiſtente $\alpha = \sqrt{\left(\frac{\sqrt{(b^2 + a^2)} - b}{2}\right)}$, $\beta = \sqrt{\left(\frac{\sqrt{(b^2 + a^2)} + b}{2}\right)} = \frac{a}{2\alpha}$.

Corollarium 1.

§. XCIV. Poſito $\frac{a}{b} = tang.\,\psi$, valores quantitatum α et β ſimplicius exprimi poſſunt. Eſt nimirum $\sqrt{(b^2 + a^2)} = b\sqrt{(1 + tang.\,\psi^2)} = \frac{b}{\text{coſ.}\,\psi}$, hinc

$\alpha = \sqrt{\frac{b(1 - \text{coſ.}\,\psi)}{2\,\text{coſ.}\,\psi}}$, $\beta = \sqrt{\frac{b(1 + \text{coſ.}\,\psi)}{2\,\text{coſ.}\,\psi}}$, vel $\alpha = (a^2 + b^2)^{\frac{1}{4}}\,\text{ſin.}\,\frac{\psi}{2}, = (b\,\text{ſec.}\,\psi)^{\frac{1}{2}}.\,\text{ſin.}\,\frac{\psi}{2}$, $\beta = (a^2 + b^2)^{\frac{1}{4}}\,\text{coſ.}\,\frac{\psi}{2} = (b\,\text{ſec.}\,\psi)^{\frac{1}{2}}.\,\text{coſin.}\,\frac{\psi}{2}$.

Corollarium 2.

§. XCV. Sit $b = 0$, erit $\alpha = \sqrt{\frac{a}{2}} = \beta$, $A.\,tang.\,\frac{a}{b} = \frac{\pi}{2}$. Inde obtinetur haec ſummatio: $A.\,tang.\,\frac{2\alpha\alpha}{1} + A.\,t.\,\frac{2\alpha\alpha}{4} + A.\,t.\,\frac{2\alpha\alpha}{9} \ldots + A.\,t.\,\frac{2\alpha\alpha}{xx} + \text{in inf.} =$

$$\pi\left(\alpha - \frac{1}{4}\right) + A.\,tang.\left\{\frac{\text{ſin.}\,2\pi\alpha}{e^{2\pi\alpha} - \text{coſ.}\,2\pi\alpha}\right\}.$$

Corol-

Corollarium 3.

Si b negatiuum valorem obtineat, vel terminus generalis feriei fummandae fit A. tang. $\frac{a}{x^2 - b}$, tum A. tang. $-\frac{a}{b}$ abit in $\pi -$ A. tang. $\frac{a}{b}$, porro $\sqrt{\left(\frac{\sqrt{(b^2+a^2)} \pm b}{2}\right)}$ in $\sqrt{\left(\frac{\sqrt{(b^2+a^2)}+b}{2}\right)}$. Quare retentis valoribus quantitatum α et β, quales §. XCIII. 4. exhibiti funt, eae in expreffione fummae permutari inuicem debent. Hine prodit fequens fummatio: A. t. $\frac{a}{1-b}$ + A. t. $\frac{a}{4-b}$ + A. t. $\frac{a}{9-b}$ + ... + A. t. $\frac{a}{xx-b}$ + etc. $= \pi(\beta - \frac{1}{2}) + \frac{1}{2}$ A. tang. $\frac{a}{b}$ + A. t. $\left\{\frac{\text{fin. } 2\beta\pi}{e^{2\alpha\pi} - \text{cof. } 2\beta\pi}\right\}$, pofito, vti fupra, $\alpha = \sqrt{\left(\frac{\sqrt{(b^2+a^2)}-b}{2}\right)}$, $\beta = \sqrt{\left(\frac{\sqrt{(b^2+a^2)}+b}{2}\right)}$. Si inter terminos huius feriei occurrunt arcus, quorum tangentes negatiuae funt $= -$ h, ii quadrante maiores accipiendi funt, $= \pi -$ A. tang. h.

Corollarium 4.

§. XCVII. 1) Seriei: A. tang. $\frac{a}{4+b}$ + A. t. $\frac{a}{16+b}$ + A. t. $\frac{a}{36+b}$ + ... + A. t. $\frac{a}{4x^2+b}$ + in inf. terminus generalis eft = A. tang. $\frac{\frac{a}{4}}{x^2 + \frac{b}{4}}$; hinc pro ea fummanda ponendum eft loco a (§. XCIII.), $\frac{a}{4}$; loco b, $\frac{b}{4}$; vnde $\sqrt{\left(\frac{\sqrt{(b^2+a^2)} \pm b}{2}\right)}$ iam abit in $\sqrt{\left(\frac{\sqrt{(b^2+a^2)} \pm b}{8}\right)}$, i. e. pro α fupponendum eft $\frac{\alpha}{2}$; pro β, $\frac{\beta}{2}$. Quare fumma illius feriei fit $= \frac{\pi\alpha}{2} - \frac{1}{2}$ A. tang. $\left\{\frac{\text{fin. } \alpha\pi}{e^{\beta\pi} - \text{cof. } \pi\alpha}\right\}$. Ea iam ferie a priori (§. XCIII.) fubducta, remanet A. tang. $\frac{a}{1+b}$ + A. t. $\frac{a}{9+b}$ + A. t. $\frac{a}{25+b}$ + ... + A. t. $\frac{a}{(2x-1)^2+b}$ + in inf. $= \frac{\pi\alpha}{2}$ + A. t. $\left\{\frac{\text{fin. } 2\alpha\pi}{e^{2\beta\pi} - \text{cof. } 2\alpha\pi}\right\}$ − A. t. $\left\{\frac{\text{fin. } \alpha\pi}{e^{\beta\pi} - \text{cof. } \alpha\pi}\right\}$ $= \frac{\pi\alpha}{2}$ − A. t. $\left\{\frac{\text{fin. } \alpha\pi}{e^{\beta\pi} + \text{cof. } \alpha\pi}\right\}$, quoniam A. t. $\left\{\frac{\text{fin. } \alpha\pi}{e^{\beta\pi} - \text{cof. } \alpha\pi}\right\}$ − A. t. $\left\{\frac{\text{fin. } \alpha\pi}{e^{\beta\pi} + \text{cof. } \alpha\pi}\right\}$

eft

eſt $= \text{A. tang.} \dfrac{2 \,\text{ſin.}\, \alpha\pi \,.\, \text{coſ.}\, \alpha\pi}{e^{2\beta\pi} - (\text{coſ.}\, \alpha\pi)^2 + (\text{ſin.}\, \alpha\pi)^2} = \text{A. tang.} \left\{ \dfrac{\text{ſin.}\, 2\alpha\pi}{e^{2\beta\pi} - \text{coſ.}\, 2\alpha\pi} \right\}$.

2) Eadem ſummatio ſic inueniri poteſt: Pro termino generali $\text{A. t.} \dfrac{a}{(2x-1)^2+b}$ $= \text{A. tang.}\, t^x$, eſt productum indefinitum (§. VII.) $P\left(\dfrac{1+t^x\gamma-1}{1-t^x\gamma-1}\right) =$ $P\left(\dfrac{(2x-1)^2+(\beta+\alpha\gamma-1)^2}{(2x-1)^2+(\beta-\alpha\gamma-1)^2}\right)$, quod, ob $P\left(\dfrac{(2x-1)^2+z^2}{(2x-1)^2}\right) = \dfrac{e^{\frac{\pi z}{2}} + e^{-\frac{\pi z}{2}}}{2}$

(§. XCII. 2.), abit in $\dfrac{e^{\frac{\pi(\beta+\alpha\gamma-1)}{2}} + e^{-\frac{\pi(\beta+\alpha\gamma-1)}{2}}}{e^{\frac{\pi(\beta-\alpha\gamma-1)}{2}} + e^{-\frac{\pi(\beta-\alpha\gamma-1)}{2}}} =$

$\dfrac{e^{\frac{\pi\beta}{2}}\left(\text{coſ.}\,\frac{\pi\alpha}{2} + \gamma-1\,.\,\text{ſin.}\,\frac{\alpha\pi}{2}\right) + e^{-\frac{\pi\beta}{2}}\left(\text{coſ.}\,\frac{\alpha\pi}{2} - \gamma-1\,.\,\text{ſin.}\,\frac{\alpha\pi}{2}\right)}{e^{\frac{\pi\beta}{2}}\left(\text{coſ.}\,\frac{\pi\alpha}{2} - \gamma-1\,.\,\text{ſin.}\,\frac{\pi\alpha}{2}\right) + e^{-\frac{\pi\beta}{2}}\left(\text{coſ.}\,\frac{\alpha\pi}{2} + \gamma-1\,.\,\text{ſin.}\,\frac{\alpha\pi}{2}\right)}$. Hinc ex (§. VII.)

ſumma prodit $= \text{A. tang.} \left\{ \dfrac{\left(e^{\frac{\pi\beta}{2}} - e^{-\frac{\pi\beta}{2}}\right)}{e^{\frac{\pi\beta}{2}} + e^{-\frac{\pi\beta}{2}}} \,.\, \text{tang.}\, \dfrac{\pi\alpha}{2} \right\}$. Ex qua formula altera (1) ſine negotio deriuatur. Quare iam demonſtrata eſt ſequens

Summatio.

$\text{A. tang.} \dfrac{a}{1+b} + \text{A. t.} \dfrac{a}{9+b} + \text{A. t.} \dfrac{a}{25+b} + \ldots + \text{A. t.} \dfrac{a}{(2x-1)^2+b} + \text{etc.} =$ $\dfrac{\alpha\pi}{2} - \text{A. tang.} \left\{ \dfrac{\text{ſin.}\, \alpha\pi}{e^{\beta\pi} + \text{coſ.}\, \alpha\pi} \right\}$, ſiue $= \text{A. tang.} \left\{ \dfrac{e^{\beta\pi} - 1}{e^{\beta\pi} + 1} \,.\, \text{tang.}\, \dfrac{\alpha\pi}{2} \right\}$, poſito $\gamma(b + a\gamma - 1) = \beta + \alpha\gamma - 1$, vel habentibus α et β valores ſupra (§. XCII.) aſſignatos.

Corol-

Corollarium 5.

§. XCVIII. 1) Pro b = o est $\alpha = \beta = \sqrt{\frac{a}{2}}$, hinc fit A. tang. $\frac{2\alpha\alpha}{2}$ + A. tang. $\frac{2\alpha\alpha}{9}$ + A. t. $\frac{2\alpha\alpha}{25}$ + ... + A. t. $\frac{2\alpha\alpha}{(2\lambda-1)^2}$ + in inf. = $\frac{\alpha\pi}{2}$ — A. tang. $\left\{\frac{\text{fin. }\alpha\pi}{e^{\alpha\pi}+\text{cof. }\alpha\pi}\right\}$.

2) Si b negatiuum valorem habeat, tum vti §. XCVI, obtinetur:

A. tang. $\frac{a}{1-b}$ + A. t. $\frac{a}{9-b}$ + A. t. $\frac{a}{25-b}$ + etc. = $\frac{\beta\pi}{2}$ — A. t. $\left\{\frac{\text{fin. }\beta\pi}{e^{\alpha\pi}+\text{cof. }\beta\pi}\right\}$, manentibus valoribus quantitatum α et β iisdem.

Corollarium 6.

§. XCIX. A fummae feriei (§. XCVII. duplo fubtracta fumma feriei (§. XCII.) remanet $\alpha\pi$ — 2 A. t. $\left\{\frac{\text{fin. }\alpha\pi}{e^{\beta\pi}+\text{cof. }\alpha\pi}\right\}$ — $\alpha\pi$ + $\frac{1}{2}$ A. t. $\frac{a}{b}$ — A. t. $\left\{\frac{\text{fin. }2\pi\alpha}{e^{2\beta\pi}-\text{cof. }2\alpha\pi}\right\}$ = $\frac{1}{2}$ A. t. $\frac{a}{b}$ — 2 A. t. $\left\{\frac{\text{fin. }\alpha\pi}{e^{\beta\pi}+\text{cof. }\alpha\pi}\right\}$ + A. t. $\left\{\frac{\text{fin. }\alpha\pi}{e^{\beta\pi}-\text{cof. }\pi\alpha}\right\}$ + A. t. $\left\{\frac{\text{fin. }\alpha\pi}{e^{\beta\pi}+\text{cof. }\alpha\pi}\right\}$ = $\frac{1}{2}$ A. t. $\frac{a}{b}$ — A. t. $\left\{\frac{\text{fin. }\alpha\pi}{e^{\beta\pi}-\text{cof. }\alpha\pi}\right\}$ — A. t. $\left\{\frac{\text{fin. }\alpha\pi}{e^{\beta\pi}+\text{cof. }\alpha\pi}\right\}$ = $\frac{1}{2}$ A. t. $\frac{a}{b}$ — A. tang. $\left\{\frac{2e^{\beta\pi}\text{fin. }\alpha\pi}{e^{2\beta\pi}-1}\right\}$. Eft autem 2 (A. t. $\frac{a}{1+b}$ + A. t. $\frac{a}{9+b}$ + A. t. $\frac{a}{25+b}$ + ...) — (A. tang. $\frac{a}{1+b}$ + A. tang. $\frac{a}{4+b}$ + A. tang. $\frac{a}{9+b}$ + ...) = A. tang. $\frac{a}{1+b}$ — A. t. $\frac{a}{4+b}$ + A. t. $\frac{a}{9+b}$ — etc. Inde huius feriei fignis alternantibus inftructae haec obtinetur Summatio.

A. tang. $\frac{a}{1+b}$ — A. t. $\frac{a}{4+b}$ + A. t. $\frac{a}{9+b}$ — A. t. $\frac{a}{16+b}$ + etc. ± A. t. $\frac{a}{x^2+b}$ ∓ in inf. = $\frac{1}{2}$ A. t. $\frac{a}{b}$ — A. t. $\left\{\frac{2e^{\beta\pi}\text{fin. }\alpha\pi}{e^{2\beta\pi}-1}\right\}$, valoribus τῶν α et β ex §. XCII. cognitis. Si terminus generalis eft = ± A. tang. $\frac{a}{x^2-b}$, tum fumma abit in $\frac{\pi}{2}$ — $\frac{1}{2}$ A. t. $\frac{a}{b}$ —

A. tang.

$$\text{A. tang.}\left\{\frac{2e^{\alpha\pi}\sin\beta\pi}{e^{2\alpha\pi}-1}\right\} = \text{A. tang.}\left\{\frac{e^{2\alpha\pi}-1}{2e^{\alpha\pi}\sin.\beta\pi}\right\} - \tfrac{1}{2}\,\text{A. tang.}\,\frac{a}{b}.$$

Corollarium 7.

§. C. Ob $\dfrac{a}{(2x-1)^2+b} = \dfrac{a}{4x^2-4x+1+b} = \dfrac{a:4}{x^2-x+\dfrac{1+b}{4}}$ ex §. XCVII, posito $a = 4A$, $\dfrac{1+b}{4} = B$, haec fluit Summatio:

$$\text{A. tang.}\frac{A}{2+B}+\text{A. t.}\frac{A}{6+B}+\text{A. t.}\frac{A}{12+B}+\text{etc.}+\text{A. t.}\frac{A}{xx+x+B}+\text{etc.} = \frac{\pi\alpha}{2} -$$

$$\text{A. tang.}\frac{A}{B}-\text{A. t.}\left\{\frac{\sin.\alpha\pi}{e^{\beta\pi}+\cos.\alpha\pi}\right\},\ \text{supposito}\ \alpha = \sqrt{\left(\frac{\sqrt{((4B-1)^2+16A^2)}-4B+1}{2}\right)},$$

$$\beta = \sqrt{\left(\frac{\sqrt{((4B-1)^2+16A^2)}+4B-1}{2}\right)}.$$

Corollarium 8.

§. CI. Cum quantitates α et β, quae in summis serierum (§§. XCII. XCVII. XCIX.) occurrunt, inuoluant quantitatem radicalem $\sqrt{(a^2+b^2)}$, vt haec rationalis prodeat, ponendum est $\dfrac{b}{a} = \dfrac{A^2-B^2}{2AB}$; tumque fit $\sqrt{(b^2+a^2)} = a\left(\dfrac{A^2+B^2}{2AB}\right)$, $\alpha = \sqrt{\left(\dfrac{a(A^2+B^2)-a(A^2-B^2)}{4AB}\right)} = \sqrt{\dfrac{aB}{2A}}$, $\beta = \sqrt{\dfrac{aA}{2B}}$. Quo iam quantitates α et β ipsae ab irrationalitate liberentur, ponatur $\dfrac{A}{2B} = \dfrac{a}{m^2}$, eritque $\beta = \dfrac{a}{m}$, $\alpha = \dfrac{m}{2}$. Exinde summationes (§§. XCII. XCVII. XCIX.) in has transformantur:

1) $$\text{A. tang.}\frac{a}{1+b}+\text{A. t.}\frac{a}{4+b}+\text{A. t.}\frac{a}{9+b}+\ldots+\text{A. t.}\frac{a}{xx+b}+\text{etc.}$$

$$= \frac{m\pi}{2}-\text{A. t.}\frac{m^2}{2a}+\text{A. t.}\left\{\frac{\sin.m\pi}{e^{\frac{2a\pi}{m}}-\cos.m\pi}\right\}.$$

2) $$\text{A. tang.}\frac{a}{1+b}+\text{A. t.}\frac{a}{9+b}+\text{A. t.}\frac{a}{25+b}+\ldots+\text{A. t.}\frac{a}{(2x-1)^2+b}+\text{etc.}$$

$$= \frac{m\pi}{4} - \text{A. t.}\left\{\frac{\sin.\frac{m\pi}{2}}{e^{\frac{a\pi}{m}}+\cos.\frac{m\pi}{2}}\right\}.$$

3) A.

3) $A.\,tang.\,\frac{a}{1+b} - A.\,t.\,\frac{a}{4+b} + A.\,t.\,\frac{a}{9+b} - \ldots \pm A.\,tang.\,\frac{a}{xx+b} \mp etc.$

$$= A.\,tang.\,\frac{mm}{2a} - A.\,tang.\left\{\frac{2e^{\frac{a\pi}{m}}\,fin.\,\frac{m\pi}{2}}{e^{\frac{2a\pi}{m}}-1}\right\},$$ posito pro tribus hisce seriebus $b = \frac{a^2}{m^2} - \frac{m^2}{4}$.

Corollarium 9.

§. CII. Posito m = numero integro = r, fin. $r\pi$ est = o, nec non fin. $\frac{r\pi}{2}$, si r fuerit numerus par. Hinc ex formulis summatoriis (§. CI.), ob euanescentem summarum vltimam partem, sequentes fluunt summationes:

$$A.\,tang.\,\frac{a}{1+b} + A.\,t.\,\frac{a}{4+b} + A.\,t.\,\frac{a}{9+b} + etc. = \frac{\pi r}{2} - A.\,t.\,\frac{rr}{2a},$$

$$A.\,tang.\,\frac{a}{1+b} + A.\,t.\,\frac{a}{9+b} + A.\,t.\,\frac{a}{25+b} + etc. = \frac{\pi r}{4},$$

$$A.\,tang.\,\frac{a}{1+b} - A.\,t.\,\frac{a}{4+b} + A.\,t.\,\frac{a}{9+b} - etc. = A.\,t.\,\frac{rr}{2a},$$

si ponatur $b = \frac{a^2}{r^2} - \frac{r^2}{4}$, et r pro prima serie = cuiuis numero integro, pro secunda et tertia = numero *pari*. Inde fit, sumto b = o seu $a = \frac{r^2}{2}$,

$$A.\,tang.\,\frac{r^2}{2} + A.\,t.\,\frac{r^2}{2.4} + A.\,t.\,\frac{r^2}{2.9} + etc. = \frac{(2r-1)\pi}{4},$$

$$A.\,tang.\,\frac{r^2}{2.1} + A.\,t.\,\frac{r^2}{2.9} + A.\,t.\,\frac{r^2}{2.25} + etc. = \frac{\pi r}{4},$$

$$A.\,tang.\,\frac{r^2}{2} - A.\,t.\,\frac{r^2}{2.4} + A.\,t.\,\frac{r^2}{2.9} - etc. = \frac{\pi}{4}.$$

Quae summationes cum supra (§§. XXIV. XXVI. XXVIII.) aliunde inuentis apprime conspirant.

Corollarium 10.

§. CIII. 1) Sit in prima serie (§. CI.) $m = \frac{2k-1}{2}$, erit fin. $m\pi = \pm 1$, prouti k fuerit numerus impar vel par; cof. $m\pi$ = o. Inde fit

$$A.\,tang.\,\frac{a}{1+b} + A.\,t.\,\frac{a}{4+b} + A.\,t.\,\frac{a}{9+b} + etc. = \frac{(2k-1)\pi}{4} - A.\,t.\,\frac{(2k-1)^2}{8a}$$

$$\pm A.\,tang.\,\frac{1}{e^{\frac{4a\pi}{2k-1}}},$$ posito $b = \frac{4a^2}{(2k-1)^2} - \frac{(2k-1)^2}{16}$.

2) Hinc

2) Hinc pro $b = 0$, vel $a = \frac{(2k-1)^2}{8}$, erit

$$\text{A. t. } \frac{(2k-1)^2}{8.1} + \text{A. t. } \frac{(2k-1)^2}{8.4} + \text{A. t. } \frac{(2k-1)^2}{8.9} + \text{etc.} = \frac{(k-1)\pi}{2} \pm \text{A. t. } \frac{1}{e^{\frac{(2k-1)\pi}{2}}}.$$

Exempli gratia pro $k = 1$, et $= 2$ hae prodeunt summationes:

$$\text{A. t. } \frac{1}{8.1} + \text{A. t. } \frac{1}{8.4} + \text{A. t. } \frac{1}{8.9} + \ldots + \text{A. t. } \frac{1}{8xx} + \text{etc.} = \text{A. t. } \left(e^{-\frac{\pi}{2}}\right) \ldots$$

$$\text{A. t. } \frac{9}{8.1} + \text{A. t. } \frac{9}{8.4} + \text{A. t. } \frac{9}{8.9} + \ldots + \text{A. t. } \frac{9}{8xx} + \text{etc.} = \text{A. t. } \left(e^{\frac{3\pi}{2}}\right) \ldots$$

3) In serie secunda et tertia (§. CI.) fit $m = 2k - 1$, eritque sin. $\frac{m\pi}{2} = \pm 1$, cos. $\frac{m\pi}{2} = 0$: Hinc fit

$$\text{A. t. } \frac{a}{1+b} + \text{A. t. } \frac{a}{9+b} + \text{A. t. } \frac{a}{25+b} + \text{etc.} = \frac{(2k-1)\pi}{4} \mp \text{A. t. } \frac{1}{e^{\frac{a\pi}{2k-1}}},$$

$$\text{A. t. } \frac{a}{1+b} - \text{A. t. } \frac{a}{4+b} + \text{A. t. } \frac{a}{9+b} - \text{A. t. } \frac{a}{16+b} + \text{etc.} = \text{A. t. } \frac{(2k-1)^2}{2a} \mp$$

2 A. t. $\frac{1}{e^{\frac{a}{2k-1}}}$, posito $b = \frac{a^2}{(2k-1)^2} - \frac{(2k-1)^2}{4}$, et sumto signo superiori pro impari k, inferiori pro pari.

4) Inde obtinetur pro $b = 0$, seu $a = \frac{(2k-1)^2}{2}$,

$$\text{A. t. } \frac{(2k-1)^2}{2.1} + \text{A. t. } \frac{(2k-1)^2}{2.9} + \text{A. t. } \frac{(2k-1)^2}{2.25} + \text{etc.} = \frac{(2k-1)\pi}{4} \mp \text{A. t. } \frac{1}{e^{\frac{(2k-1)\pi}{2}}};$$

$$\text{A. t. } \frac{(2k-1)^2}{2.1} - \text{A. t. } \frac{(2k-1)^2}{2.4} + \text{A. t. } \frac{(2k-1)^2}{2.9} - \text{etc.} = \frac{\pi}{4} \mp 2 \text{ A. t. } \frac{1}{e^{\frac{(2k-1)\pi}{2}}}.$$

Exempli gratia pro $k = 1$, et $= 2$ erit:

$$\text{A. t. } \frac{1}{2.1} + \text{A. t. } \frac{1}{2.9} + \text{A. t. } \frac{1}{2.25} + \ldots + \text{A. t. } \frac{1}{2(2x-1)^2} + \text{etc.} = \frac{\pi}{4} - \text{A. cot. } e^{\frac{\pi}{2}},$$

$$\text{A. t. } \frac{1}{2.1} - \text{A. t. } \frac{1}{2.4} + \text{A. t. } \frac{1}{2.9} + \text{etc. } \pm \text{A. t. } \frac{1}{2xx} = \frac{\pi}{4} - 2 \text{ A. cot. } e^{\frac{\pi}{2}},$$

$$\text{A. t.}\,\frac{9}{2.1}+\text{A. t.}\,\frac{9}{2.9}+\text{A. t.}\,\frac{9}{2.25}+\text{etc.}=\frac{3\pi}{4}+\text{A. cot. e}^{\frac{3\pi}{2}},$$

$$\text{A. t.}\,\frac{9}{2.1}-\text{A. t.}\,\frac{9}{2.4}+\text{A. t.}\,\frac{9}{2.9}-\text{A. t.}\,\frac{9}{2.16}+\text{etc.}=\frac{\pi}{4}+2\,\text{A. cot. e}^{\frac{3\pi}{2}}.$$

Scholion 1.

§. CIV. 1) Casus Coroll. 2. Probl. 8. alia iam occasione solutionem publicaui, inuenique: $\text{A. tang.}\,\frac{2\alpha^2}{1}+\text{A. tang.}\,\frac{2\alpha^2}{4}+\text{A. tang.}\,\frac{2\alpha^2}{9}+\text{etc.}=\frac{\pi}{4}-$ $\text{A. tang.}\left\{\frac{e^{2\alpha\pi}-1}{e^{2\alpha\pi}+1}.\,\text{cotang.}\,\alpha\pi\right\}$, (*) quae expressio ad supra (§. XCV.) inuentam reduci potest. Est nimirum ea $=\text{A. t.}\left\{\frac{e^{2\alpha\pi}+1}{e^{2\alpha\pi}-1}.\,\text{tang.}\,\alpha\pi\right\}-\frac{\pi}{4}=\alpha\pi+$ $\text{A. tang.}\left\{\frac{\text{sin. }2\alpha\pi}{e^{2\alpha\pi}-\text{cos. }2\alpha\pi}\right\}-\frac{\pi}{4}$ (§. XCIII. 2.). Quanquam ita consensus inter vtramque formulam appareat, posterior tamen hic tradita priori praeferenda est; quin haec, nisi caute adhibeatur, ad errores deducit, qui quomodo euitandi sint, haud statim in aperto est. Etenim ponatur $\alpha=\frac{r\pi}{2}$, existente r = numero integro, tum erit cot. $\alpha\pi=0$ pro r impari, $=\infty$ pro r pari. Hinc summa ex priori expressione, pro r impari, $=\frac{\pi}{4}-\text{A. tang. }0=\frac{\pi}{4}$ prodire videtur: quod cum summatione supra (§. CII. XXIV.) inuenta pugnat, quippe summa $\frac{\pi}{4}$ casui tantum r = 1 conuenit, pro maiore autem r maior esse debet, et quidem $=\frac{(2r-1)\pi}{4}$.

2) Ad hanc difficultatem tollendam praemisi obseruationem Scholii (§. XI.). Quanquam nimirum prior expressio praebeat summam $=\frac{\pi}{4}-\text{A. tang. }0$, haud tamen inde concludi potest, summam semper esse $=\frac{\pi}{4}$, quippe A. tang. o non tantum est = o, verum

(*) *Versuch einer neuen Summations-Methode; nebst andern damit zusammenhängenden analytischen Bemerkungen Berlin* 1788. 8. pag 101 sqq. Methodus qua tum vsus sum, differt ab ea, cuius ope hoc loco summationes generaliores obtinui; illa nimirum nititur primo differentiatione seriei summandae, deinde integratione differentialis, ope summationis quarundam serierum, cuiusmodi §. CV. 1. commemorantur.

verum etiam $= \pm k\pi$, denotante k quemuis numerum integrum. Exinde apparet, illa expressione summam indeterminatam relinqui, quae per alteram demum determinatur $= \frac{(2r-1)\pi}{4}$: ac est reuera $\frac{(2r-1)\pi}{4} = \frac{\pi}{4} + \frac{(r-1)\pi}{2}$, vnde numerus indeterminatus $k = \frac{(r-1)}{2}$ et negatiue sumi debet. Pro r = numero pari similiter ratiocinari licet.

3) Quae ita pro casu $\alpha = \frac{r\pi}{2}$ exposita sunt, ad quemuis valorem quantitatis α patent. Cum sit A. tang. $h = H \pm k\pi$, denotante H arcum minimum, cui tangens h competat, Arcus in priori expressione occurrens sc. A. t. $\left\{\frac{e^{2\alpha\pi}-1}{e^{2\alpha\pi}+1} \cdot \text{cotang.}\,\alpha\pi\right\}$ est quantitas indeterminata, quippe pro eo Arcum minimum haud semper accipi posse iam ex (1) liquet: indeque ipsa summae expressio ambigua est. Quam ambiguitatem altera iam expressio tollit, in qua arcus $\alpha\pi$ separatus est. Pro arcu nimirum, quem ea inuoluit, A. tang. $\left\{\frac{\text{sin.}\,2\alpha\pi}{e^{2\alpha\pi}-\text{cos.}\,2\alpha\pi}\right\}$, semper Arcus minimus tangentis accipiendus est, et, si tangens negatiua sit, Arcus negatiuus seu pro A. tang. $-h$, $-$ A. tang. h.

4) Quod assertum vt extra dubium ponatur, redeundum est ad summas §. CII. exhibitas, quae iam supra (§. XXIV. XXVI. XXVIII.) ex aliis principiis sunt demonstratae, vt pro iis quidem certum sit, Arcus minimos accipiendos esse cf. §. XXIII. XXI.). Pro summatione generaliore (§. CI. 1.), (ad hanc enim omnis quaestio reducitur) est $a = m\sqrt{(b + \frac{m^2}{4})}$, cum pro serie correspondente specialiori (§. CII.) sit $a = r\sqrt{(b + \frac{r^2}{4})}$, existente r numero integro. Vtriusque nunc seriei comparatio instituenda est. Quantitas b vtrique sit communis, sumaturque ea primo affirmatiue; quantitas a pro serie generaliore designetur per A. Cum quantitas $r\sqrt{(b + \frac{r^2}{4})}$ a nihilo in infinitum crescat, crescente r a o in infinitum: quicquid sit A, semper duo numeri integri r et $r+1$ exhiberi possunt, vt valorum illis respondentium quantitatis a, sc. a et a^I, vno quantitas A maior, altero minor sit, seu $A > a$ et $< a^I$; vnde simul erit $m > r$ et $< r+1$. Iam cum quantitate b eadem manente, seriei Arcuum singuli termini = A. tang. $\frac{a}{xx+b}$ crescant, crescente a, erit summa seriei generalioris, pro assumto A, $> \frac{\pi r}{2} -$ A. t. $\frac{r}{\sqrt{(r^2+4b)}}$ et $< \frac{\pi(r+1)}{2} -$ A. tang. $\frac{r+1}{\sqrt{((r+1)^2+4b)}}$, hinc magis adhuc $> \frac{\pi r}{2} - \frac{\pi}{4}$,

$-\frac{\pi}{4}$, et $< \frac{\pi(r+1)}{2}$. Quodsi nunc summa esset $= \frac{m\pi}{2} \pm k\pi - A.\ tang.\ \frac{m^2}{2a} +$ $A.\ tang. \left\{ \frac{\sin.\ m\pi}{e^{\frac{2\alpha\pi}{m}} - \cos.\ m\pi} \right\}$, i. e. si non Arcus minimi in summae expressione acciperentur, sed additione demum vel subtractione multipli semicircumferentiae vera summa prodiret, tum, primo k affirmatiue sumto, summa foret $> \frac{m\pi}{2} + \pi - \frac{\pi}{2}$ seu $> \frac{(m+1)\pi}{2}$; est nimirum Arcus subtractus A. tang. $\frac{m^2}{2a} < \frac{\pi}{4}$ ob $\frac{m^2}{2a} < 1$, et si vel sin. $m\pi$ negatiuus hincque Arcus $A.\ tang. \left\{ \frac{\sin.\ m\pi}{e^{\frac{2\alpha\pi}{m}} - \cos.\ m\pi} \right\}$ etiam subtrahendus esset, hic quoque Arcus minor quadrante est, ob $\frac{\sin.\ m\pi}{e^{\frac{2\alpha\pi}{m}} - \cos.\ m\pi} < \frac{m^2}{2a} < 1$, id quod facile intelligitur, quippe quod $\frac{2a}{m^2}\sin.\ m\pi < \frac{2a}{m^2} . m\pi < \frac{2\alpha\pi}{m}$, cos. $m\pi < 1$, indeque $\frac{2a}{m^2}$ sin. $m\pi +$ cos. $m\pi < e^{\frac{2\alpha\pi}{m}}$ $(= 1 + \frac{2\alpha\pi}{m} +$ etc.$)$. Porro cum sit $m > r$, summa ista magis adhuc foret $> \frac{(r+1)\pi}{2}$, quod esse nequit, cum eam $< \frac{(r+1)\pi}{2}$ prodire iam vidimus. Deinde si k negatiue accipiatur, tum summa foret $< \frac{m\pi}{2} - \pi$, i. e. $< \frac{(m-2)\pi}{2}$ hinc, ob $m < r+1$, magis adhuc summa $< \frac{(r-1)\pi}{2}$, quod rursus esse nequit, cum summa sit $> \frac{\pi r}{2} - \frac{\pi}{4}$. Ex his satis perspicitur, pro A. t. $\frac{m^2}{2a}$ et $A.\ t. \left\{ \frac{\sin.\ m\pi}{e^{\frac{2\alpha\pi}{m}} - \cos.\ m\pi} \right\}$ Arcus minimos accipi debere, et posteriorem arcum negatiue, si tangens fuerit negatiua.

5) Sumatur iam b negatiue, vel consideretur series A. tang. $\frac{a}{1-b} +$ A. t. $\frac{a}{4-b}$ $+$ A. t. $\frac{a}{9-b} +$ etc., et erit $b = \frac{m^2}{4} - \frac{a^2}{m^2}$ pro serie generaliori, pro serie particulari

b =

$b = \frac{r^2}{4} - \frac{a^2}{r^2}$. Vtrique feriei communis fit quantitas a, b autem pro illa ferie defignetur per B. Cum crefcente r quantitas b in infinitum crefcat, femper duo numeri r, $r+1$ affumi poffunt, vt fit $B > b$ et $< b^{I}$, indeque $m > r$ et $< r+1$. Iam crefcente b et manente a finguli Arcus $=$ A. tang. $\frac{a}{xx - b}$ crefcunt, quoniam tangentium pofitiuarum maiorum Arcus etiam maiores funt, et pro tangentibus negatiuis A. tang. $- h = \pi -$ A. t. $+h$ crefcit, decrefcente $+h$. Hinc fumma feriei generalis erit $> \frac{\pi r}{2} -$ A. t. $\frac{r^2}{2a}$ et $< \frac{\pi(r+1)}{2} -$ A. t. $\frac{(r+1)^2}{2a}$, magisque adhuc $> \frac{\pi . r-1)}{2}$ et $< \frac{\pi(r+1)}{2}$. Quodfi nunc fumma poneretur $= \frac{m\pi}{2} \pm k\pi -$ A. tang. $\frac{m^2}{2a} +$ A. t. $\left\{ \frac{\text{fin. } m\pi}{e^{\frac{2\alpha\pi}{m}} - \text{cof. } m\pi} \right\}$, tum ea foret, pro $+k$, $> \frac{m\pi}{2} + \pi - \frac{\pi}{2}$ feu $> \frac{(m+1)\pi}{2}$ et magis $> \frac{(r+1)\pi}{2}$; pro $-k$ fumma effet $< \frac{m\pi}{2} - \pi$ feu $< \frac{(m-2)\pi}{2}$ magisque igitur $< \frac{(r-1)\pi}{2}$. Vtrumque autem contradictionem inuolueret.

6) Exinde apparet, fiue quantitas b fuerit affirmatiua fiue negatiua, in expreffione fummae (§. CI 1.) Arcus minimos intelligendos effe; et, fi cum fin. $m\pi$ tangens in negatiuum abeat, fub A. tang. $-h$, $-$ A. tang. h. Pofterior autem interpretatio nequaquam de ipfis feriei fummandae terminis obtinet, quippe hi ex regula iam fupra ftabilita affirmatiue, et tangentium cum b forte negatiuarum Arcus obtufi accipiendi funt. Id enim de ferie particulari ex fuperioribus conftat, indeque transfertur ad generalem. Quae omnia etiam de reliquis fummationibus (§. CI. 2. 3.), quippe primae corollariis, obferuanda effe manifeftum eft.

Scholion 2.

§. CV. In extricandis difficultatibus, quarum expofitionem et refolutionem praecedens Scholion continet, occupatus, incidi in aliam folutionem problematis fundamentalis 8 (§. XCIII.), quae nititur refolutione fingulorum feriei fummandae terminorum in feries infinitas, cuius quidem methodi alio loco (*) compluria exhibui fpecimina, illud autem, quod iam tradam, tum nondum animaduerteram.

1) Eft nimirum terminus generalis feriei (§. XCIII.) A. tang. $\left(\frac{a}{xx+b}\right) = \frac{a}{xx+b} - \frac{1}{3}\left(\frac{a}{xx+b}\right)^3 + \frac{1}{5}\left(\frac{a}{xx+b}\right)^5 -$ etc. Hinc prodit fumma feriei feu

S. A.

(*) l. c.

S. A. tang. $\frac{a}{xx+b} = a\text{S.}\frac{1}{xx+b} - \frac{a^3}{3}\text{S.}\frac{1}{(xx+b)^3} + \frac{a^5}{5}\text{S.}\left(\frac{1}{xx+b}\right)^5 + \text{etc.}$ Est autem $\frac{1}{1+b} + \frac{1}{4+b} + \frac{1}{9+b} +$ in inf. vel S. $\frac{1}{xx+b} = -\frac{1}{2b} + \left\{\frac{e^{2\pi\sqrt{b}}+1}{e^{2\pi\sqrt{b}}-1}\right\}.\frac{\pi}{2\sqrt{b}}$. (*) Ex hac summa reliquae summae sc. S. $\frac{1}{(x^2+b)^3}$, S. $\frac{1}{(x^2+b)^5}$ ... S. $\frac{1}{(x^2+b)^{2n-1}}$ ... ope differentiationis elici possunt. Posito enim:

$$\frac{1}{1+b} + \frac{1}{4+b} + \frac{1}{9+b} + \ldots + \frac{1}{xx+b} + \ldots = y,$$

erit, quantitatem b pro variabili habendo,

$$\frac{1}{(1+b)^2} + \frac{1}{(4+b)^2} + \frac{1}{(9+b)^2} \ldots + \frac{1}{(xx+b)^2} + \ldots = -\frac{dy}{db},$$

$$\frac{1}{(1+b)^3} + \frac{1}{(4+b)^3} + \frac{1}{(9+b)^3} \ldots + \frac{1}{(xx+b)^3} + \ldots = +\frac{1}{2}\frac{d^2y}{db^2},$$

$$\frac{1}{(1+b)^4} + \frac{1}{(4+b)^4} + \frac{1}{(9+b)^4} \ldots + \frac{1}{(xx+b)^4} + \ldots = -\frac{1}{2.3}\frac{d^3y}{db^3},$$

. . . .

$$\frac{1}{(1+b)^m} + \frac{1}{(4+b)^m} + \frac{1}{(9+b)^m} \ldots + \frac{1}{(xx+b)^m} + \ldots = \pm\frac{1}{1.2.3\ldots m-1}\cdot\frac{d^{m-1}y}{db^{m-1}}.$$

2) Quibus summarum expressionibus suppositis, prouenit S. A. tang. $\frac{1}{xx+b} = ay - \frac{a^3}{1.2.3}\frac{d^2y}{db^2} + \frac{a^5}{1.2.3.4.5}\frac{d^4y}{db^4} - \frac{a^7}{1\ldots7}\frac{d^7y}{db^7} + \text{etc.}$

Designetur iam summa y, ceu a quantitate b pendens, per φ b, erit ex theoremate Tayloriano

$$\varphi(b+a) = y + \frac{a\,dy}{db} + \frac{a^2d^2y}{1.2\,db^2} + \frac{a^3d^3y}{1.2.3\,db^3} + \frac{a^4d^4y}{1\ldots4\,db^4} + \text{etc.}$$

$$\varphi(b-a) = y - \frac{a\,dy}{db} + \frac{a^2d^2y}{1.2\,db^2} - \frac{a^3d^3y}{1.2.3\,db^3} + \frac{a^4d^4y}{1\ldots4\,db^4} - \text{etc.}$$

ac addendo: $\frac{\varphi(b+a)+\varphi(b-a)}{2} = y + \frac{a^2d^2y}{1.2\,db^2} + \frac{a^4d^4y}{1.2.3.4\,db^4} + \frac{a^6d^6y}{1\ldots6\,db^6} + \text{etc.}$

et ponendo $a\sqrt{-1}$ pro a, $\frac{\varphi(b+a\sqrt{-1})+\varphi(b-a\sqrt{-1})}{2} = y - \frac{a^2d^2y}{1.2\,db^2} + \frac{a^4d^4y}{1.2.3.4\,db^4} - \frac{a^6d^6y}{1\ldots6\,db^6} + \text{etc.}$, vnde multiplicando per da et integrando (quantitate

(*) cf. EVLERI Introduct. T. I. Cap. IX. §. 183.

tate b pro constante habita) fit: $ay - \frac{a^3 d^2 y}{1.2.3\, db^2} + \frac{a^5 d^4 y}{1\ldots5\, db^4} - \frac{a^7}{1\ldots7}\frac{d^6 y}{db^6} + \text{etc.}$
$= \frac{1}{2}\left(\int da.\varphi(b+a\sqrt{-1}) + \int da.\varphi(b-a\sqrt{-1})\right).$

3) Hinc pro summa quaesita haec obtinetur formula:

$$\text{S. A. tang. } \frac{a}{xx+b} = \frac{1}{2}\int da.\varphi(b+a\sqrt{-1}) + \frac{1}{2}\int da.\varphi(b-a\sqrt{-1}),$$

vbi quid signo functionali φ denotetur, ex (1) constat. Est nimirum $\varphi(b+a\sqrt{-1})$
$= -\frac{1}{2(b+a\sqrt{-1})} + \frac{e^{2\pi\sqrt{-1}(b+a\sqrt{-1})}+1}{e^{2\pi\sqrt{-1}(b+a\sqrt{-1})}-1}\cdot\frac{\pi}{2\sqrt{-1}(b+a\sqrt{-1})}$. Binorum integralium vnum tantum definiendum est, ex quo alterum sequitur permutando $\sqrt{-1}$ cum $-\sqrt{-1}$. Est autem $\int da.\varphi(b+a\sqrt{-1}) = -\frac{1}{2}\int\frac{da}{b+a\sqrt{-1}} +$
$\frac{\pi}{2}\int\frac{da\left(e^{2\pi\sqrt{-1}(b+a\sqrt{-1})}+1\right)}{\sqrt{-1}(b+a\sqrt{-1})\left(e^{2\pi\sqrt{-1}(b+a\sqrt{-1})}-1\right)}$. Cuius integralis vt pars secunda integrari queat, ponatur $e^{2\pi\sqrt{-1}(b+a\sqrt{-1})} = u$, eritque $2\pi\sqrt{-1}(b+a\sqrt{-1}) = \log. u$, $\frac{\pi\sqrt{-1}.da}{\sqrt{-1}(b+a\sqrt{-1})} = \frac{du}{u}$, hinc fit illa pars =

$$= \frac{1}{2\sqrt{-1}}\int\frac{du}{u}\left(\frac{u+1}{u-1}\right) = \frac{1}{2\sqrt{-1}}\int\left(\frac{2du}{u-1} - \frac{du}{u}\right)$$

$$= \frac{1}{\sqrt{-1}}\left(\log.\left(e^{2\pi\sqrt{-1}(b+a\sqrt{-1})}-1\right) - \pi\sqrt{-1}(b+a\sqrt{-1})\right)$$

$$= \frac{1}{\sqrt{-1}}\left(\log.\left(e^{2\pi(\beta+a\sqrt{-1})}-1\right) - \pi(\beta+a\sqrt{-1})\right),$$

posito, vti supra §. XCIII. $\sqrt{-1}(b+a\sqrt{-1}) = \beta + a\sqrt{-1}$. Eadem ratione prodit pro altero integrali $\int da\,\varphi(b-a\sqrt{-1})$ pars correspondens =
$-\frac{1}{\sqrt{-1}}\left(\log.\left(e^{2\pi(\beta-\alpha\sqrt{-1})}-1\right) - \pi(\beta-\alpha\sqrt{-1})\right)$; vtriusque summa per 2 diuisa est $= \frac{1}{2\sqrt{-1}}\log.\left\{\frac{e^{2\pi(\beta+\alpha\sqrt{-1})}-1}{e^{2\pi(\beta-\alpha\sqrt{-1})}-1}\right\} - \pi\alpha =$

$$\frac{1}{2\sqrt{-1}}\log.\left\{\frac{e^{2\pi\beta}(\cos. 2\pi\alpha + \sqrt{-1}.\sin. 2\pi\alpha)-1}{e^{2\pi\beta}(\cos. 2\pi\alpha - \sqrt{-1}.\sin. 2\pi\alpha)-1}\right\} - \alpha\pi =$$

$$\text{A. tang.}\left\{\frac{e^{2\pi\beta}\sin. 2\pi\alpha}{e^{2\pi\beta}\cos. 2\pi\alpha - 1}\right\} - \pi\alpha.$$

Primae

Primae partes vtriusque integralis $\int da\,\Phi(b+a\sqrt{-1})$, $\int da\,\Phi(b-a\sqrt{-1})$, dimidiatae et additae praebent $-\frac{1}{4}\int\left(\frac{da}{b+a\sqrt{-1}}+\frac{da}{b-a\sqrt{-1}}\right)=-\frac{1}{2}\int\frac{b\,da}{b^2+a^2}$ $=-\frac{1}{2}$ A. tang. $\frac{a}{b}$. Hinc tandem prodit summa S. A. t. $\frac{a}{xx+b}=-\frac{1}{2}$ A. t. $\frac{a}{b}$ $+$ A. t. $\left\{\frac{e^{2\pi\beta}\sin 2\pi\alpha}{e^{2\pi\beta}\cos 2\pi\alpha-1}\right\}-\alpha\pi$, quae abit in summam supra §. XCIII. inuentam $-\frac{1}{2}$ A. tang. $\frac{a}{b}+\pi\alpha+$ A. tang. $\left\{\frac{\sin 2\pi\alpha}{e^{2\pi\beta}-\cos 2\pi\alpha}\right\}$, quoniam est quicunque arcus $\gamma=$ A. tang. $\left(\frac{z\sin\gamma}{z\cos\gamma-1}\right)-$ A. t. $\left(\frac{\sin}{z-\cos\gamma}\right)$.

Problema IX.

§. CVI. Summare seriem infinitam:

$$\text{A. tang. }\frac{a}{1+b}+\text{A. t. }\frac{a}{(2p-1)^2+b}+\text{A. t. }\frac{a}{(2p+1)^2+b}+\text{A. t. }\frac{a}{(4p-1)^2+b}+$$
$$\text{A. tang. }\frac{a}{(4p+1)^2+b}+\text{A. t. }\frac{a}{(6p-1)^2+b}+\text{A. t. }\frac{a}{(6p+1)^2+b}+\text{etc.}$$

Solutio.

1) Productum indefinitum (§. VII.) $P\left(\frac{1+t^x\sqrt{-1}}{1-t^x\sqrt{-1}}\right)$ est

$$=\frac{\left[1+\frac{a}{1+b}\sqrt{-1}\right]\left[1+\frac{a\sqrt{-1}}{(2p-1)^2+b}\right]\left[1+\frac{a\sqrt{-1}}{(2p+1)^2+b}\right]\left[1+\frac{a\sqrt{-1}}{(4p-1)^2+b}\right]\ldots}{\left[1-\frac{a}{1+b}\sqrt{-1}\right]\left[1-\frac{a\sqrt{-1}}{(2p-1)^2+b}\right]\left[1-\frac{a\sqrt{-1}}{(2p+1)^2+b}\right]\left[1-\frac{a\sqrt{-1}}{(4p-1)^2+b}\right]\ldots}$$

$$=\frac{(1+b+a\sqrt{-1})\left[1+\frac{b+a\sqrt{-1}}{(2p-1)^2}\right]\left[1+\frac{b+a\sqrt{-1}}{(2p+1)^2}\right]\left[1+\frac{b+a\sqrt{-1}}{(4p-1)^2}\right]\ldots}{(1+b-a\sqrt{-1})\left[1+\frac{b-a\sqrt{-1}}{(2p-1)^2}\right]\left[1+\frac{b-a\sqrt{-1}}{(2p+1)^2}\right]\left[1+\frac{b-a\sqrt{-1}}{(4p-1)^2}\right]\ldots}$$

cuius valor reperitur ope Lemmatis (§. XCII. 3.), (posito $\sqrt{(b+a\sqrt{-1})}=$ $\beta+\alpha\sqrt{-1}$)

$$=\frac{e^{\frac{(\beta+\alpha\sqrt{-1})\pi}{m}}-2\cos\frac{\pi}{m}+e^{-\frac{(\beta+\alpha\sqrt{-1})\pi}{m}}}{e^{\frac{(\beta-\alpha\sqrt{-1})\pi}{m}}-2\cos\frac{\pi}{m}+e^{-\frac{(\beta-\alpha\sqrt{-1})\pi}{m}}}.$$

Exinde, cum numerator

merator fit =

$$e^{\frac{\beta\pi}{m}}\left(\text{cof.}\,\frac{\alpha\pi}{m}+\sqrt{-1}\,.\,\text{fin.}\,\frac{\alpha\pi}{m}\right)-2\,\text{cof.}\,\frac{\pi}{m}+e^{-\frac{\beta\pi}{m}}\left(\text{cof.}\,\frac{\alpha\pi}{m}-\sqrt{-1}\,.\,\text{fin.}\,\frac{\alpha\pi}{m}\right);$$

prodit fumma feriei Arcuum = A. tang. $\frac{N}{M}$ (§. VIII.) =

$$\text{A. t.}\left\{\frac{\left\{e^{\frac{\beta\pi}{m}}-e^{-\frac{\beta\pi}{m}}\right\}\text{fin.}\,\frac{\alpha\pi}{m}}{\left\{e^{\frac{\beta\pi}{m}}+e^{-\frac{\beta\pi}{m}}\right\}\text{cof.}\,\frac{\alpha\pi}{m}-2\,\text{cof.}\,\frac{\pi}{m}}\right\}=\text{A. t.}\left\{\frac{\left\{e^{\frac{2\beta\pi}{m}}-1\right\}\text{fin.}\,\frac{\alpha\pi}{m}}{\left\{e^{\frac{2\beta\pi}{m}}+1\right\}\text{cof.}\,\frac{\alpha\pi}{m}-2e^{\frac{\beta\pi}{m}}\,\text{cof.}\,\frac{\pi}{m}}\right\}.$$

2) Haec fummae expreffio in aliam transformari poteft, quae ob rationes in Scholio 1. §. CIV. expofitas praeferenda eft. Subtrahatur nimirum ab illa Arcus $\frac{\pi\alpha}{m}$ = A. tang. $\frac{\text{fin.}\,\frac{\pi\alpha}{m}}{\text{cof.}\,\frac{\pi\alpha}{m}}$, eritque differentia =

$$\text{A. t.}\,\frac{\left\{e^{\frac{2\beta\pi}{m}}-1\right\}\text{fin.}\,\frac{\alpha\pi}{m}\,\text{cof}\,\frac{\alpha\pi}{m}-\left\{e^{\frac{2\beta\pi}{m}}+1\right\}\text{cof.}\,\frac{\alpha\pi}{m}\,\text{fin.}\,\frac{\alpha\pi}{m}+2e^{\frac{\beta\pi}{m}}\,\text{cof.}\,\frac{\pi}{m}\,\text{fin.}\,\frac{\alpha\pi}{m}}{\left\{e^{\frac{2\beta\pi}{m}}+1\right\}\left(\text{cof.}\,\frac{\alpha\pi}{m}\right)^2-2e^{\frac{\beta\pi}{m}}\,\text{cof.}\,\frac{\alpha\pi}{m}\,\text{cof.}\,\frac{\alpha\pi}{m}+\left\{e^{\frac{2\beta\pi}{m}}-1\right\}\left(\text{fin.}\,\frac{\alpha\pi}{m}\right)^2}$$

$$=\text{A. t.}\left\{\frac{2e^{\frac{\beta\pi}{m}}\,\text{cof.}\,\frac{\pi}{m}\,\text{fin.}\,\frac{\alpha\pi}{m}-\text{fin.}\,\frac{2\alpha\pi}{m}}{e^{\frac{2\beta\pi}{m}}-2e^{\frac{\beta\pi}{m}}\,\text{cof.}\,\frac{\pi}{m}\,\text{cof.}\,\frac{\pi\alpha}{m}+\text{cof.}\,\frac{2\alpha\pi}{m}}\right\}.$$ Quare fumma feriei etiam fic exprimi poteft: A. tang. $\frac{a}{1+b}$ + A. t. $\frac{a}{(2m-1)^2+b}$ + A. t. $\frac{a}{(2m+1)^2+b}$ + etc. = $\frac{\alpha\pi}{m}$

$$+\ \text{A. tang.}\left\{\frac{2e^{\frac{\beta\pi}{m}}\,\text{cof.}\,\frac{\pi}{m}\,\text{fin.}\,\frac{\alpha\pi}{m}-\text{fin.}\,\frac{2\alpha\pi}{m}}{e^{\frac{2\beta\pi}{m}}-2e^{\frac{\beta\pi}{m}}\,\text{cof.}\,\frac{\pi}{m}\,\text{cof.}\,\frac{\pi\alpha}{m}+\text{cof.}\,\frac{2\alpha\pi}{m}}\right\},\ \text{fine} = \frac{\alpha\pi}{m}$$

+A.

$$+\text{A. t.}\left\{\frac{\text{fin.}\frac{(\alpha+1)\pi}{m}}{e^{\frac{\beta\pi}{m}}-\text{cof.}\frac{(\alpha+1)\pi}{m}}\right\}+\text{A. t.}\left\{\frac{\text{fin.}\frac{(\alpha-1)\pi}{m}}{e^{\frac{\beta\pi}{m}}-\text{cof.}\frac{(\alpha-1)\pi}{m}}\right\}.$$

Valores quantitatum α et β ex aequatione $\sqrt{}(b+a\sqrt{}-1)=\beta+\alpha\sqrt{}-1$ definiendi, et fupra iam §. XCIII. affignati funt. Si b negatiuum valorem habeat $=-b$, tum in fummae expreffione β et α inuicem permutari oportet (cf. §. XCVI.).

Corollarium 1.

§. CVII. Series generalior $\text{A. t.}\frac{\mathfrak{A}}{\nu^2+\mathfrak{B}}+\text{A. t.}\frac{\mathfrak{A}}{(\lambda-\nu)^2+\mathfrak{B}}+\text{A. t.}\frac{\mathfrak{A}}{(\lambda+\nu)^2+\mathfrak{B}}$ $+\text{A. t.}\frac{\mathfrak{A}}{(2\lambda-\nu)^2+\mathfrak{B}}+\text{A. t.}\frac{\mathfrak{A}}{(2\lambda+\nu)^2+\mathfrak{B}}+\text{A. t.}\frac{\mathfrak{A}}{(3\lambda-\nu)^2+\mathfrak{B}}+\text{A. t.}\frac{\mathfrak{A}}{(3\lambda+\nu)^2+\mathfrak{B}}$ $+\text{A. t.}\frac{\mathfrak{A}}{(4\lambda-\nu)^2+\mathfrak{B}}+\text{A. t.}\frac{\mathfrak{A}}{(4\lambda+\nu)^2+\mathfrak{B}}+$ in inf. ad priorem (§. CVI.) facile reducitur: ponendo $\mathfrak{A}=a\nu^2$, $\mathfrak{B}=b\nu^2$, $\lambda=2\nu m$. Proinde, pofito $\lambda=2l$, haec oritur

Summatio.

$$\text{A. tang.}\frac{\mathfrak{A}}{\nu^2+\mathfrak{B}}+\text{A. t.}\frac{\mathfrak{A}}{(2l-\nu)^2+\mathfrak{B}}+\text{A. t.}\frac{\mathfrak{A}}{(2l+\nu)^2+\mathfrak{B}}+\text{A. t.}\frac{\mathfrak{A}}{(4l-\nu)^2+\mathfrak{B}}$$

$$+\text{A. t.}\frac{\mathfrak{A}}{(4l+\nu)^2+\mathfrak{B}}+\text{etc.}=\frac{\alpha\pi}{l}+\text{A. t.}\left\{\frac{2e^{\frac{\beta\pi}{l}}\cdot\text{cof.}\frac{\nu\pi}{l}.\text{fin.}\frac{\alpha\pi}{l}-\text{fin.}\frac{2\alpha\pi}{l}}{e^{\frac{2\beta\pi}{l}}-2e^{\frac{\beta\pi}{l}}\text{cof.}\frac{\nu\pi}{l}\text{cof.}\frac{\alpha\pi}{l}+\text{cof.}\frac{2\alpha\pi}{l}}\right\},$$

literis α et β ex aequatione $\sqrt{}(\mathfrak{B}+\mathfrak{A}\sqrt{}-1)=\beta+\alpha\sqrt{}-1$ determinatis.

Corollarium 2.

§. CVIII. Ex hac fummatione fummae fupra inuentae (§. XCIII. XCVII.) veluti cafus particulares deduci poffunt.

1) Pofito nimirum §. CVI, $2m=1$, erit $\text{cof.}\frac{\pi}{m}=1$, hinc fumma abit in $2\alpha\pi$

$$+\text{A. t.}\left\{\frac{2e^{2\beta\pi}\text{fin.}2\alpha\pi-\text{fin.}4\alpha\pi}{e^{4\beta\pi}-2e^{2\beta\pi}\text{cof.}2\alpha\pi+\text{cof.}4\alpha\pi}\right\}=2\alpha\pi+\text{A. t.}\frac{2\,\text{fin.}2\alpha\pi.\left(e^{2\beta\pi}-\text{cof.}2\alpha\pi\right)}{\left(e^{2\beta\pi}-\text{cof.}2\alpha\pi\right)^2-\left(\text{fin.}2\alpha\pi\right)^2}$$

$$=2\alpha\pi$$

$= 2\alpha\pi + 2\,\text{A. tang.}\,\frac{\sin. 2\alpha\pi}{e^{2\beta\pi} - \cos. 2\alpha\pi}$. Eſt autem ſeries ipſa $= \text{A. tang.}\,\frac{a}{1+b}$ $+ \text{A. t.}\,\frac{a}{b} + \text{A. t.}\,\frac{a}{2^2+b} + \text{A. t.}\,\frac{a}{1^2+b} + \text{A. t.}\,\frac{a}{3^2+b} + \text{A. t.}\,\frac{a}{2^2+b} + \text{etc.} = \text{A. t.}\,\frac{a}{b}$ $+ 2\left(\text{A. tang.}\,\frac{a}{1+b} + \text{A. t.}\,\frac{a}{4+b} + \text{A. t.}\,\frac{a}{9+b} + \text{etc.}\right)$, vnde ſummatio eadem prodit, quae §. XCIII. eſt demonſtrata.

2) Ponatur $m = 2$, erit (§. CVI.) $\cos. \frac{\pi}{m} = 0$, hinc $\text{A. t.}\,\frac{a}{1+b} + \text{A. t.}\,\frac{a}{3^2+b}$ $+ \text{A. t.}\,\frac{a}{5^2+b} + \text{etc.} = \frac{\alpha\pi}{2} - \text{A. t.}\,\frac{\sin. \alpha\pi}{e^{\beta\pi} + \cos. \alpha\pi}$, vti §. XCVII. inuentum eſt.

3) Pro aliis valoribus quantitatis m nouae obtinentur ſummationes. E. g. ſit $m = \frac{3}{2}$, erit $\cos. \frac{\pi}{m} = \cos. \frac{2\pi}{3} = -\cos. \frac{\pi}{3} = -\frac{1}{2}$, hinc ſit:

$\text{A. tang.}\,\frac{a}{1+b} + \text{A. t.}\,\frac{a}{2^2+b} + \text{A. t.}\,\frac{a}{4^2+b} + \text{A. t.}\,\frac{a}{5^2+b} + \text{A. t.}\,\frac{a}{7^2+b} + \text{A. t.}\,\frac{a}{8^2+b}$ $+ \text{etc.} = \frac{2}{3}\alpha\pi - \text{A. tang.}\left\{\frac{e^{\frac{2}{3}\beta\pi} \cdot \sin. \frac{2}{3}\alpha\pi + \sin. \frac{4}{3}\alpha\pi}{e^{\frac{4}{3}\beta\pi} + e^{\frac{2}{3}\beta\pi} \cos. \frac{2}{3}\alpha\pi + \cos. \frac{4}{3}\alpha\pi}\right\}$. Pro $m = 3$, prodit ob $\cos. \frac{\pi}{m} = \frac{1}{2}$, $\text{A. tang.}\,\frac{a}{1+b} + \text{A. t.}\,\frac{a}{5^2+b} + \text{A. t.}\,\frac{a}{7^2+b} + \text{A. t.}\,\frac{a}{11^2+b} +$ $\text{A. tang.}\,\frac{a}{13^2+b} + \text{A. t.}\,\frac{a}{17^2+b} + \text{A. t.}\,\frac{a}{19^2+b} + \text{etc.} = \frac{\alpha\pi}{3} +$

$$\text{A. tang.}\left\{\frac{e^{\frac{\beta\pi}{3}} \cdot \left(\sin. \frac{\alpha\pi}{3} - \sin. \frac{2\alpha\pi}{3}\right)}{e^{\frac{2\beta\pi}{3}} - e^{\frac{\beta\pi}{3}}\left(\cos. \frac{\alpha\pi}{3} + \cos. \frac{2\alpha\pi}{3}\right)}\right\}$$

; vbi numeri 1, 7, 13, 19 ... et 5, 11, 17... in ſerie arithmetica differentiae 6 procedunt.

Corollarium 2.

§. CIX. 1) Ponatur §. CVI. $\frac{\alpha\pi}{m} = r =$ numero integro, erit $\sin. \frac{\alpha\pi}{m} = 0 = \sin. \frac{2\alpha\pi}{m}$. Hinc eſt $\text{A. tang.}\,\frac{a}{1+b} + \text{A. t.}\,\frac{a}{(2m-1)^2+b} + \text{A. t.}\,\frac{a}{(2m+1)^2+b} +$

 A. t.

$A.\,t.\frac{a}{(4m-1)^2+b}+A.\,t.\frac{a}{(4m+1)^2+b}+$ etc. $=\pi r$, dum fuerit $b=\frac{a^2}{4r^2m^2}-r^2m^2$. Quae fummatio binas fupra §. XXIV. XXVI. inuentas tanquam corollaria pro $m=\frac{1}{2}$ et $m=2$ complectitur.

2) Pofito $\frac{\alpha}{m}=\frac{2k-1}{2}$ aliae fummationes iis, quae §. CIV. demonftratae funt, analogae reperiuntur, ob fin. $\frac{\alpha\pi}{m}=\pm 1$, cof. $\frac{\alpha\pi}{m}=0$, fin. $\frac{2\alpha\pi}{m}=0$, cof. $\frac{2\alpha\pi}{m}=-1$. Eft nimirum: $A.\,t.\frac{a}{1+b}+A.\,t.\frac{a}{(2m-1)^2+b}+A.\,t.\frac{a}{(2m+1)^2+b}+$ $A.\,t.\frac{a}{(4m-1)^2+b}+A.\,t.\frac{a}{(4m+1)^2+b}+$ etc. $=\frac{(2k-1)\pi}{2}\pm$

$$A.\,tang.\left\{\frac{2e^{\frac{\alpha\pi}{(2k-1)mm}}\cdot cof.\frac{\pi}{m}}{e^{\frac{2\alpha\pi}{(2k-1)mm}}-1}\right\},\ \text{pofito } b=\frac{a^2}{m^2(2k-1)^2}-\frac{(2k-1)^2m^2}{4}.$$

Problema X.

§. CX. Summare feriem infinitam:

$$A.\,t.\frac{a}{1}-A.\,t.\frac{a}{3}+A.\,t.\frac{a}{5}-A.\,t.\frac{a}{7}+A.\,t.\frac{a}{9}\ldots\pm A.\,tang.\frac{a}{2x-1}\mp \text{etc.}$$

Solutio.

Coniunctis terminis feriei primo et fecundo, et reliquorum binis fibi inuicem proximis, ea in aliam transformatur, cuius terminus x^{tus} (ob ferierum 1, 5, 9 . . .; et 3, 7, 11 . . . terminos $x^{tos}=4x-3$; $4x-1$) reperitur $=A.\,tang.\frac{a}{4x-3}-A.\,t.\frac{a}{4x-1}=A.\,t.\frac{2a}{16x^2-16x+3+a^2}=A.\,t.\frac{a:2}{(2x-1)^2+\frac{a^2-1}{4}}$. Cuius feriei transformatae: $A.\,tang.\frac{a:2}{1+\frac{a^2-1}{4}}+A.\,t.\frac{a:2}{9+\frac{a^2-1}{4}}+A.\,t.\frac{a:2}{25+\frac{a^2-1}{4}}+$ etc. ex §. CI. 2, pofito loco a, $\frac{a}{2}$, et $m=1$, prodit fumma $=\frac{\pi}{4}-A.\,t.\frac{1}{e^{\frac{\alpha\pi}{2}}}=$ $A.\,tang.\left(e^{\frac{\alpha\pi}{2}}\right)-\frac{\pi}{4}$. Inde haec obtinetur

Sum-

Summatio.

$$\text{A. tang.}\ \frac{a}{1} - \text{A. t.}\ \frac{a}{3} + \text{A. t.}\ \frac{a}{5} - \text{A. t.}\ \frac{a}{7} + \text{A. t.}\ \frac{a}{9} - \text{etc.} = \text{A. t.}\left(e^{\frac{\pi\alpha}{2}}\right) - \frac{\pi}{4}$$

$$= \text{A. tang.}\left\{\frac{e^{\frac{\pi\alpha}{2}} - 1}{e^{\frac{\pi\alpha}{2}} + 1}\right\}.$$

Scholion.

§. CXI. Eadem summatio etiam ope differentiationis et integrationis demonstrari potest. Differentiando nimirum seriem (§. CX.), cuius summa sit $= S$, spectata quantitate a ceu variabili, obtinetur $dS = da\left(\frac{1}{1+a^2} - \frac{3}{9+a^2} + \frac{5}{25+a^2} - \frac{7}{49+a^2} + \text{etc.}\right)$.

Est autem $\frac{1}{1+a^2} - \frac{3}{9+a^2} + \frac{5}{25+a^2} - \text{etc.} = \frac{\pi}{2\left(e^{\frac{\pi\alpha}{2}} + e^{-\frac{\pi\alpha}{2}}\right)}$, vti ex serie pro secante (*) facile deriuatur. Hinc fit $dS = \frac{\pi\, da . e^{\frac{\pi\alpha}{2}}}{2\left(e^{\pi\alpha} + 1\right)} = \frac{d\left(e^{\frac{\pi\alpha}{2}}\right)}{1 + \left(e^{\frac{\pi\alpha}{2}}\right)^2}$, indeque $S = \text{A. tang. } e^{\frac{\pi\alpha}{2}} + C$, vbi ex casu $a = 0$ et $S = 0$, C prodit $= -$ A. tang. 1 $= -\frac{\pi}{4}$.

PROBLEMA XI.

§. CXII. Summare seriem infinitam:

$$\text{A. tang.}\ \frac{a}{1} - \text{A. t.}\ \frac{a}{2m-1} + \text{A. t.}\ \frac{a}{2m+1} - \text{A. t.}\ \frac{a}{4m-1} + \text{A. t.}\ \frac{a}{4m+1} - \text{etc. in inf.}$$

Solutio.

Seriei summandae coniunctis duobus terminis sibi inuicem proximis, excepto primo, oritur series, cuius terminus generalis seu x^{tus} est $=$

$$-\left(\text{A. t.}\ \frac{a}{2mx-1} - \text{A. t.}\ \frac{a}{2mx+1}\right) = -\text{A. t.}\ \frac{2a}{4m^2x^2 - 1 + a^2} = -\text{A.t.}\frac{a:2m^2}{x^2 + \frac{a^2-1}{4m^2}},$$

cuiusque

(*) L. EVLER Institutiones calculi differentialis. Petropoli 1755. P. II. Cap. VIII. §. 225. pag. 543.

cuiusque summa ex §. CI. 1. (posito ibi $\frac{a}{2m^2}$ loco a, et $\frac{1}{m}$ loco m,) prodit

$$= -\left\{\frac{\pi}{2m} - A.t.\frac{1}{a} + A.t.\frac{\sin.\frac{\pi}{m}}{e^{\frac{a\pi}{m}} - \cos.\frac{\pi}{m}}\right\}.$$ Hinc erit, addito termino primo $A.t.\frac{a}{1}$, seriei in problemate propositae summa $= A.t.a + A.t.\frac{1}{a} - \frac{\pi}{2m} - A.t.\frac{\sin.\frac{\pi}{m}}{e^{\frac{a\pi}{m}} - \cos.\frac{\pi}{m}}$

$$= \frac{\pi(m-1)}{2m} - A.t.\left\{\frac{\sin.\frac{\pi}{m}}{e^{\frac{a\pi}{m}} - \cos.\frac{\pi}{m}}\right\}.$$ Eadem summa sic exprimi potest:

$$A.\,tang.\left\{\frac{e^{\frac{a\pi}{m}} - 1}{e^{\frac{a\pi}{m}} + 1}\,.\,\cotang.\frac{\pi}{2m}\right\}.$$ Prior autem expressio praeferenda est (cf. §. CIV.).

Scholion.

§. CXIII. Series praecedens etiam methodo §. CXI. adhibita summari potest. Differentiando obtinetur $dS =$

$$da\left(\frac{a}{1+a^2} - \frac{(2m-1)}{(2m-1)^2+a^2} + \frac{(2m+1)}{(2m+1)^2+a^2} - \frac{(4m-1)}{(4m-1)^2+a^2} + \frac{(4m+1)}{(4m+1)^2+a^2} - \text{etc.}\right).$$

Est autem

$$\frac{1}{1+a^2} - \frac{(2m-1)}{(2m-1)^2+a^2} + \frac{2m+1}{(2m+1)^2+a^2} - \text{etc.} = \frac{\frac{\pi}{m}\sin.\frac{\pi}{m}}{e^{\frac{a\pi}{m}} + e^{-\frac{a\pi}{m}} - 2\cos.\frac{\pi}{m}} \text{ (*)}.$$

Hinc fit $dS = \frac{\pi}{m}\sin.\frac{\pi}{m}\cdot\frac{da}{e^{\frac{a\pi}{m}} + e^{-\frac{a\pi}{m}} - 2\cos.\frac{\pi}{m}}$. Ad quam formulam integrandam

(*) Haec summatio sponte sequitur ex ea, quam l. c. p. 56. §. XIII. 2. demonstraui, posito illic loco λ, $\frac{1}{m}$, et pro μ, $\frac{a}{m}\sqrt{-1}$.

dam ponatur $e^{\frac{a\pi}{m}} = u$, eritque $da \cdot \frac{\pi}{m} e^{\frac{a\pi}{m}} = du$, vnde $dS = \frac{\text{fin.}\frac{\pi}{m}\cdot du}{u^2 - 2u\,\text{cof.}\frac{\pi}{m} + 1} =$

$\frac{\text{fin.}\frac{\pi}{m}\cdot d\left(u - \text{cof.}\frac{\pi}{m}\right)}{\left(u - \text{cof.}\frac{\pi}{m}\right)^2 + \left(\text{fin.}\frac{\pi}{m}\right)^2}$, et, fumendo integralia, $S = A.t.\frac{u - \text{cof.}\frac{\pi}{m}}{\text{fin.}\frac{\pi}{m}} + \text{Conft.}$

Ad definiendam Conftantem pofito $a = 0$, habetur $u = 1$, hinc $S = 0 = A.t.\frac{1 - \text{cof.}\frac{\pi}{m}}{\text{fin.}\frac{\pi}{m}}$

$+ C = \frac{\pi}{2m} + C$; $C = -\frac{\pi}{2m}$. Quare erit $S = -\frac{\pi}{2m} + A.t.\frac{e^{\frac{a\pi}{m}} - \text{cof.}\frac{\pi}{m}}{\text{fin.}\frac{\pi}{m}}$

$= \frac{\pi(m-1)}{2m} - A.t.\frac{\text{fin.}\frac{\pi}{m}}{e^{\frac{a\pi}{m}} - \text{cof.}\frac{\pi}{m}}$.

Corollarium 1.

§. CXIV. Summatio feriei:

$A.\text{tang.}\frac{a}{\nu} - A.t.\frac{a}{2l-\nu} + A.t.\frac{a}{2l+\nu} - A.t.\frac{a}{4l-\nu} + A.t.\frac{a}{4l+\nu} - \text{etc.}$

facile ex praecedentibus deriuatur, pofito $\frac{a}{\nu}$ loco a, $\frac{l}{\nu}$ loco m. Eft nimirum fumma

$= A.\text{tang.}\left\{\frac{e^{\frac{a\pi}{l}} - 1}{e^{\frac{a\pi}{l}} + 1}\cdot\text{cot.}\frac{\pi\nu}{2l}\right\} = \frac{\pi(l-\nu)}{2l} - A.\text{tang.}\left\{\frac{\text{fin.}\frac{\pi\nu}{l}}{e^{\frac{a\pi}{l}} - \text{cof.}\frac{\pi\nu}{l}}\right\}$. Eadem ratione obtinetur, pofito $a l$ loco a, νl loco ν:

$A.\text{tang.}\frac{a}{\nu} + A.t.\frac{a}{\nu-2} + A.t.\frac{a}{\nu+2} + A.t.\frac{a}{\nu-4} + A.t.\frac{a}{\nu+4} + \text{etc.}$

= A.

$= A.t. \left\{\frac{e^{a\pi} - 1}{e^{a\lambda} + 1} . \cot \frac{\pi\nu}{2}\right\} = \frac{\pi(1-\nu)}{2} - A.t \left\{\frac{\sin. \nu\pi}{e^{a\lambda} - \cos \nu\pi}\right\}$. E. g. pro $\nu = l - 1$,

fit ex priori ferie: $A.t. \frac{a}{l-1} - A.t. \frac{a}{l+1} + A.t. \frac{a}{3l-1} - A.t. \frac{a}{3l+1} + A.t. \frac{a}{5l-1}$

$- \text{etc.} = \frac{\pi}{2l} - A. \text{tang.} \left\{\frac{\sin. \frac{\pi}{l}}{e^{\frac{a\pi}{l}} + \cos. \frac{\pi}{l}}\right\}.$

Corollarium 2.

§. CXV. 1) A ferie dupla:

$$2\left(A.t.\ a - A.t. \frac{a}{4m-1} + A.t. \frac{a}{4m+1} - A.t. \frac{a}{8m-1} + A.t. \frac{a}{8m+1} - \text{etc.}\right)$$

fubtrahatur feries

$$A.\ \text{tang.}\ a - A.t. \frac{a}{2m-1} + A.t. \frac{a}{2m+1} - A.t. \frac{a}{4m-1} + A.t. \frac{a}{4m+1} - \text{etc.},$$

remanebitque feries

$$A.t.\ a + A.t. \frac{a}{2m-1} - A.t \frac{a}{2m+1} - A.t. \frac{a}{4m-1} + A.t. \frac{a}{4m+1} + A.t. \frac{a}{6m-1}$$

— etc. quae cum proxime praecedente feu fecunda terminos communes, at figna non alternantia, fed duobus fignis affirmatiuis duo negatiua fuccedentia habet. Seriei primae fumma ex §. CXII, pofito 2 m loco m, prodit $= \frac{\pi(2m-1)}{4m} - A.t. \left\{\frac{\sin. \frac{\pi}{2m}}{e^{\frac{a\pi}{2m}} - \cos. \frac{\pi}{2m}}\right\}$.

Hinc erit fumma tertiae feriei ex fubtractione genitae =

$$\frac{\pi(2m-1)}{2m} - 2A.t. \left\{\frac{\sin. \frac{\pi}{2m}}{e^{\frac{a\pi}{2m}} - \cos. \frac{\pi}{2m}}\right\} - \frac{\pi(m-1)}{2m} + A.t. \left\{\frac{\sin. \frac{\pi}{m}}{e^{\frac{a\pi}{m}} - \cos. \frac{\pi}{m}}\right\} = \frac{\pi}{2}$$

$$- A.t. \left\{\frac{\sin. \frac{\pi}{m}}{e^{\frac{a\pi}{2m}} - \cos. \frac{\pi}{2m}}\right\} - A.t. \left\{\frac{\sin. \frac{\pi}{2m}}{e^{\frac{a\pi}{2m}} + \cos. \frac{\pi}{2m}}\right\} \text{(cf. §. XCVII. 1.)} = \frac{\pi}{2}$$

— A.

$$-\text{A. tang.}\left\{\frac{2e^{\frac{a\pi}{2m}}\text{ fin. }\frac{\pi}{2m}}{e^{\frac{a\pi}{m}}-1}\right\}=\text{A. t.}\left\{\frac{e^{\frac{a\pi}{m}}-1}{2e^{\frac{a\pi}{2m}}\text{ fin. }\frac{\pi}{2m}}\right\}.$$ Quare iam haec demonſtrata eſt ſummatio:

$$\text{A. t. a}+\text{A. t.}\frac{a}{2m-1}-\text{A. t.}\frac{a}{2m+1}-\text{A. t.}\frac{a}{4m-1}+\text{A. t.}\frac{a}{4m+1}+\text{A. t.}\frac{a}{6m-1}$$

$$-\text{etc.}=\text{A. tang.}\left\{\frac{e^{\frac{a\pi}{m}}-1}{2e^{\frac{a\pi}{2m}}\text{ fin. }\frac{\pi}{2m}}\right\}.$$ Ita igitur aſſignata eſt ſumma binarum ſerierum:

$$\text{A. tang. a}-\text{A. t.}\frac{a}{2m+1}+\text{A. t.}\frac{a}{4m+1}-\text{A. t.}\frac{a}{6m+1}+\text{etc.}$$

$$\text{et A. tang.}\frac{a}{2m-1}-\text{A. t}\frac{a}{4m-1}+\text{A. t.}\frac{a}{6m-1}-\text{etc. in inf.}$$

quarum neutra ſeorſim ſummari poteſt. Simili ratione Probl. II. (§. CXII.) exhibet differentiam binarum ſerierum, per ſe haud ſummabilium.

2) *Exempli* gratia pro $m=2$ ex §. CXII. conſequitur ſummatio §. CX; eſtque etiam: $\text{A. t. a}+\text{A. t.}\frac{a}{3}-\text{A. t.}\frac{a}{5}-\text{A. t.}\frac{a}{7}+\text{A. t.}\frac{1}{9}+\text{A. t.}\frac{1}{11}-\text{etc.}$

$$=\text{A. tang.}\left\{\frac{e^{\frac{a\pi}{2}}-1}{e^{\frac{a\pi}{4}}\cdot\sqrt{2}}\right\}.$$ Pro $m=\frac{3}{2}$ habetur:

$$\text{A. t. a}-\text{A. t.}\frac{a}{2}+\text{A. t.}\frac{a}{4}-\text{A. t.}\frac{a}{5}+\text{A. t.}\frac{a}{7}-\text{A. t.}\frac{a}{8}+\text{etc.}$$

$$=\frac{\pi}{6}-\text{A. tang.}\left\{\frac{\sqrt{3}}{2e^{\frac{2a\pi}{3}}+1}\right\};$$

$$\text{A. tang. a}+\text{A. tang.}\frac{a}{2}-\text{A. t.}\frac{a}{4}-\text{A. t.}\frac{a}{5}+\text{A. t.}\frac{a}{7}+\text{A. t.}\frac{a}{8}-\text{etc.}$$

 $=$ A.

$$= \text{A. tang.} \left\{ \frac{e^{\frac{2}{3} a \pi} - 1}{e^{\frac{a\pi}{3}} . \sqrt{3}} \right\}.$$ Pro $m = 3$ eſt

$$\text{A. tang. } a - \text{A. t.} \frac{a}{5} + \text{A. t.} \frac{a}{7} - \text{A. t.} \frac{a}{11} + \text{A. t.} \frac{a}{13} - \text{A. t.} \frac{a}{15} + \text{etc.}$$

$$= \frac{\pi}{3} - \text{A. tang.} \left\{ \frac{\sqrt{3}}{2 e^{\frac{a\pi}{3}} - 1} \right\};$$

$$\text{A. tang. } a + \text{A. t.} \frac{a}{5} - \text{A. t.} \frac{a}{7} - \text{A. t.} \frac{a}{11} + \text{A. t.} \frac{a}{13} + \text{A. t.} \frac{a}{15} - \text{etc.}$$

$$= \text{A. tang.} \left\{ \frac{e^{\frac{a\pi}{3}} - 1}{e^{\frac{a\pi}{6}}} \right\}.$$

Corollarium 3.

§. CXVI. Series (§. CXV. 1.) ſummata additione binorum terminorum ſibi inuicem proximorum in aliam transformatur, cuius terminus generalis ſeu x^{tus} eſt

$$= \pm \left(\text{A. t.} \frac{a}{2(x-1)m+1} + \text{A. t.} \frac{a}{2xm-1} \right) = \pm \text{A. t.} \frac{2am(2x-1)}{4m^2x^2 - 1 - 2m(2xm-1) - a^2}$$

$$= \pm \text{A. t.} \frac{2am(2x-1)}{m^2(2x-1)^2 - (m-1)^2 - a^2}.$$ Exinde haec oritur ſummatio;

$$\text{A. t.} \frac{2am.1}{1^2.m^2 - (m-1)^2 - a^2} - \text{A. t} \frac{2am.3}{3^2 m^2 - (m-1)^2 - a^2} + \text{A. t.} \frac{2am.5}{5^2 m^2 - (m-1)^2 - a^2}$$

$$- \text{etc.} = \text{A. tang.} \left\{ \frac{e^{\frac{a\pi}{m}} - 1}{2 e^{\frac{a\pi}{2m}} \text{ſin.} \frac{\pi}{2m}} \right\}.$$ *Exempli* gratia pro $m = 2$, eſt:

$$\text{A. t.} \frac{a}{1 - \left(\frac{1+a^2}{4}\right)} - \text{A. t.} \frac{3a}{4 - \left(\frac{1+a^2}{4}\right)} + \text{A. t.} \frac{5a}{25 - \left(\frac{1+a^2}{4}\right)} - \text{etc.} = \text{A. t.} \left\{ \frac{e^{\frac{a\pi}{2}} - 1}{e^{\frac{a\pi}{4}} . \sqrt{2}} \right\}.$$

Corol-

Corollarium 4.

§. CXVII. 1) Hinc deriuari poteſt ſummatio ſeriei, cuius terminus generalis eſt $= \pm$ A. tang. $\frac{(2x-1)f}{(2x-1)^2-g}$. Comparatio huius ſeriei cum modo ſummata praebet: $f = \frac{2a}{m}$, $g = \frac{(m-1)^2+a^2}{m^2}$. Inde fit $g - \frac{f^2}{4} = \left(\frac{m-1}{m}\right)^2$, ſeu $1 - \frac{1}{m} = \surd\left(g - \frac{f^2}{4}\right)$; $\frac{1}{m} = 1 - \surd\left(g - \frac{f^2}{4}\right)$; quare ſin. $\frac{\pi}{2m} = $ coſ. $\left(\frac{\pi}{2}\surd\left(g - \frac{f^2}{4}\right)\right) = $ coſ. $\frac{\pi F}{4}$. Quibus ſuppoſitis §. CXVI. haec obtinetur ſummatio:

$$\text{A. tang. } \frac{f}{1-g} - \text{A. t. } \frac{3f}{9-g} + \text{A. t. } \frac{5f}{25-g} - \text{A. t. } \frac{7f}{49-g} + \text{etc.} = \text{A. t. } \left\{\frac{e^{\frac{\pi f}{2}} - 1}{2e^{\frac{\pi f}{4}} \text{ coſ. } \frac{\pi F}{4}}\right\},$$

poſito $F = \surd(4g - f^2)$.

2) Si $4g < f^2$, tum quantitas F fit imaginaria. Sit igitur pro hoc caſu $F = \surd(f^2 - 4g)$, vel loco F (1) ponatur $F\surd-1$, eritque coſ. $\left(\frac{\pi F \surd -1}{4}\right) = \frac{e^{\frac{\pi F}{4}} + e^{-\frac{\pi F}{4}}}{2}$. Quare ſumma (1) abit in: A. tang. $\left\{\frac{e^{\frac{\pi f}{2}} - 1}{e^{\frac{\pi f}{4}}\left(e^{\frac{\pi F}{4}} + e^{-\frac{\pi F}{4}}\right)}\right\} =$

$= \text{A. t. } e^{\frac{\pi(F+f)}{2}} - \text{A. t. } e^{\frac{\pi(F-f)}{2}}$. Summationi igitur (1) haec adiungenda eſt:

$$\text{A. tang. } \frac{f}{1-g} - \text{A. t. } \frac{3f}{9-g} + \text{A. t. } \frac{5f}{25-g} - \text{A. t. } \frac{7f}{49-g} + \text{etc.}$$

$$= \text{A. tang. } \left(e^{\frac{\pi(F+f)}{2}}\right) - \text{A. t. } \left(e^{\frac{\pi(F-f)}{2}}\right),$$ poſito $F = \surd(f^2 - 4g)$. Quam adhibere oportet, ſi $f^2 > 4g$, vel etiam ſi g negatiuum valorem habeat.

PROBLEMA XII.

§. CXVIII. Summare ſeriem infinitam:

$$\text{A. tang. } \frac{a}{1^4+b} + \text{A. t. } \frac{a}{2^4+b} + \text{A. t. } \frac{a}{3^4+b} + \text{etc.} + \text{A. t. } \frac{a}{x^4+b} + \ldots \text{ in inf.}$$

Solu-

Solutio.

1) Productum indefinitum (§. VII.) $P\left(\frac{1+t^x\sqrt{-1}}{1-t^x\sqrt{-1}}\right)$ est $= P\left(\frac{x^4+b+a\sqrt{-1}}{x^4+b-a\sqrt{-1}}\right)$; quod, posito $b+a\sqrt{-1} = -(\mathfrak{B}+\mathfrak{A}\sqrt{-1})^4$, abit in
$P\left(\frac{(x^2-(\mathfrak{B}+\mathfrak{A}\sqrt{-1})^2)(x^2+(\mathfrak{B}+\mathfrak{A}\sqrt{-1})^2)}{(x^2-(\mathfrak{B}-\mathfrak{A}\sqrt{-1})^2)(x^2+(\mathfrak{B}-\mathfrak{A}\sqrt{-1})^2)}\right)$. Adhibito Lemmate §. XCII. 1. valor huius producti reperitur =

$$\frac{(\mathfrak{B}-\mathfrak{A}\sqrt{-1}).\left(e^{\pi(\mathfrak{B}+\mathfrak{A}\sqrt{-1})}-e^{-\pi(\mathfrak{B}+\mathfrak{A}\sqrt{-1})}\right).(\mathfrak{B}\sqrt{-1}+\mathfrak{A})}{(\mathfrak{B}+\mathfrak{A}\sqrt{-1}).\left(e^{\pi(\mathfrak{B}-\mathfrak{A}\sqrt{-1})}-e^{-\pi(\mathfrak{B}-\mathfrak{A}\sqrt{-1})}\right).(\mathfrak{A}-\mathfrak{B}\sqrt{-1})}\cdot\frac{\left(e^{\pi(\mathfrak{A}-\mathfrak{B}\sqrt{-1})}-e^{-\pi(\mathfrak{A}-\mathfrak{B}\sqrt{-1})}\right)}{\left(e^{\pi(\mathfrak{A}+\mathfrak{B}\sqrt{-1})}-e^{-\pi(\mathfrak{A}+\mathfrak{B}\sqrt{-1})}\right)}.$$

2) Producto itaque ad formam Coroll. 2. §. VIII. reuocato, eiusque numeratore habente quatuor factores, prodit summa seriei =

$$A.\,tang.\frac{N^{I}}{M^{I}}+A.\,t.\frac{N^{II}}{M^{II}}+A.\,t.\frac{N^{III}}{M^{III}}+A.\,t.\frac{N^{IV}}{M^{IV}} = A.\,t.-\frac{\mathfrak{A}}{\mathfrak{B}}+A.\,t.\frac{\mathfrak{B}}{\mathfrak{A}}+$$

$$A.\,tang.\frac{\left(e^{\pi\mathfrak{B}}+e^{-\pi\mathfrak{B}}\right)\,fin.\,\mathfrak{A}\pi}{\left(e^{\pi\mathfrak{B}}-e^{-\pi\mathfrak{B}}\right)\,cof.\,\mathfrak{A}\pi}+A.\,tang.\frac{\left(e^{\pi\mathfrak{A}}+e^{-\pi\mathfrak{A}}\right).-fin.\,\mathfrak{B}\pi}{\left(e^{\pi\mathfrak{A}}-e^{-\pi\mathfrak{A}}\right)\,cof.\,\mathfrak{B}\pi} =$$

$$A.\,tang.\frac{\mathfrak{B}}{\mathfrak{A}}-A.\,t.\frac{\mathfrak{A}}{\mathfrak{B}}+A.\,t.\frac{\left(e^{2\pi\mathfrak{B}}+1\right)}{\left(e^{2\pi\mathfrak{B}}-1\right)}.\,tang.\,\mathfrak{A}\pi-A.\,t.\left\{\frac{e^{2\pi\mathfrak{A}}+1}{e^{2\pi\mathfrak{A}}-1}\right\}.\,tang.\,\mathfrak{B}\pi$$

$$= 2\,A.\,t.\frac{\mathfrak{B}}{\mathfrak{A}}-\frac{\pi}{2}+\pi(\mathfrak{A}-\mathfrak{B})+A.\,t.\left(\frac{fin.\,2\pi\mathfrak{A}}{e^{2\pi\mathfrak{B}}-cof.\,2\pi\mathfrak{A}}\right)$$

$$-A.\,t.\left(\frac{fin.\,2\pi\mathfrak{B}}{e^{2\pi\mathfrak{A}}-cof.\,2\pi\mathfrak{B}}\right)$$ (cf. §. XCIII. 2.).

3) Quod iam ad quantitates $\mathfrak{A}$ et $\mathfrak{B}$ attinet, eae ex aequatione: $\sqrt[4]{(-b-a\sqrt{-1})} = \mathfrak{B}+\mathfrak{A}\sqrt{-1}$ (1) definiendae sunt. Est igitur $\mathfrak{B}^2-\mathfrak{A}^2+2\mathfrak{A}\mathfrak{B}\sqrt{-1} = \sqrt{(-b-a\sqrt{-1})} = \sqrt{-1}.\sqrt{(b+a\sqrt{-1})} = \sqrt{-1}.(\beta+\alpha\sqrt{-1}) = -\alpha+\beta\sqrt{-1}$, vbi quantitates α et β ex §. XCIII. 3. cognitae sunt. Inde prodit $\mathfrak{B}^2-\mathfrak{A}^2 = -\alpha$, $2\mathfrak{A}\mathfrak{B} = \beta$, feu $\mathfrak{A}^2 = \frac{\sqrt{(\beta^2+\alpha^2)}+\alpha}{2}$, $\mathfrak{B}^2 = \frac{\sqrt{(\beta^2+\alpha^2)}-\alpha}{2}$; porro eft $2\,A.\,tang.\frac{\mathfrak{B}}{\mathfrak{A}} = A.\,t.\frac{2\mathfrak{A}\mathfrak{B}}{\mathfrak{A}^2-\mathfrak{B}^2} = A.\,t.\frac{\beta}{\alpha}$, et $2\,A.\,t.\frac{\mathfrak{B}}{\mathfrak{A}}-\frac{\pi}{2} = -A.\,t.\frac{\alpha}{\beta} = -\frac{1}{2}A.\,t.\frac{a}{b}$ (§. XCIII. 3.). Quibus combinatis folutionem problematis fequens complectitur

Summa-

Summatio.

$$\text{A. tang.}\ \frac{a}{1+b}+\text{A. t.}\ \frac{a}{2^4+b}+\text{A. t.}\ \frac{a}{3^4+b}+\ldots+\text{A. t.}\ \frac{a}{x^4+b}+\text{in inf.}$$

$$=\pi(\mathfrak{A}-\mathfrak{B})-\tfrac{1}{2}\ \text{A. tang.}\ \frac{a}{b}+\text{A. t.}\left(\frac{\sin 2\pi\mathfrak{A}}{e^{2\pi\mathfrak{B}}-\cos 2\pi\mathfrak{A}}\right)-$$

$$\text{A. tang.}\left(\frac{\sin 2\pi\mathfrak{B}}{e^{2\pi\mathfrak{A}}-\cos 2\pi\mathfrak{B}}\right),\ \text{poſito}\ \sqrt[4]{(-b-a\sqrt{-1})}=\mathfrak{B}+\mathfrak{A}\sqrt{-1},\ \text{ſeu}$$

$$\mathfrak{A}^2=\frac{\sqrt{(\beta^2+\alpha^2)}+\alpha}{2},\ \mathfrak{B}^2=\frac{\sqrt{(\beta^2+\alpha^2)}-\alpha}{2};\ \beta^2=\frac{\sqrt{(b^2+a^2)}+b}{2},\ \alpha^2=\frac{\sqrt{(b^2+a^2)}-b}{2}.$$

Corollarium 1.

§. CXIX. Aſſumto, vti §. XCIV, angulo ψ cuius tangens $=\frac{a}{b}$, quantitates $\mathfrak{A}$ et $\mathfrak{B}$ ſimpliciorem formam induunt. Eſt nimirum $\alpha=(a^2+b^2)^{\frac{1}{4}}\sin\frac{\psi}{2}$, $\beta=(a^2+b^2)^{\frac{1}{4}}\cos\frac{\psi}{2}$; hinc $\mathfrak{A}^2=(a^2+b^2)^{\frac{1}{4}}\left(\frac{1+\sin\frac{1}{2}\psi}{2}\right)$, $\mathfrak{B}^2=(a^2+b^2)^{\frac{1}{4}}\left(\frac{1-\sin\frac{1}{2}\psi}{2}\right)$, vnde $\mathfrak{A}=(a^2+b^2)^{\frac{1}{8}}\cos\left(\frac{\pi-\psi}{4}\right)$, $\mathfrak{B}=(a^2+b^2)^{\frac{1}{8}}\sin\left(\frac{\pi-\psi}{4}\right)$, $\mathfrak{A}-\mathfrak{B}=$

$$2(a^2+b^2)^{\frac{1}{8}}\sin\frac{\psi}{4}\cdot\sin\frac{\pi}{4}=(a^2+b^2)^{\frac{1}{8}}\frac{\sin\frac{\psi}{4}}{\sin\frac{\pi}{4}}=(a^2+b^2)^{\frac{1}{8}}\sin\frac{\psi}{4}\cdot\sqrt{2}.$$

Corollarium 2.

§. CXX. Si b negatiuum valorem habeat, ſeu terminus generalis ſeriei ſummandae ponatur $=\text{A. t.}\ \frac{a}{x^4-b}$, tum β et α permutari inuicem oportet (§. XCVI.), eritque $2\ \text{A. tang.}\ \frac{\mathfrak{B}}{\mathfrak{A}}=\text{A. t.}\ \frac{\alpha}{\beta}=\frac{1}{2}\text{A. t.}\ \frac{a}{b}$. Hinc ſumma prodit $=\pi(\mathfrak{A}-\mathfrak{B}-\frac{1}{2})+\frac{1}{2}\ \text{A. tang.}\ \frac{a}{b}+\text{A. t.}\left(\frac{\sin 2\pi\mathfrak{A}}{e^{2\pi\mathfrak{B}}-\cos 2\pi\mathfrak{A}}\right)-\text{A. t.}\left(\frac{\sin 2\pi\mathfrak{B}}{e^{2\pi\mathfrak{A}}-\cos 2\pi\mathfrak{B}}\right)$, vbi iam eſt $\mathfrak{A}^2=\frac{\sqrt{(\beta^2+\alpha^2)}+\beta}{2}$, $\mathfrak{B}^2=\frac{\sqrt{(\beta^2+\alpha^2)}-\beta}{2}$, valoribus τῶν α et β iisdem manentibus; vel etiam eſt, poſito, ob tangentem $\frac{a}{-b}$ negatiuam, $\pi-\psi$ loco ψ, $\mathfrak{A}=(a^2+b^2)^{\frac{1}{8}}\cos\frac{\psi}{4}$, $\mathfrak{B}=(a^2+b^2)^{\frac{1}{8}}\sin\frac{\psi}{4}$, exiſtente ψ, vti antea $=\text{A. t.}\ \frac{a}{b}$.

Corol-

Corollarium 3.

§. CXXI. Posito §. CXVIII. $b = 0$ erit $\beta^2 = \frac{a}{2} = \alpha^2$; $\beta = \alpha = \sqrt{\frac{a}{2}}$; hinc $\mathfrak{A}^2 = \frac{\sqrt{(a)}+\sqrt{\frac{a}{2}}}{2}$, $\mathfrak{B}^2 = \frac{\sqrt{a}-\sqrt{\frac{a}{2}}}{2}$, seu $\mathfrak{A} = \left(\frac{a}{2}\right)^{\frac{1}{4}}\left(\frac{\sqrt{2}+1}{2}\right)^{\frac{1}{2}}$, $\mathfrak{B} = \left(\frac{a}{2}\right)^{\frac{1}{4}}\left(\frac{\sqrt{2}-1}{2}\right)^{\frac{1}{2}}$. Ex §. CIX. ob $A.\,t.\,\frac{a}{b} = \frac{\pi}{2}$ hae quantitates etiam sic exprimi possunt: $\mathfrak{A} = a^{\frac{1}{4}} \cos.\frac{\pi}{8}$, $\mathfrak{B} = a^{\frac{1}{4}} \sin.\frac{\pi}{8}$. Inde posito $a = k^4$, haec obtinetur summatio: $A.\,t.\,\frac{k^4}{1^4} + A.\,t.\,\frac{k^4}{2^4} + A.\,t.\,\frac{k^4}{3^4} + A.\,t.\,\frac{k^4}{4^4} + \ldots + A.\,t.\,\frac{k^4}{x^4} + \text{etc.}$

$$= \frac{\pi}{2}\left\{\frac{k-\frac{1}{2}}{\cos.\frac{\pi}{8}}\right\} + A.\,t.\left\{\frac{\sin.\left(2\pi k \cos.\frac{\pi}{8}\right)}{e^{2\pi k \sin.\frac{\pi}{8}} - \cos.\left(2\pi k \cos.\frac{\pi}{8}\right)}\right\} - A.\,tang.\left\{\frac{\sin.\left(2\pi k \sin.\frac{\pi}{8}\right)}{e^{2\pi k \cos.\frac{\pi}{8}} - \cos.\left(2\pi k \sin.\frac{\pi}{8}\right)}\right\}.$$

Est autem $\cos.\frac{\pi}{8} = \sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}$, $\sin.\frac{\pi}{8} = \sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}$.

Corollarium 4.

§. CXXII. Ex aequatione (§. CXVIII. 1.): $-b - a\sqrt{-1} = (\mathfrak{B}+\mathfrak{A}\sqrt{-1})^4 = \mathfrak{B}^4 - 6\,\mathfrak{A}^2\mathfrak{B}^2 + \mathfrak{A}^4 + (4\mathfrak{B}^3\mathfrak{A} - 4\mathfrak{B}\mathfrak{A}^3)\sqrt{-1}$ sequitur: $a = 4\mathfrak{B}\mathfrak{A}(\mathfrak{A}^2 - \mathfrak{B}^2)$, $b = 6\,\mathfrak{A}^2\mathfrak{B}^2 - \mathfrak{A}^4 - \mathfrak{B}^4 = 4\,\mathfrak{A}^2\mathfrak{B}^2 - (\mathfrak{A}^2 - \mathfrak{B}^2)^2$. Ponatur iam $4\,\mathfrak{A}\mathfrak{B} = r$, erit $b = \frac{r^2}{4} - \frac{a^2}{r^2}$. Hac litera r introducta quantitates $\mathfrak{A}$ et $\mathfrak{B}$ ex aequationibus: $4\,\mathfrak{A}\mathfrak{B} = r$, et $\mathfrak{A}^2 - \mathfrak{B}^2 = \frac{a}{r}$ definiuntur. Assumta igitur aequatione ista inter b et a, $b = \frac{r^2}{4} - \frac{a^2}{r^2}$ (aequationi §. CI. analoga) summatio §. CXVIII. ad aliam formam reuocari potest. Quo iam bini Arcus posteriores in expressione summae seu $A.\ tang.\ \frac{\sin.\,2\pi\mathfrak{A}}{e^{2\pi\mathfrak{B}} - \cos.\,2\pi\mathfrak{A}}$ et $A.\,tang.\,\frac{\sin.\,2\pi\mathfrak{B}}{e^{2\pi\mathfrak{A}} - \cos.\,2\pi\mathfrak{B}}$ euanescant, $2\,\mathfrak{A}$ et $2\,\mathfrak{B}$ numeris

meris

meris integris aequari debent. Hinc primo $r = 2\mathfrak{A} \cdot 2\mathfrak{B}$ numero integro aequetur necesse est. Posito $\mathfrak{A} = \frac{s}{2}$, $\mathfrak{B} = \frac{t}{2}$, erit $a = \frac{st(s^2 - t^2)}{4}$. Quare haec obtinetur summatio: A. tang. $\frac{a}{1+b}$ + A. t. $\frac{a}{2^4+b}$ + A. t. $\frac{a}{3^4+b}$ + etc. + A. t. $\frac{a}{x^4+b}$ + etc. $= \frac{\pi}{2}(s - t - 1) + 2$ A. t. $\frac{t}{s}$, dum fuerit 1) $b = \frac{s^2 t^2}{4} - \frac{a^2}{s^2 t^2}$, 2) $a = \frac{st(s^2 - t^2)}{4}$, denotantibus s et t numeros integros.

Scholion.

§. CXXIII. Summatio §. XCVIII. etiam per resolutionem termini generalis in Arcus simpliciores inuestigari potest. Sit nimirum A. tang. $\frac{a}{x^4+b}$ = A. t. $\frac{F}{x^2+G}$ + A. tang. $\frac{f}{x^2+g}$ = A. t. $\left(\frac{(F+f)x^2 + Fg + fG}{x^4 + (G+g)x^2 + Gg - Ff}\right)$. Hinc quatuor obtinentur aequationes: 1) $F + f = 0$; 2) $Fg + fG = a$; 3) $G + g = 0$; 4) $Gg - Ff = b$. Vnde prodit $f = -F$; $g = -G$; $-2FG = a$; $F^2 - G^2 = b$. Ob $-2FG = a$ quantitatum F et G vna F affirmatiue, altera G negatiue $= -g$ accipi debet, vt habeatur $2Fg = a$, $F^2 - g^2 = b$. Exinde apparet, quantitates F et g cum supra inuentis β et α (§. XCIII. XCVIII.) consentire, esseque $F^2 = \beta^2 = \frac{\sqrt{(b^2+a^2)}+b}{2}$, $g^2 = \alpha^2 = \frac{\sqrt{(b^2+a^2)}-b}{2}$. Ex quibus valoribus prodit A. tang. $\frac{a}{x^4+b}$ = A. t. $\frac{\beta}{x^2-\alpha}$ — A. t. $\frac{\beta}{x^2+\alpha}$. Quare series summanda in duas dispescitur, quae ex Probl. VIII. (§. XCIII et XCVI.) summabiles sunt. Reperitur itaque S. A. tang. $\frac{a}{x^4+b}$ = S. A. t. $\frac{\beta}{x^2-\alpha}$ — S. A. t. $\frac{\beta}{x^2+\alpha}$ $= \pi(\mathfrak{A} - \frac{1}{2}) + \frac{1}{2}$ A. t. $\frac{\beta}{\alpha}$ + A. t. $\left(\frac{\sin. 2\pi\mathfrak{A}}{e^{2\pi\mathfrak{A}} - \cos. 2\pi\mathfrak{B}}\right)$ $- \pi\mathfrak{B} + \frac{1}{2}$ A. tang. $\frac{\beta}{\alpha}$ — A. t. $\left(\frac{\sin. 2\pi\mathfrak{B}}{e^{2\pi\mathfrak{A}} - \cos. 2\pi\mathfrak{B}}\right)$, posito $\mathfrak{A}^2 = \frac{\sqrt{(\beta^2+\alpha^2)}+\alpha}{2}$, $\mathfrak{B}^2 = \frac{\sqrt{(\beta^2+\alpha^2)}-\alpha}{2}$. Quae summa cum supra (§. CXVIII.) inuenta conspirat, ob $-\frac{\pi}{2}$ + A. t. $\frac{\beta}{\alpha}$ = — A. t. $\frac{\alpha}{\beta}$ = — $\frac{1}{2}$ A. t. $\frac{2\alpha\beta}{\beta^2 - \alpha^2}$ = — $\frac{1}{2}$ A. t. $\frac{a}{b}$.

PRO-

Problema XIII.

§. CXXIV. Summare feriem infinitam:

$$A.\,t.\,\frac{a}{x^4+b}+A.\,t.\,\frac{a}{(2m-1)^4+b}+A.\,t.\,\frac{a}{(2m+1)^4+b}+A.\,t.\,\frac{a}{(4m-\;)^4+b}+$$
$$A.\,tang.\,\frac{a}{(4m+1)^4+b}+etc.$$

Solutio.

Ex refolutione in praecedenti Scholio (§. CXXIII.) tradita habetur $A.\,t.\,\frac{a}{(2mx\pm 1)^4+b}$ $=A.\,t.\,\frac{\beta}{(2mx\pm 1)^2-\alpha}-A.\,t.\,\frac{\beta}{(2mx\pm 1)^2+\alpha}$, vbi eft $\beta^2=\frac{\surd(b^2+a^2)+b}{2}$, $\alpha^2=\frac{\surd(b^2+a^2)-b}{2}$. Hinc feries in duas difpefcitur, quarum fummatio Probl. IX. (§. CVI.) expofita eft. Quare illius fumma prodit =

$$(\mathfrak{A}-\mathfrak{B})\frac{\pi}{m}+A.\,t.\left\{\frac{2e^{\frac{\mathfrak{B}\pi}{m}}\,cof.\frac{\pi}{m}\,fin.\frac{\mathfrak{A}\pi}{m}-fin.\frac{2\mathfrak{A}\pi}{m}}{e^{\frac{2\mathfrak{B}\pi}{m}}-2e^{\frac{\mathfrak{B}\pi}{m}}\,cof.\frac{\pi}{m}\,cof.\frac{\pi\mathfrak{A}}{m}+cof.\frac{2\mathfrak{A}\pi}{m}}\right\}$$
$$-A.\,t.\left\{\frac{2e^{\frac{\mathfrak{A}\pi}{m}}\,cof.\frac{\pi}{m}\,fin.\frac{\mathfrak{B}\pi}{m}-fin.\frac{2\mathfrak{B}\pi}{m}}{e^{\frac{2\mathfrak{A}\pi}{m}}-2e^{\frac{\mathfrak{A}\pi}{m}}\,cof.\frac{\pi}{m}\,cof.\frac{\mathfrak{B}\pi}{m}+cof.\frac{2\mathfrak{B}\pi}{m}}\right\},$$ pofito, vti fupra,

$\mathfrak{A}^2=\frac{\surd(\beta^2+\alpha^2)+\alpha}{2}$, $\mathfrak{B}^2=\frac{\surd(\beta^2+\alpha^2)-\alpha}{2}$; feu $\mathfrak{A}=(a^2+b^2)^{\frac{1}{8}}\,cof.\left(\frac{\pi-\psi}{4}\right)$,

$\mathfrak{B}=(a^2+b^2)^{\frac{1}{8}}\,fin.\left(\frac{\pi-\psi}{4}\right)$; $\mathfrak{A}-\mathfrak{B}=(a^2+b^2)^{\frac{1}{8}}\frac{fin\,\frac{\psi}{4}}{fin.\,\frac{\psi}{4}}$, fumto angulo $\psi=$

$A.\,tang.\,\frac{a}{b}$.

Corollarium 1.

§. CXXV. Pofito $m=2$, ob $cof.\,\frac{\pi}{m}=0$, ex praecedente haec deducitur fummatio:

A. tang.

$$\text{A. t.}\frac{a}{1+b}+\text{A. t.}\frac{a}{3^4+b}+\text{A. t.}\frac{a}{5^4+b}+\text{etc.}+\text{A. t.}\frac{a}{(2x-1)^4+b}+\text{in inf.}$$
$$=(\mathfrak{A}-\mathfrak{B})\frac{\pi}{2}-\text{A. t.}\left(\frac{\text{fin. }\mathfrak{A}\pi}{e^{\mathfrak{B}\pi}+\text{cof. }\mathfrak{A}\pi}\right)+\text{A. t.}\left(\frac{\text{fin. }\mathfrak{B}\pi}{e^{\mathfrak{A}\pi}+\text{cof. }\mathfrak{B}\pi}\right).$$

Corollarium 2.

§. CXXVI. A cuius feriei duplo fubtracta ferie §. CXVIII. prodit reliquae feriei fumma $=-2\,\text{A. t.}\left(\frac{\text{fin. }\mathfrak{A}\pi}{e^{\mathfrak{B}\pi}+\text{cof. }\mathfrak{A}\pi}\right)+2\,\text{A. t.}\left(\frac{\text{fin. }\mathfrak{B}\pi}{e^{\mathfrak{A}\pi}+\text{cof. }\mathfrak{B}\pi}\right)$ $+\frac{1}{2}\text{A. t.}\frac{a}{b}-\text{A. t.}\left(\frac{\text{fin. }2\mathfrak{A}\pi}{e^{2\mathfrak{B}\pi}-\text{cof. }2\mathfrak{A}\pi}\right)+\text{A. t.}\left(\frac{\text{fin. }2\mathfrak{B}\pi}{e^{2\mathfrak{A}\pi}-\text{cof. }2\mathfrak{B}\pi}\right)$. Inde facta reductione vti §. XCIX. haec obtinetur

Summatio.

$$\text{A. tang.}\frac{a}{1+b}-\text{A. t.}\frac{a}{2^4+b}+\text{A. t.}\frac{a}{3^4+b}-\text{A. t.}\frac{a}{4^4+b}+\text{etc.}\pm\text{A. t.}\frac{a}{x^4+b}\mp\ldots$$
$$=\frac{1}{2}\text{A. t.}\frac{a}{b}+\text{A. t.}\left\{\frac{2e^{\mathfrak{A}\pi}\text{fin. }\mathfrak{B}\pi}{e^{2\mathfrak{A}\pi}-1}\right\}-\text{A. t.}\left\{\frac{2e^{\mathfrak{B}\pi}\text{fin. }\mathfrak{A}\pi}{e^{2\mathfrak{B}\pi}-1}\right\}.$$

PROBLEMA XIV.

§. CXXVII. Summare feriem infinitam:

$$\text{A. tang.}\frac{a}{1^3}-\text{A. t.}\frac{a}{3^3}+\text{A. t.}\frac{a}{5^3}-\text{A. t.}\frac{a}{7^3}+\ldots\pm\text{A. t.}\frac{a}{(2x-1)^3}\mp\text{etc.}$$

Solutio.

1) Ponatur $\text{A. t.}\frac{a}{(2x-1)^3}=\text{A. t.}\frac{\alpha}{2x-1}+\text{A. t.}\frac{\beta}{2x-1}+\text{A. t.}\frac{\gamma}{2x-1}=$ $\text{A. tang.}\frac{(\alpha+\beta)(2x-1)}{(2x-1)^2-\alpha\beta}+\text{A. t.}\frac{\gamma}{2x-1}=\text{A. t.}\left(\frac{(\alpha+\beta+\gamma)(2x-1)^2-\alpha\beta\gamma}{(2x-1)^3-(\alpha\beta+\alpha\gamma+\beta\gamma)(2x-1)}\right)$, tum pro determinandis α,β,γ tres habentur aequationes: $\alpha+\beta+\gamma=0$; $-\alpha\beta\gamma=a$; $\alpha\beta+\alpha\gamma+\beta\gamma=0$. Hinc erunt α, β, γ radices aequationis cubicae: $u^3+a=(u-\alpha)(u-\beta)(u-\gamma)=0$. Quare erit $\alpha=-\sqrt[3]{a}=-A$; $\beta=\frac{A+A\sqrt{-3}}{2}$; $\gamma=\frac{A-A\sqrt{-3}}{2}$.

2) Refoluta itaque ferie in tres feries, adhibito Probl. X. (§. CX.) prodit illius fumma $=\frac{\pi}{4}-\text{A. t. }e^{\frac{\pi A}{2}}+\frac{\pi}{2}-\text{A. t. }e^{-\frac{\pi\beta}{2}}-\text{A. t. }e^{-\frac{\pi\gamma}{2}}=\frac{\pi}{4}$

 —A.

$$-A.\,t.\,e^{\frac{\pi A}{2}}+\frac{\pi}{2}-A.\,t.\left\{\frac{e^{-\frac{\pi\beta}{2}}+e^{-\frac{\pi\gamma}{2}}}{1-e^{-\frac{\pi(\beta+\gamma)}{2}}}\right\}=\frac{\pi}{4}-A.\,t.\,e^{\frac{\pi A}{2}}+\frac{\pi}{2}$$

$$-A.\,t.\,\frac{e^{-\frac{\pi A}{4}}\left(e^{-\frac{\pi A\sqrt{-3}}{4}}+e^{\frac{\pi A\sqrt{-3}}{4}}\right)}{1-e^{-\frac{\pi A}{2}}}=\frac{\pi}{4}-A.\,t.\,e^{\frac{\pi A}{2}}+\frac{\pi}{2}$$

$$-A.\,t.\,\frac{2\,e^{\frac{\pi A}{4}}\cos.\frac{\pi A\sqrt{3}}{4}}{e^{\frac{\pi A}{2}}-1}.$$ Inde haec iam inuenta est

Summatio.

$$A.\,t.\frac{A^3}{1^3}-A.\,t.\frac{A^3}{3^3}+A.\,t.\frac{A^3}{5^3}-\text{etc.}\pm A.\,t.\frac{A^3}{(2x-1)^3}\mp\ldots=\frac{\pi}{4}-A.\,t.\,e^{\frac{\pi A}{2}}$$
$$+A.\,t.\left\{\frac{e^{\frac{\pi A}{2}}-1}{2\,e^{\frac{\pi A}{4}}\cos.\left(\frac{\pi A\sqrt{3}}{4}\right)}\right\}.$$

Corollarium I.

§. CXXVIII. Alia resolutio termini generalis eandem summationem praebet. Est nimirum $A.\ \text{tang.}\ \frac{A^3}{(2x-1)^3}=A.\,t.\,\frac{(2x-1)A}{(2x-1)^2-A^2}-A.\,t.\,\frac{A}{2x-1}$. Serierum, in quas ita series summanda dispescitur, primae summa ex §. CXVII. prodit $=$ $A.\,\text{tang.}\left\{\frac{e^{\frac{\pi A}{2}}-1}{2\,e^{\frac{\pi A}{4}}\cos.\frac{\pi A\sqrt{3}}{4}}\right\}$, alterius summa ex §. CX. $= A.\,\text{tang.}\,e^{\frac{\pi A}{2}}-\frac{\pi}{4}$. Vnde summae expressio prius inuenta (§. CXXVII.) confirmatur. Quae etiam sic exhiberi potest:

potest: $S. \pm A. \text{tang.} \frac{A^3}{(2x-1)^3} = \frac{3\pi}{4} - A.\,t.\, e^{\frac{\pi A}{2}} - A.\,t. \left\{ \frac{2 e^{\frac{\pi A}{4}} \text{cos.}\left(\frac{\pi A \sqrt{3}}{4}\right)}{e^{\frac{\pi A}{2}} - 1} \right\}$.

Haec posterior expressio tum adhibenda est, cum cos. $\left(\frac{\pi A \sqrt{3}}{4}\right)$ negatiuum valorem habeat, ac pro A. tang. — h ponendum — A. t. h.

Corollarium 2.

§. CXXIX. 1) Sit $A\sqrt{3} = 2(2k-1)$, denotante $2k-1$ quemuis numerum imparem, tum cos. $\frac{\pi A \sqrt{3}}{4}$ euanescet. Inde haec obtinetur Summatio:

$$A.\,\text{tang.}\frac{a}{1^3} - A.\,t.\frac{a}{3^3} + A.\,t.\frac{a}{5^3} - A.\,t.\frac{a}{7^3} + \text{etc.} \pm A.\,t.\frac{a}{(2x-1)^3} \mp \ldots$$

$$= \frac{3\pi}{4} - A.\,t.\left(e^{\frac{(2k-1)\pi}{\sqrt{3}}}\right), \text{ si fuerit } a = \frac{8(2k-1)^3}{3\sqrt{3}}.$$

2) Posito $A\sqrt{3} = 4r$, denotante r quemuis numerum integrum, erit cos. $\left(\frac{\pi A \sqrt{3}}{4}\right) = \text{cos. } \pi r = \mp 1$. Inde haec nascitur Summatio:

$$A.\,\text{tang.}\frac{a}{1^3} - A.\,t.\frac{a}{3^3} + A.\,t.\frac{a}{5^3} - A.\,t.\frac{a}{7^3} + \ldots \pm A.\,t.\frac{a}{(2x-1)^3} \mp \text{etc.}$$

$$= \frac{\pi}{4} + A.\,t.\frac{1}{e^{\frac{2\pi r}{\sqrt{3}}}} \pm 2\,A.\,t.\frac{1}{e^{\frac{\pi r}{\sqrt{3}}}},$$ posito $a = \frac{64 r^3}{3\sqrt{3}}$, et sumto in summa signo superiori pro impari r, contra inferiori.

Scholion.

§. CXXX. Series Problematis XIV. etiam per differentiationem et integrationem, methodo §. CXI. CXIII. adhibita, summari potest.

1) Sit: $A.\,\text{tang. } a - A.\,t.\frac{a}{3^3} + A.\,t.\frac{a}{5^3} - \text{etc.} = y$, erit $dy =$

$da\left(\frac{1}{1+a^2} - \frac{3^3}{3^6+a^2} + \frac{5^3}{5^6+a^2} - \ldots \pm \frac{(2x-1)^3}{(2x-1)^6+a^2} \mp \ldots\right)$. Quare series in da ducta summanda est. Posito $(2x-1)^2 = u$, $a = A^3$, est $\frac{(2x-1)^3}{(2x-1)^6+a^2} =$

 (2x — 1)

$$(2x-1)\cdot\frac{u}{u^3+A^6} = \frac{(2x-1)}{3A^2}\left(-\frac{1}{u+A^2}+\frac{\beta}{u-A^2\left(\cos.\frac{\pi}{3}+\Upsilon-1\,.\sin.\frac{\pi}{3}\right)}\right.$$

$$\left.+\frac{\gamma}{u-A^2\left(\cos.\frac{\pi}{3}-\Upsilon-1\,.\sin.\frac{\pi}{3}\right)}\right),$$ sumto $\beta = \cos.\frac{\pi}{3}-\Upsilon-1\sin.\frac{\pi}{3}$, $\gamma =$ $\cos.\frac{\pi}{3}+\Upsilon-1\,.\sin.\frac{\pi}{3}$. Hinc erit $dy =$

$$dA\left(-\Sigma\pm\frac{(2x-1)}{(2x-1)^2+A^2}+\beta\Sigma\pm\frac{(2x-1)}{(2x-1)^2-\gamma A^2}+\gamma\Sigma\pm\frac{(2x-1)}{(2x-1)^2-\beta A^2}\right).$$

Summae, quas hoc differentiale inuoluit, ex §. CXI. innotescunt. Est nimirum

$$\Sigma\pm\frac{(2x-1)}{(2x-1)^2+\xi^2} = \frac{\pi}{2\left(e^{\frac{\pi\xi}{2}}+e^{-\frac{\pi\xi}{2}}\right)}.$$ Hinc fit $dy =$

$$dA\left\{-\frac{\pi}{2\left(e^{\frac{\pi A}{2}}+e^{-\frac{\pi A}{2}}\right)}+\frac{\beta\pi}{2\left(e^{\frac{\pi A\Upsilon-\gamma}{2}}+e^{-\frac{\pi A\Upsilon-\gamma}{2}}\right)}+\frac{\gamma\pi}{2\left(e^{\frac{\pi A\Upsilon-\beta}{2}}+e^{-\frac{\pi A\Upsilon-\beta}{2}}\right)}\right\}.$$

2) Tria differentialia ita inuenta integrantur ope formulae: $\int\frac{dq}{e^q+e^{-q}} = \int\frac{dq\,.\,e^q}{e^{2q}+1}$ $=$ A. tang. e^q. Quare est $y =$

$$A.\ tang.\ e^{\frac{\pi A}{2}}+\frac{\beta}{\Upsilon-\gamma}.A.t.e^{\frac{\pi A\Upsilon-\gamma}{2}}+\frac{\gamma}{\Upsilon-\beta}.A.t.e^{\frac{\pi A\Upsilon-\beta}{2}}+\text{Const.}$$

Est autem $\gamma = \frac{1}{\beta}$, hinc $\frac{\beta}{\Upsilon-\gamma} = \frac{\beta^{\frac{3}{2}}}{\Upsilon-1} = \frac{\cos.\frac{\pi}{2}-\Upsilon-1\,.\sin.\frac{\pi}{2}}{\Upsilon-1} = -1$; eodem modo $\frac{\gamma}{\Upsilon-\beta} = 1$; porro $\Upsilon-\gamma = \cos.\frac{\pi}{6}\Upsilon-1-\sin.\frac{\pi}{6}$, $\Upsilon-\beta = \Upsilon-1\,.\Upsilon\beta$ $= \cos.\frac{\pi}{6}\Upsilon-1+\sin.\frac{\pi}{6}$. Hinc prodit $y =$

$$-A.t.e^{\frac{\pi A}{2}}+A.t.e^{\frac{\pi A}{2}\left(\cos.\frac{\pi}{6}\Upsilon-1+\sin.\frac{\pi}{6}\right)}-A.t.e^{\frac{\pi A}{2}\left(\cos.\frac{\pi}{6}\Upsilon-1-\sin.\frac{\pi}{6}\right)}+\text{Cst.}$$

$= -$ A.

$$= -\text{A. t. } e^{\frac{\pi A}{2}} + \text{A. t.}\left\{\frac{e^{\frac{\pi A \cos\frac{\pi}{6}\sqrt{-1}}{2}}\left(e^{\frac{\pi A}{4}} - e^{-\frac{\pi A}{4}}\right)}{1 + e^{\pi A \cos\frac{\pi}{6}\sqrt{-1}}}\right\} + \text{Const.}$$

$$= -\text{A. t. } e^{\frac{\pi A}{2}} + \text{A. t.}\left\{\frac{e^{\frac{\pi A}{4}} - e^{-\frac{\pi A}{4}}}{2\cos\left(\frac{\pi A}{2}\cos\frac{\pi}{6}\right)}\right\} + \text{Const.}$$

Constans ex valore $y = 0$, pro $A = 0$ determinatur, indeque ea est $= \frac{\pi}{4}$. Sic igitur haec summatio cum prius inuenta consentit.

Problema XV.

§. CXXXI. Summare seriem infinitam:

$$\text{A. t. a} - \text{A. t.}\frac{a}{(2m-1)^3} + \text{A. t.}\frac{a}{(2m+1)^3} - \text{A. t.}\frac{a}{(4m-1)^3} + \text{A. t.}\frac{a}{(4m+1)^3} - \text{etc.}$$

Solutio.

1) Ex resolutione §. CXXVII. (1) tradita, est quisque huius seriei terminus $\text{A. t.}\frac{a}{(2xm \pm 1)^3} = \text{A. t.}\frac{\alpha}{2xm \pm 1} + \text{A. t.}\frac{\beta}{2xm \pm 1} + \text{A. t.}\frac{\gamma}{2xm \pm 1}$, posito $\alpha = -\sqrt[3]{a} = -A$, $\beta = \frac{A + A\sqrt{-3}}{2}$, $\gamma = \frac{A - A\sqrt{-3}}{2}$.

2) Hinc summa seriei, adhibito Probl. XI. §. CXII., triplici parte constat: Primo $\frac{\pi(m-1)}{2m} - \text{A. t.}\left\{\frac{\sin\frac{\pi}{m}}{e^{\frac{\alpha\pi}{m}} - \cos\frac{\pi}{m}}\right\}$; deinde $\frac{\pi(m-1)}{2m} - \text{A. t.}\left\{\frac{\sin\frac{\pi}{m}}{e^{\frac{\beta\pi}{m}} - \cos\frac{\pi}{m}}\right\}$; tertio $\frac{\pi(m-1)}{2m} - \text{A. t.}\left\{\frac{\sin\frac{\pi}{m}}{e^{\frac{\gamma\pi}{m}} - \cos\frac{\pi}{m}}\right\}$. Est autem $\text{A. t.}\left\{\frac{\sin\frac{\pi}{m}}{e^{\frac{\beta\pi}{m}} - \cos\frac{\pi}{m}}\right\} +$

$$\text{A. tang.}\left\{\frac{\sin\frac{\pi}{m}}{e^{\frac{\gamma\pi}{m}} - \cos\frac{\pi}{m}}\right\} = \text{A. t.}\left\{\frac{\left(e^{\frac{\gamma\pi}{m}} + e^{\frac{\beta\pi}{m}}\right)\sin\frac{\pi}{m} - \sin\frac{2\pi}{m}}{e^{\frac{(\beta+\gamma)\pi}{m}} - \left(e^{\frac{\gamma\pi}{m}} + e^{\frac{\beta\pi}{m}}\right)\cos\frac{\pi}{m} + \cos\frac{2\pi}{m}}\right\} =$$

A. t.

$$\text{A. t.}\left\{\frac{e^{\frac{A\pi}{2m}}\left(e^{\frac{A\pi}{2m}\sqrt{-3}}+e^{-\frac{A\pi}{2m}\sqrt{-3}}\right)\left(\sin.\frac{\pi}{m}-\sin.\frac{2\pi}{m}\right)}{e^{\frac{A\pi}{m}}-e^{\frac{A\pi}{2m}}\left(e^{\frac{A\pi}{2m}\sqrt{-3}}+e^{-\frac{A\pi}{2m}\sqrt{-3}}\right)\left(\cos.\frac{\pi}{m}+\cos.\frac{2\pi}{m}\right)}\right\}=$$

$$\text{A. t.}\left\{\frac{2e^{\frac{A\pi}{2m}}\cos.\frac{A\pi\sqrt{3}}{2m}.\left(\sin.\frac{\pi}{m}-\sin.\frac{2\pi}{m}\right)}{e^{\frac{A\pi}{m}}-2e^{\frac{A\pi}{2m}}\cos.\frac{A\pi\sqrt{3}}{2m}.\left(\cos.\frac{\pi}{m}+\cos.\frac{2\pi}{m}\right)}\right\}.$$

Exinde prodit summa seriei in problemate propositae $=$

$$\frac{3\pi(m-1)}{2m}-\text{A. t.}\left\{\frac{\sin.\frac{\pi}{m}}{e^{-\frac{A\pi}{m}}-\cos.\frac{\pi}{m}}\right\}-\text{A. t.}\left\{\frac{2e^{\frac{A\pi}{2m}}\cos.\left(\frac{A\pi\sqrt{3}}{2m}\right)\left(\sin.\frac{\pi}{m}-\sin.\frac{2\pi}{m}\right)}{e^{\frac{A\pi}{m}}-2e^{\frac{A\pi}{2m}}\cos.\left(\frac{A\pi\sqrt{3}}{2m}\right)\left(\cos.\frac{\pi}{m}+\cos.\frac{2\pi}{m}\right)}\right\},$$

vbi est $A=\sqrt[3]{a}$. Loco partium primae et secundae huius expressionis poni etiam potest

$$\frac{\pi(m-1)}{2m}+\text{A. t.}\left\{\frac{\sin.\frac{\pi}{m}}{e^{\frac{A\pi}{m}}-\cos.\frac{\pi}{m}}\right\}.$$

Corollarium I.

§. CXXXII. Posito $m=\frac{1}{r}$, denotante r numerum integrum, ob $\sin.\frac{\pi}{m}$ et $\sin.\frac{2\pi}{m}$ euanescentes, summatio modo inuenta in hanc abit:

$$\text{A. t. } a-\text{A. t.}\frac{ar^3}{(2-r)^3}+\text{A. t.}\frac{ar^3}{(2+r)^3}-\text{A. t.}\frac{ar^3}{(4-r)^3}+\text{A. t.}\frac{ar^3}{(4+r)^3}-\text{etc.}$$

$=-\frac{\pi(r-1)}{2}$. Pro $m=\frac{2}{2k-1}$ summa euadit $=\frac{\pi(m-1)}{2m}\pm\text{A. t.}\frac{1}{e^{\frac{A\pi}{m}}}\mp$

$$\text{A. t.}\left\{\frac{2e^{\frac{A\pi}{2m}}\cos.\left(\frac{A\pi\sqrt{3}}{2m}\right)}{e^{\frac{A\pi}{m}}-1}\right\},$$

vbi signa superiora pro impari k obtinent.

Corollarium 2.

§. CXXXIII. Simili ratione ac §. CXV. ſummari poteſt ſeries:

$$\text{A. t. } a + \text{A. t.} \frac{a}{(2m-1)^3} - \text{A. t.} \frac{a}{(2m+1)^3} - \text{A. t.} \frac{a}{(4m-1)^3} + \text{A. t.} \frac{a}{(4m+1)^3} + \text{etc.}$$

vel dum haec ſeries ad binas formae praecedentis (§. CXXXI.) reuocatur, vel ita, vt illius terminus generalis in tres Arcus reſoluatur, vbi ſummatio §. CXV. ter adhibenda eſt. Cuius ſeriei ſi bini termini inuicem proximi addantur, noua oritur ſeries itidem ſummabilis, vti §. CXVI. Quae tamen cum ex hactenus demonſtratis repeti queant, ea amplius euoluere ſuperfluum, ſatiusque videtur, ad problemata generaliora progreſſum facere.

CAP. II.

SVMMATIONES GENERALIORES.

PROBLEMA XVI.

§. CXXXIV. Reſoluere Arcum, cuius tangens eſt functio fracta quantitatis z, ſeu A. tang. $\frac{P}{Q}$, in tot Arcus, quorum tangentes ſunt fractiones ſimplices, ſc. A. t. $\frac{a^{I}}{z+b^{I}}$ $+$ A. t. $\frac{a^{II}}{z+b^{II}}$ $+$ etc. $+$ A. t. $\frac{a^{N}}{z+b^{N}}$, ad quot gradus aſſurgit denominator Q.

Solutio.

2) Ex aequatione aſſumta: $\text{A. t.} \frac{P}{Q} = \text{A. t.} \frac{a^{I}}{z+b^{I}} + \text{A. t.} \frac{a^{II}}{z+b^{II}} + \ldots + \text{A. t.} \frac{a^{N}}{z+b^{N}}$

ſequitur, adhibendo ea quae §. VII. demonſtrata ſunt: $\frac{Q+P\sqrt{-1}}{Q-P\sqrt{-1}} = P \left\{ \frac{1+\frac{a^{N}}{z+b^{N}}.\sqrt{-1}}{1-\frac{a^{N}}{z+b^{N}}.\sqrt{-1}} \right\}$,

$$\text{id eſt} = \frac{(z+b^{I}+a^{I}\sqrt{-1})(z+b^{II}+a^{II}\sqrt{-1})\ldots(z+b^{N}+a^{N}\sqrt{-1})}{(z+b^{I}-a^{I}\sqrt{-1})(z+b^{II}-a^{II}\sqrt{-1})\ldots(z+b^{N}-a^{N}\sqrt{-1})}.$$

2) Iam ponatur factorum numeratoris quicunque $z+b^{R}+a^{R}\sqrt{-1}=0$, etiam alterius fractionis numerator $Q+P\sqrt{-1}$ euaneſcat neceſſe eſt. Aequatio $Q+P\sqrt{-1}=0$, n radices imaginarias habet. Realis nulla eſſe poteſt, quippe pro radice

radice reali valor realis Q foret $= - P\sqrt{-1} =$ quantitati imaginariae: at P et Q nec euanescere simul possunt, cum in fractione $\frac{P}{Q}$ numerator et denominator a factore communi liberi supponantur. Hinc apparet, aequationis $Q + P\sqrt{-1} = 0$ n radices esse $z = -b^{I} - a^{I}\sqrt{-1}$; $-b^{II} - a^{II}\sqrt{-1}$; ... $-b^{N} - a^{N}\sqrt{-1}$.

3) Quibus ita consideratis sequens oritur problematis solutio: Resoluatur aequatio $Q + P\sqrt{-1} = 0$, sintque huius quantitatis factores imaginarii $z + b^{I} + a^{I}\sqrt{-1}$; ... $z + b^{R} + a^{R}\sqrt{-1}$; ... $z + b^{N} + a^{N}\sqrt{-1}$; tum ex horum quouis formandus est Arcus A. t. $\frac{a^{R}}{z + b^{R}}$, qui Arcus inuicem additi conficient A. t. $\frac{P}{Q}$.

Corollarium 1.

§. CXXXV. Posito coëfficiente τῷ z^{II} in denominatore $Q = 1$, erit $Q + P\sqrt{-1}$ $= (z + b^{I} + a^{I}\sqrt{-1}) \ldots (z + b^{R} + a^{R}\sqrt{-1}) \ldots (z + b^{N} + a^{N}\sqrt{-1})$; vnde permutando $\sqrt{-1}$ cum $-\sqrt{-1}$, sponte consequitur: $Q - P\sqrt{-1} =$ $(z + b^{I} - a^{I}\sqrt{-1}) \ldots (z + b^{R} - a^{R}\sqrt{-1}) \ldots (z + b^{N} - a^{N}\sqrt{-1})$. Hinc fit multiplicando: $Q^2 + P^2 = (z^2 + 2b^{I}z + (a^{I})^2 + (b^{I})^2) \ldots$ $(z^2 + 2b^{R}z + (a^{R})^2 + (b^{R})^2) \ldots (z^2 + 2b^{N}z + (a^{N})^2 + (b^{N})^2)$. Quare quantitatis $Q^2 + P^2$ factor quilibet trinomialis erit $z^2 + 2b^{R}z + (a^{R})^2 + (b^{R})^2$, compositus ex factoribus imaginariis $z + b^{R} + a^{R}\sqrt{-1}$, $z + b^{R} - a^{R}\sqrt{-1}$. Aequationem $Q^2 + P^2 = 0$ radices tantum imaginarias habere, exinde apparet, quod pro realibus haud esse posset $Q^2 = -P^2$. Sic igitur resolutio aequationis $Q + P\sqrt{-1} = 0$, hinc quoque solutio probl. XVI. (§. CXXXIV.), reducitur ad resolutionem aequationis $Q^2 + P^2 = 0$.

Corollarium 2.

§. CXXXVI. 1) Si $\frac{P}{Q}$ est functio spuria fracta, i. e. dimensio numeratoris maior dimensione denominatoris, eiue aequalis, tum a $\frac{P}{Q}$ per diuisionem separari potest functio integra vel Constans. Quae sit $= Z$, ac $\frac{P}{Q} = Z + \frac{\mathfrak{P}}{Q}$; tum erit A. t. $\frac{P}{Q}$ — A.t.Z $=$ A. t. $\dfrac{\frac{\mathfrak{P}}{Q}}{1 + Z\frac{P}{Q}}$, hincque A. t. $\frac{P}{Q}$ = A. t. Z + A. t. $\frac{\mathfrak{P}}{PZ + Q}$, vbi est $\frac{\mathfrak{P}}{PZ + Q}$ functio ve-

ra

ra fracta. Proinde Arcus, cuius tangens est fractio fracta spuria. in binos Arcus resolui potest, quorum vnius tangens est functio integra ex diuisione numeratoris functionis spurae per denominatorem orta.

2) *Exempli* gratia sit $P = Az^n + Bz^{n-1} + Cz^{n-2} + \text{etc.}$, $Q = z^n + \mathfrak{B}z^{n-1} + \mathfrak{C}z^{n-2} + \text{etc.}$, diuidendo prodit $\frac{P}{Q} = A + \frac{(B - A\mathfrak{B})z^{n-1} + (C - A\mathfrak{C})z^{n-2} + (D - A\mathfrak{D})z^{n-3} + \ldots}{z^n + \mathfrak{B}z^{n-1} + \mathfrak{C}z^{n-2} + \ldots}$. Hinc fit

$$A.\,\text{tang.}\left\{\frac{Az^n + Bz^{n-1} + Cz^{n-2} + \ldots}{z^n + \mathfrak{B}z^{n-1} + \mathfrak{C}z^{n-2} + \ldots}\right\} = A.\,\text{tang.}\,A +$$

$$A.\,\text{tang.}\left\{\frac{(B - A\mathfrak{B})z^{n-1} + (C - A\mathfrak{C})z^{n-2} + (D - A\mathfrak{D})z^{n-3} + \ldots}{(1 + AA)z^n + (\mathfrak{B} + AB)z^{n-1} + (\mathfrak{C} + AC)z^{n-2} + \ldots}\right\}$$ (cf. supra §. XXX.)

Scholion.

§. CXXXVII. Solutio problematis XVI. (§. CXXXIV.) sequenti etiam ratione inuestigari potest:

1) Sit $A.\,\text{tang.}\,\frac{P}{Q} = A.\,t.\,\frac{Az^{n-1} + Bz^{n-2} + Cz^{n-3} + \ldots}{z^n + \mathfrak{B}z^{n-1} + \mathfrak{C}z^{n-2} + \ldots} =$

$A.\,\text{tang.}\,\frac{a^I}{z + b^I} + A.\,t.\,\frac{a^{II}}{z + b^{II}} \ldots + A.\,t.\,\frac{a^R}{z + b^R} \ldots + A.\,t.\,\frac{a^N}{z + b^N}$, erit differentiando

$$\frac{Q^2 d\frac{P}{Q}}{Q^2 + P^2} = \frac{QdP - PdQ}{Q^2 + P^2} = -\left\{\frac{a^I}{(z + b^I)^2 + (a^I)^2} + \frac{a^{II}}{(z + b^{II})^2 + (a^{II})^2} + \ldots + \frac{a^R}{(z + b^R) + (a^R)^2} + \ldots\right\} dz.$$

2) Resoluenda igitur est functio fracta $\frac{QdP - PdQ}{(Q^2 + P^2)dz} = \frac{M}{N}$ in fractiones simplices. Ex denominatoris $N = Q^2 + P^2$ quibusuis binis factoribus imaginariis $z + \beta^R + \alpha^R\sqrt{-1}$, $z + \beta^R - \alpha^R\sqrt{-1}$, oriuntur fractiones simplices $\frac{\mathfrak{A}}{z + \beta^R + \alpha^R\sqrt{-1}}$, $\frac{\mathfrak{A}^I}{z + \beta^R - \alpha^R\sqrt{-1}}$, vbi numeratores $\mathfrak{A}$, $\mathfrak{A}^I$ sunt $= \frac{Mdz}{dN}$, posito pro illo $z + \beta^R + \alpha^R\sqrt{-1} = o$, pro hoc $z + \beta^R - \alpha^R\sqrt{-1} = o$, hinc vtrinque

que $Q^2+P^2=0$ (*). Est autem $N=Q^2\left(1+\frac{P^2}{Q^2}\right)$; $dN=Q^2\cdot 2\frac{P}{Q}\cdot d\left(\frac{P}{Q}\right)+\left(1+\frac{P^2}{Q^2}\right)2QdQ$, hinc $\frac{Mdz}{dN}=\frac{Q^2 d\frac{P}{Q}}{2PQd\left(\frac{P}{Q}\right)+\left(1+\frac{P^2}{Q^2}\right)2QdQ}=\frac{Q}{2P}$; pro $Q^2+P^2=0$. Iam pro $z+\beta^R+\alpha^R\sqrt{-1}=0$ fit $\frac{P}{Q}=\sqrt{-1}$, seu $z+\beta^R+\alpha^R\sqrt{-1}$ sit factor quantitatis imaginariae $Q+P\sqrt{-1}$, et erit pro $z+\beta^R-\alpha^R\sqrt{-1}=0$, $\frac{P}{Q}=-\sqrt{-1}$. Quare prouenit $\mathfrak{A}=\frac{1}{2\sqrt{-1}}$, $\mathfrak{A}^I=-\frac{1}{2\sqrt{-1}}$, et fractionum simplicium $\frac{\mathfrak{A}}{z+\beta^R+\alpha^R\sqrt{-1}}$, $\frac{\mathfrak{A}^I}{z+\beta^R-\alpha^R\sqrt{-1}}$ summa $=\frac{1}{2\sqrt{-1}}\cdot$ $-\frac{2\mathfrak{A}\alpha^R\sqrt{-1}}{(z+\beta^R)^2+(\alpha^R)^2}$. Hinc erit functio fracta $\frac{M}{N}=-\frac{\alpha^I}{(z+\beta^I)^2+(\alpha^I)^2}-\frac{\alpha^{II}}{(z+\beta^{II})^2+(\alpha^{II})^2}-\ldots-\frac{\alpha^R}{(z+\beta^R)^2+(\alpha^R)^2}-$ etc.

3) Qua aequatione comparata cum aequatione (1) sponte consequitur, quantitates assumtas b^I, a^I; b^{II}, a^{II}; ... b^R, a^R; etc. conuenire cum quantitatibus inuentis β^I, α^I; β^{II}, α^{II}; ... β^R, α^R; ... quae quidem ita sunt comparatae, vt $z+\beta^R+\alpha^R\sqrt{-1}$ sit factor functionis $Q+P\sqrt{-1}$. Hinc igitur solutio prius inuenta (§. CXXXIV.) confirmatur.

Corollarium 3.

§. CXXXVIII. Quodsi $\frac{P}{Q}$ fuerit functio *par* quantitatis z, seu functio quadrati z^2, tum poni potest $z^2=u$, et resoluetur A. t. $\frac{P}{Q}$ in Arcus sequentes:

$$\text{A. tang.}\frac{a^I}{z^2+b^I}+\text{A. t.}\frac{a^{II}}{z^2+b^{II}}+\ldots+\text{A. t.}\frac{a^N}{z^2+b^N},$$

existentibus $u+b^I+a^I\sqrt{-1}$, $u+b^{II}+a^{II}\sqrt{-1}$, ... $u+b^N+a^N\sqrt{-1}$ factoribus simplicibus quantitatis $Q+P\sqrt{-1}$.

Corol-

(*) KAESTNER *Analysis des Unendlichen*, §. 386. pag. 318.

Corollarium 4.

§. CXXXIX. 1) Sit $\frac{P}{Q}$ functio *fracta impar* quantitatis z, tum quoniam vel numerator vel denominator par, ac alteruter impar esse debet, poni poterit $\frac{P}{Q} = \frac{\mathfrak{A} z^{n-1} - \mathfrak{C} z^{n-3} + \mathfrak{E} z^{n-5} - \text{etc.}}{z^n - \mathfrak{B} z^{n-2} + \mathfrak{D} z^{n-4} - \text{etc.}}$. Hinc aequatio $Q + P\sqrt{-1} = 0$ abit in hanc: $z^n - \mathfrak{B} z^{n-2} + \mathfrak{D} z^{n-4} - \text{etc.} + (\mathfrak{A} z^{n-1} - \mathfrak{C} z^{n-3} + \mathfrak{E} z^{n-5} - \text{etc.})\sqrt{-1} = 0$. Posito $z = -\zeta\sqrt{-1}$, erit $z^2 = -\zeta^2$, $z^3 = +\zeta^3\sqrt{-1}$, $z^4 = \zeta^4$, ... $z^{2r} = \mp \zeta^{2r}$, $z^{2r+1} = \pm \zeta^{2r+1}\sqrt{-1}$, signis superioribus pro impari r, inferioribus pro pari r sumtis. Hinc prodit: $\zeta^n - \mathfrak{A}\zeta^{n-1} + \mathfrak{B}\zeta^{n-2} - \mathfrak{C}\zeta^{n-3} + \mathfrak{D}\zeta^{n-4} - \text{etc.} = 0$. Huius aequationis radicum aliqua denotetur per c^R, eritque $z = -c^R\sqrt{-1}$, seu $z + c^R\sqrt{-1}$ factor functionis $Q + P\sqrt{-1}$. Quare in solutione §. CXXXIV. poni possunt $b^I = b^{II} = b^{III} \ldots b^N = 0$; a^I, a^{II}, $a^{III} \ldots a^N = c^I, c^{II}, c^{III} \ldots c^N$.

2) Exinde pro hoc casu, cum sit $\frac{P}{Q}$ functio fracta impar quantitatis z, haec oritur resolutio Arcus compositi in Arcus simplices:

$$\text{A. tang.} \frac{P}{Q} = \text{A. t.} \frac{c^I}{z} + \text{A. t.} \frac{c^{II}}{z} + \text{A. t.} \frac{c^{III}}{z} \ldots + \text{A. t.} \frac{c^N}{z},$$

denotantibus c^I, c^{II}, $c^{III} \ldots c^N$ radices aequationis: $\zeta^n - \mathfrak{A}\zeta^{n-1} + \mathfrak{B}\zeta^{n-2} - \mathfrak{C}\zeta^{n-3} + \mathfrak{D}\zeta^{n-4} - \text{etc.} = 0$, posito scilicet $\frac{P}{Q} = \frac{\mathfrak{A} z^{n-1} - \mathfrak{C} z^{n-3} + \mathfrak{E} z^{n-5} - \ldots}{z^n - \mathfrak{B} z^{n-2} + \mathfrak{D} z^{n-4} - \ldots}$.

Scholion.

§. CXL. 1) Eiusdem resolutionis alia quoque demonstratio ex formula supra (§. IV.) tradita deduci potest. Est nimirum $\text{A. tang.} \frac{c^I}{z} + \text{A. t.} \frac{c^{II}}{z} \ldots + \text{A. t.} \frac{c^N}{z} = \text{A. tang.} \frac{A - C + E - G \ldots}{1 - B + D - F \ldots}$, denotantibus A, B, C, D ... summas Vnionum, Binionum, Ternionum etc. ex quantitatibus $\frac{c^I}{z}, \frac{c^{II}}{z} \ldots \frac{c^N}{z}$ conflatarum. Quodsi iam $\mathfrak{A}$,

$\mathfrak{B}, \mathfrak{C} \ldots$ idem denotent, quoad quantitates $c^I, c^{II} \ldots c^N$, erit $A = \frac{\mathfrak{A}}{z^1}$, $B = \frac{\mathfrak{B}}{z^2}$,

$C = \frac{\mathfrak{C}}{z^3} \ldots$ Hinc fit summa ista $= A.\,t. \left\{ \frac{\frac{\mathfrak{A}}{z} - \frac{\mathfrak{C}}{z^3} + \frac{\mathfrak{E}}{z^5} - \text{etc.}}{1 - \frac{\mathfrak{B}}{z^2} + \frac{\mathfrak{D}}{z^4} - \frac{\mathfrak{F}}{z^6} + \text{etc.}} \right\} =$

$A.\,\text{tang.} \left\{ \frac{\mathfrak{A} z^{n-1} - \mathfrak{C} z^{n-3} + \mathfrak{E} z^{n-5} \ldots}{z^n - \mathfrak{B} z^{n-2} + \mathfrak{D} z^{n-4} - \mathfrak{F} z^{n-6} \ldots} \right\}$. Quodsi vero $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \ldots$ complectantur vniones, biniones etc. quantitatum $c^I, c^{II} \ldots c^N$, sponte apparet, aequationis: $z^n - \mathfrak{A} z^{n-1} + \mathfrak{B} z^{n-2} - \mathfrak{C} z^{n-3} + \mathfrak{D} z^{n-4} -$ etc. radices esse $c^I, c^{II} \ldots c^N$, indeque prior resolutio recurrit.

2) Si radicum $c^I, c^{II} \ldots c^N$ aliqua est imaginaria, $= f + g\sqrt{-1}$, simul altera aderit $= f - g\sqrt{-1}$; tumque erit summa binorum Arcuum simplicium his radicibus respondentium $= A.\,\text{tang.} \frac{f + g\sqrt{-1}}{z} + A.\,t. \frac{f - g\sqrt{-1}}{z} = A.\,t. \frac{2fz}{z^2 - f^2 - g^2}$, quae summa iam ad formam realem reducta etiam realiter in Arcus simplices resolui potest, scilicet $A.\,\text{tang.} \frac{f}{z+g} + A.\,t. \frac{f}{z-g}$.

Problema XVII.

§. CXLI. Summare seriem infinitam:

$A.\,\text{tang.} \frac{A}{B} + A.\,t. \frac{C}{D} + A.\,t. \frac{E}{F} \ldots + A.\,t. \frac{P}{Q} +$ etc. existente $\frac{P}{Q} =$ functioni *fractae pari* indicis x.

Solutio.

Resoluatur terminus generalis $A.\,\text{tang.} \frac{P}{Q}$ in plures Arcus, modo §. CXXXVIII. exposito. Ex quolibet nimirum quantitatis $Q + P\sqrt{-1}$ factore $= x^2 + b + a\sqrt{-1}$ oritur istiusmodi Arcus $= A.\,t. \frac{a}{x^2 + b}$. Qua ratione dispescitur series summanda in plures series, formae: $A.\,t. \frac{a}{1+b} + A.\,t. \frac{a}{4+b} + A.\,t. \frac{a}{9+b} \ldots + A.\,t. \frac{a}{xx+b} +$ etc. quae singulae ex Probl. VIII. (§. XCIII.) summari possunt. Talis quippe seriei summa reperitur $= \alpha\pi - A.\,t. \frac{\alpha}{\beta} + A.\,t. \left(\frac{\sin.\, 2\pi\alpha}{e^{2\pi\beta} - \cos.\, 2\pi\alpha} \right)$, posito $\alpha^2 = \frac{\sqrt{(b^2+a^2)} - b}{2}$,

et

et $\beta^2 = \frac{\sqrt{(b^2+a^2)}+b}{2}$, seu $\beta = \frac{a}{2\alpha}$; loco A. t. $\frac{\alpha}{\beta}$ etiam poni potest $\frac{1}{2}$ A. t. $\frac{a}{b}$, vel si b negatiuum valorem habeat, $\frac{\pi}{2} - \frac{1}{2}$ A. t. $\frac{a}{-b}$. Hac expressione ad singulas series partiales adhibita prodit seriei problematis ex istis compositae summa.

Corollarium 1.

§. CXLII. 1) Eadem ratione summari potest series, pro cuius termino generali seu x^{to}, A. tang. $\frac{P}{Q}$, aequatur $\frac{P}{Q}$ *functioni fractae pari* x^{ti} *numeri imparis* $= 2x - 1$; quippe tum A. t. $\frac{P}{Q}$ resoluitur in Arcus, quorum quilibet est $=$ A. t. $\frac{a}{(2x-1)^2+b}$, indeque series in series partiales ex §. XCVII. summabiles.

2) Nec minus summabitur series problematis praecedentis, si *signa alternantur*, seu A. t. $\frac{A}{B}$ — A. t. $\frac{C}{D}$ + A. t. $\frac{E}{F}$ — . . . + A. t. $\frac{P}{Q} \mp$ etc., adhibita simili resolutione termini generalis, et summatione §. XCIX.

Corollarium 2.

§. CXLIII. 1) Si in serie A. tang. $\frac{A}{B}$ + A. t. $\frac{C}{D}$ + . . . + A. t. $\frac{P}{Q}$ + A. t. $\frac{R}{S}$ + etc. fuerit $\frac{P}{Q}$ functio fracta par quantitatis $2mx - 1$, et $\frac{R}{S}$ similis functio quantitatis $2mx + 1$, pertinentibus A. t. $\frac{P}{Q}$ et A. t. $\frac{R}{S}$ ad x^{tam} combinationem duorum seriei terminorum, tum hi Arcus resolui possunt in Arcus simplices formae A. t. $\frac{a}{(2mx-1)^2+b}$ et A. t. $\frac{a}{(2mx+1)^2+b}$. Hinc series ista in plures dispescitur, quae singulae ex probl. IX. §. CVI. summabiles sunt.

2) Cum sint $\frac{P}{Q}$ et $\frac{R}{S}$ functiones similes quadratorum $(2mx-1)^2$ et $(2mx+1)^2$, signo φ denotandae, erit A. t. $\frac{P}{Q}$ + A. t. $\frac{R}{S}$ = A. t. $\left(\frac{\varphi(2mx-1)^2+\varphi(2mx+1)^2}{1-\varphi(2mx-1)^2 \cdot \varphi(2mx+1)^2}\right)$ = A. t. $\frac{\Psi}{\Omega}$. Est autem $\varphi(2mx-1)^2 + \varphi(2mx+1)^2$ functio τῦ x^2, quoniam in additione quarumuis potestatum $(2mx-1)^{2p} + (2mx-1)^{2p}$ impares potestates τῦ x se mutuo tollunt; at $\varphi(2mx+1)^2 - \varphi(2mx-1)^2$ est functio impar τῦ x.

Hinc

Hinc productum $\varphi(2mx-1)^2 \cdot \varphi(2mx+1)^2 = \frac{(\varphi(2mx+1)^2+\varphi(2mx-1)^2)^2-(\varphi(2mx+1)^2-\varphi(2mx-1)^2)^2}{4}$ est functio par. Quare $\frac{\mathfrak{P}}{\mathfrak{Q}}$ etiam est functio par. Idem exinde apparet, quod cum $\mathfrak{P}$ tum $\mathfrak{Q}$ valorem non mutent, cum x abeat in $-x$. Ex his efficitur, coniungendo feriei (1) binos terminos sibi inuicem proximos eam in aliam transformari, ex probl. XVII. (§. CXLI.) summabilem.

PROBLEMA XVIII.

§. CXLIV. Summare seriem infinitam signis alternantibus praeditam: $A.\ tang.\ \frac{A}{B} - A.\ t.\ \frac{C}{D} + A.\ t.\ \frac{E}{F} - \ldots \pm A.\ t.\ \frac{P}{Q} \mp$ etc., existente $\frac{P}{Q}$, tangente x^{ti} termini, $=$ functioni *impari fractae* x^{ti} numeri imparis $2x-1$.

Solutio.

1) Resoluatur terminus x^{tus} $A.\ t.\ \frac{P}{Q}$ in plures Arcus: $A.\ tang.\ \frac{c^I}{2x-1} + A.\ tang.\ \frac{c^{II}}{2x-1} + \ldots + A.\ t.\ \frac{c^N}{2x-1}$, denominatoribus c^I, $c^{II} \ldots c^N$ ad regulam §. CXXXIX. 2. determinatis; tum series summanda in plures dispescitur, quarum quaelibet huius est formae: $A.\ t.\ c - A.\ t.\ \frac{c}{3} + A.\ t.\ \frac{c}{5} - \ldots \pm A.\ t.\ \frac{c}{2x-1} \mp$ etc., et ex probl. X. §. CX. summam habet $= A.\ tang.\ e^{\frac{\pi c}{2}} - \frac{\pi}{4}$.

2) Si inter quantitates c^I, $c^{II} \ldots c^N$ binae adsint imaginariae $f + g\sqrt{-1}$, $f - g\sqrt{-1}$, tum summae partiales ex iis oriundae inuicem additae praebent $\frac{\pi}{2}$

$$- A.\ t.\ e^{\frac{-(f+g\sqrt{-1})\pi}{2}} - A.\ t.\ e^{\frac{-(f-g\sqrt{-1})\pi}{2}} = \frac{\pi}{2} -$$

$$A.\ tang.\ \frac{e^{\frac{-f\pi}{2}}\left(e^{\frac{-g\pi\sqrt{-1}}{2}} + e^{\frac{+g\pi\sqrt{-1}}{2}}\right)}{1-e^{-f\pi}} = \frac{\pi}{2} - A.\ t.\left\{\frac{2e^{\frac{f\pi}{2}}\ \text{cos}.\ \frac{g\pi}{2}}{e^{f\pi}-1}\right\},$$ siue etiam $= A.\ t.\left\{\frac{e^{f\pi}-1}{2e^{\frac{f\pi}{2}}\ \text{cos}.\ \frac{g\pi}{2}}\right\}$.

Corol-

Corollarium.

§. CXLV. 1) Eadem ratione ſeries infinita:

$$\text{A. tang.}\frac{A}{B} - \text{A. t.}\frac{C}{D} + \text{A. t.}\frac{E}{F} - \text{A. t.}\frac{G}{H} \ldots \pm \text{A. t.}\frac{P}{Q} \mp \text{A. t.}\frac{R}{S} \pm \text{etc.}$$

ſummabilis eſt, ſi fuerit $\frac{P}{Q}$ functio fracta impar quantitatis $2mx - 1$, $\frac{R}{S}$ ſimilis functio quantitatis $2mx + 1$. Haec enim ſeries ex §. CXXXIX. in plures reſoluitur huius formae: $\text{A. t.}\frac{a}{2m-1} - \text{A. t.}\frac{a}{2m+1} + \text{A. t.}\frac{a}{4m-1} - \text{A. t.}\frac{a}{4m+1} \ldots + \text{A. t.}\frac{a}{2mx-1}$ $- \text{A. t.}\frac{a}{2mx+1} + \text{etc.}$ quarum quaelibet ex probl. XI. §. CXII. ſummabilis eſt.

2) Ceterum haec ſummatio ad probl. XVII. §. CXLI. reducitur: quippe coniungendo binos terminos oritur ſeries, cuius terminus $x^{\text{tus}} = \text{A. t.}\frac{P}{Q} - \text{A. t.}\frac{R}{S} =$

$\text{A. tang.}\frac{\psi(2mx-1) - \psi(2mx+1)}{1 + \psi(2mx-1)\,.\,\psi(2mx+1)} = \text{A. t}\frac{\mathfrak{P}}{\mathfrak{Q}}$, vbi numerator $\mathfrak{P}$ et denominator $\mathfrak{Q}$ ſunt functiones pares ſeu τȣ x^2, quippe poſito $-x$ pro x, ille eſt $= \psi(-2mx-1)$ $- \psi(-2mx+1) = \psi(2mx-1) - \psi(2mx+1)$, hic $= 1 + \psi(-2mx-1)$ $.\,\psi(-2mx+1) = 1 + \psi(2mx+1)\,.\,\psi(2mx-1)$, hincque neuter valorem mutat; quod eſt proprium functionum parium, cum functiones impares, ceu $\frac{P}{Q}$ et $\frac{R}{S}$ per ſignum ψ denotatae, ſimul cum variabili ſignum mutent.

3) Series (1) etiam complectitur ſeriem problematis XVIII. pro $m = 2$, quare hoc etiam problema reduci poteſt ad problema XVII.

Scholion.

§. CXLVI. Quae haſtenus vniuerſe expoſita ſunt, inprimis adplicanda videntur ad binos caſus, cum fuerit $\frac{P}{Q}$ (§. CXLI.) $= \frac{a}{x^{2n}+b}$, et (§. CXLIV.) $\frac{P}{Q} = \frac{a}{(2x-1)^{2n-1}}$. Deinde etiam poni poteſt (§. CXLIII.) $\frac{P}{Q} = \frac{a}{(2mx-1)^{2n}+b}$, et (§. CXLV.) $\frac{P}{Q} =$ $\frac{a}{(2mx-1)^{2n-1}}$. Haec iam problemata vt rite ſoluantur, inueſtigandi ſunt factores quantitatis imaginariae $Q + P\sqrt{-1} = x^{2n} + b + a\sqrt{-1}$. Qui quidem factores ex factoribus quantitatis $z^m + B$, ope theorematis Coteſiani inuentis, deriuari poſſunt, ponendo $B = b + a\sqrt{-1}$. Cum tamen ipſum hoc theorema ita demonſtrari ſoleat, vt ſuppo-

ſupponatur quantitas B eſſe realis: (*) cumque in tractatione quantitatum imaginariarum haud leuis attentio adhibenda ſit, vt ſphalmata euitentur, praeſtare videtur, factores quantitatis imaginariae $u^n + a + b\sqrt{-1}$ ex ipſis principiis peculiariter inueſtigare. Quorſum ſpectat ſequens Lemma.

Lemma.

§. CXLVII. **Inueſtigare factores ſimplices quantitatis imaginariae $u^n + b + a\sqrt{-1}$.**

Solutio.

1) Cum aequatio $u^n + b + a\sqrt{-1} = 0$ radices tantum imaginarias habeat, ponatur earum quaelibet $u = f\,\text{coſ.}\,\phi + f\,\text{ſin.}\,\phi\sqrt{-1}$, eritque $u^n = f^n(\text{coſ.}\,n\phi + \text{ſin.}\,n\phi.\sqrt{-1})$. Inde duplex oritur aequatio: $f^n\,\text{coſ.}\,n\phi + b = 0$; $f^n\,\text{ſin.}\,n\phi + a = 0$. Ex prima eſt $f^{2n}(\text{coſ.}\,n\phi)^2 = b^2$, ex altera $f^{2n}(\text{ſin.}\,n\phi)^2 = a^2$, hinc addendo $f^{2n} = b^2 + a^2$, et $f = \sqrt[2n]{(b^2 + a^2)}$; cuius valor vnicus realis, idemque poſitiuus accipiatur, vnde etiam $\sqrt{}.(b^2 + a^2) = f^n$ poſitiue accipi debet. Exinde ſit $\text{coſ.}\,n\phi = -\frac{b}{\sqrt{(b^2+a)^2}}$, $\text{ſin.}\,n\phi = -\frac{a}{\sqrt{(b^2+a^2)}}$. Sit iam ψ Arcus minimus affirmatiuus, cui competit ſinus $\frac{a}{\sqrt{(b^2+a^2)}}$, coſinus $\frac{b}{\sqrt{(b^2+a^2)}}$, tangens $= \frac{a}{b}$, aequationibus pro $\text{coſ.}\,n\phi$ et $\text{ſin.}\,n\phi$ ſatisfiet, ſumendo $n\phi = \psi \pm (2k+1)\pi$, quia tum eſt $\text{coſ.}\,n\phi = -\text{coſ.}\,\psi$, $\text{ſin.}\,n\phi = -\text{ſin.}\,\psi$. Quare erit $\phi = \frac{\psi \pm (2k+1)\pi}{n}$, et $u = (b^2 + a^2)^{\frac{1}{2n}}\left(\text{coſ.}\left(\frac{\psi \pm (2k+1)\pi}{n}\right) + \text{ſin.}\left(\frac{\psi \pm (2k+1)\pi}{n}\right).\sqrt{-1}\right)$.

2) Sit primo n = numero pari $= 2r$, tum pro $2k+1$ accipiendi ſunt omnes numeri impares minores quam n, i. e. ab 1 vsque ad $2r-1$; quorum multitudo cum ſit r, ob duplicitatem ſigni naſcuntur $2r$ valores quantitatis u, i. e. n radices aequationis n^{ti} gradus, vti par eſt. Erit igitur, ſumto $f = \sqrt[2n]{(b^2 + a^2)}$, et $\psi = A.t.\frac{a}{b}$.

$u^n + b + a\sqrt{-1}$

(*) KAESTNER l. c. §. 392. pag. 328.

$$u^n + b + a\sqrt{-1} = (u - f\cos.\left(\frac{\psi+\pi}{n}\right) - f\sin.\left(\frac{\psi+\pi}{n}\right).\sqrt{-1})$$
$$(u - f\cos.\left(\frac{\psi-\pi}{n}\right) - f\sin.\left(\frac{\psi-\pi}{n}\right).\sqrt{-1})$$
$$(u - f\cos.\left(\frac{\psi+3\pi}{n}\right) - f\sin.\left(\frac{\psi+3\pi}{n}\right).\sqrt{-1})$$
$$(u - f\cos.\left(\frac{\psi-3\pi}{n}\right) - f\sin.\left(\frac{\psi-3\pi}{n}\right).\sqrt{-1})$$
$$(u - f\cos.\left(\frac{\psi+5\pi}{n}\right) - f\sin.\left(\frac{\psi+5\pi}{n}\right).\sqrt{-1})$$
$$(u - f\cos.\left(\frac{\psi-5\pi}{n}\right) - f\sin.\left(\frac{\psi-5\pi}{n}\right).\sqrt{-1})$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$(u - f\cos.\left(\frac{\psi+(n-1)\pi}{n}\right) - f\sin.\left(\frac{\psi+(n-1)\pi}{n}\right).\sqrt{-1})$$
$$(u - f\cos.\left(\frac{\psi-(n-1)\pi}{n}\right) - f\sin.\left(\frac{\psi-(n-1)\pi}{n}\right).\sqrt{-1})$$

3) Sit secundo n = numero impari $= 2r - 1$, tum pro $2k + 1$ (1) omnes numeri impares ab 1 ad $2r - 1$ sumi possunt, at cum pro $2k + 1 = n$ sit $\cos.\left(\frac{\psi - n\pi}{n}\right) = \cos.\left(\frac{\psi + n\pi}{n}\right) = -\cos.\frac{\psi}{n}$; et $\sin.\left(\frac{\psi - n\pi}{n}\right) = \sin.\left(\frac{\psi + n\pi}{n}\right) = -\sin.\frac{\psi}{n}$: factor ex positione $2k + 1 = n$ oriundus non quadraticus, sed simplex accipiendus est; quo ita sumto numerus factorum $= n$ completur. Quare etiam pro numero impari n expressio (2) inuenta obtinet, hoc tantum discrimine, vt loco factorum binorum vltimorum, quos praeberet ista expressio (pro $2k + 1 = n$), quique inuicem aequales sunt, vnus tantum sumatur, siue etiam multipla τῶ π extendantur tantum ad $(n-2)\pi$, ac insuper adiiciatur factor vnicus $u + f\cos.\frac{\psi}{n} + f\sin.\frac{\psi}{n}.\sqrt{-1}$.

Corollarium.

§. CXLVIII. 1) Si b negatiuum valorem habeat, vel quantitas imaginaria sit $u^n - b + a\sqrt{-1}$, tum cum cosinus $\frac{b}{\sqrt{(b^2+a^2)}}$ (§. CXLVIII. 1.) sit negatiuus, sinu $\frac{a}{\sqrt{(b^2+a^2)}}$ eodem manente, loco ψ accipi debet $\pi - A.\,tang.\frac{a}{b} = \pi - \psi$; hincque $\psi + \pi$, $\psi + 3\pi$, $\psi + 5\pi$ .... abeunt in $2\pi - \psi$, $4\pi - \psi$, $6\pi - \psi$ ...; et $\psi - \pi$, $\psi - 3\pi$, $\psi - 5\pi$, ... in $-\psi$, $-\psi - 2\pi$, $-\psi - 4\pi$ ...

 Vnde

Vnde manifeſtum eſt, quomodo pro hoc caſu expreſſio praecedens (§. CXLVII. 2.) ſit mutanda. Transformationem huius expreſſionis tam pro affirmatiuo quam pro negatiuo valore τῦ b vlterius proſequi hoc loco ſuperfluum videtur.

2) Poſito $a = 0$, angulus ψ euaneſcit; et Lemma ſuppeditat formulas vſitatas pro reſolutione τῦ $x^m \pm b$ in factores ſimplices vel quadraticos.

Problema XIX.

§. CXLIX. Summare ſeriem infinitam:

$$\text{A. tang.}\,\frac{a}{1^{2n}+b}+\text{A. t.}\,\frac{a}{2^{2n}+b}+\text{A. t.}\,\frac{a}{3^{2n}+b}+\ldots+\text{A. t.}\,\frac{a}{x^{2n}+b}+\text{etc.}$$

Solutio.

1) Ope reſolutionis §. CXXXVIII. terminus generalis $\text{A. t.}\,\frac{a}{x^{2n}+b}$ prodit =

$$\text{A. tang.}\left\{\frac{f\,\text{ſin.}\left(\frac{\pi-\psi}{n}\right)}{x^2-f\,\text{coſ.}\left(\frac{\pi-\psi}{n}\right)}\right\}-\text{A. t.}\left\{\frac{f\,\text{ſin.}\left(\frac{\pi+\psi}{n}\right)}{x^2-f\,\text{coſ.}\left(\frac{\pi+\psi}{n}\right)}\right\}$$

$$+\text{A. tang.}\left\{\frac{f\,\text{ſin.}\left(\frac{3\pi-\psi}{n}\right)}{x^2-f\,\text{coſ.}\left(\frac{3\pi-\psi}{n}\right)}\right\}-\text{A. t.}\left\{\frac{f\,\text{ſin.}\left(\frac{3\pi+\psi}{n}\right)}{x^2-f\,\text{coſ.}\left(\frac{3\pi+\psi}{n}\right)}\right\}$$

$$+\text{A. tang.}\left\{\frac{f\,\text{ſin.}\left(\frac{5\pi-\psi}{n}\right)}{x^2-f\,\text{coſ.}\left(\frac{5\pi-\psi}{n}\right)}\right\}-\text{A. t.}\left\{\frac{f\,\text{ſin.}\left(\frac{5\pi+\psi}{n}\right)}{x^2-f\,\text{coſ.}\left(\frac{5\pi+\psi}{n}\right)}\right\}$$

$$+\text{etc.}\qquad\qquad -\text{etc.}$$

qui termini continuantur, donec in ſerie 1, 3, 5 . . . perueniatur ad numerum imparem, qui ſit = n, vel proxime minor pari n; dummodo illo caſu loco terminorum binorum vltimorum aequalium vnicus tantum ſumatur (§. CXLVII.).

2) Adhibito probl. VIII. Coroll. 3. (§. XCVI.), ex quauis parte termini generalis

$$\text{A. tang.}\left\{\frac{f\,\text{ſin.}\left(\frac{(2k+1)\pi\pm\psi}{n}\right)}{x^2-f\,\text{coſ.}\left(\frac{(2k+1)\pi\pm\psi}{n}\right)}\right\}$$ oritur pars ſummae ſeu ſumma partialis =

$\alpha\pi$ —

$\alpha\pi - A.\,t.\,\frac{\alpha}{\beta} + A.\,t.\left(\frac{\text{fin.}\,2\alpha\pi}{e^{2\beta\pi} - \text{cof.}\,2\alpha\pi}\right)$, exiftente $\alpha = \sqrt{\left(\frac{\sqrt{(b^2+a^2)} - b}{2}\right)}$

$= \sqrt{\left\{\frac{f + f\,\text{cof.}\left(\frac{(2k+1)\pi \pm \psi}{n}\right)}{2}\right\}} = f^{\frac{1}{2}}\,\text{cof.}\left(\frac{(2k+1)\pi \pm \psi}{2n}\right)$; $\beta = \sqrt{\left(\frac{\sqrt{(b^2+a^2)}+b}{2}\right)}$

$= f^{\frac{1}{2}}\,\text{fin.}\left(\frac{(2k+1)\pi \pm \psi}{2n}\right)$; vnde eft $A.\,t.\,\frac{\alpha}{\beta} = \frac{\pi}{2} - \left(\frac{(2k+1)\pi \pm \psi}{2n}\right)$.

3) Iam duo cafus funt difcernendi. Sit nimirum primo n par $= 2r$, tum pofito $f^{\frac{1}{2}} = c$, prodit fumma feriei problematis $=$

$$\pi c.\,\text{cof.}\left(\frac{\pi-\psi}{2n}\right) - \left(\frac{\pi}{2} - \left(\frac{\pi-\psi}{2n}\right)\right) + A.\,t.\left\{\frac{\text{fin.}\left(2c\pi\,\text{cof.}\left(\frac{\pi-\psi}{2n}\right)\right)}{e^{2c\pi\,\text{fin.}\left(\frac{\pi-\psi}{2n}\right)} - \text{cof.}\left(2c\pi\,\text{cof.}\left(\frac{\pi-\psi}{2n}\right)\right)}\right\}$$

$$-\pi c.\,\text{cof.}\left(\frac{\pi+\psi}{2n}\right) + \left(\frac{\pi}{2} - \left(\frac{\pi+\psi}{2n}\right)\right) - A.\,t.\left\{\frac{\text{fin.}\left(2c\pi\,\text{cof.}\left(\frac{\pi+\psi}{2n}\right)\right)}{e^{2c\pi\,\text{fin.}\left(\frac{\pi+\psi}{2n}\right)} - \text{cof.}\left(2c\pi\,\text{cof.}\left(\frac{\pi+\psi}{2n}\right)\right)}\right\}$$

$$+\pi c\;\text{cof.}\left(\frac{3\pi-\psi}{2n}\right) - \left(\frac{\pi}{2} - \left(\frac{3\pi-\psi}{2n}\right)\right) + A.\,t.\left\{\frac{\text{fin.}\left(2c\pi\,\text{cof.}\left(\frac{3\pi-\psi}{2n}\right)\right)}{e^{2c\pi\,\text{fin.}\left(\frac{3\pi-\psi}{2n}\right)} - \text{cof.}\left(2c\pi\,\text{cof.}\left(\frac{3\pi-\psi}{2n}\right)\right)}\right\}$$

$$-\pi c.\,\text{cof.}\left(\frac{3\pi+\psi}{2n}\right) + \left(\frac{\pi}{2} - \left(\frac{3\pi+\psi}{2n}\right)\right) - A.\,t.\left\{\frac{\text{fin.}\left(2c\pi\,\text{cof.}\left(\frac{3\pi+\psi}{2n}\right)\right)}{e^{2c\pi\,\text{fin.}\left(\frac{3\pi+\psi}{2n}\right)} - \text{cof.}\left(2c\pi\,\text{cof.}\left(\frac{3\pi+\psi}{2n}\right)\right)}\right\}$$

etc. etc. etc.

Quae expreffio continuatur, vsque dum $(n-1)\pi = (2r-1)\pi$ feu r^{tum} multiplum τȣ π occurrat. Quo ea ad formam fimpliciorem redigatur, confiderandae funt fummarum partialium (2) partes primae et fecundae. Ob $\text{cof.}\left(\frac{(2k+1)\pi-\psi}{2n}\right) - \text{cof.}\left(\frac{(2k+1)\pi+\psi}{2n}\right) = 2\,\text{fin.}\,\frac{\psi}{2n}\,.\,\text{fin.}\,\frac{(2k+1)\pi}{2n}$, partes primae

mae $(\alpha\pi)$ fimul fumtae praebent: $2c\pi\,\text{fin.}\frac{\psi}{2n}\left(\text{fin.}\frac{\pi}{2n}+\text{fin.}\frac{3\pi}{2n}+\ldots+\text{fin.}\frac{(2r-1)\pi}{2n}\right)$

$$=2c\pi\,\text{fin}\frac{\psi}{2n}\cdot\frac{\left(\text{fin.}\frac{\pi r}{2n}\right)^2}{\text{fin.}\frac{\pi}{2n}}\ (*)=2c\pi\ \text{fin.}\frac{\psi}{2n}\cdot\left\{\frac{1-\text{cof.}\frac{\pi r}{n}}{2\ \text{fin.}\frac{\pi}{2n}}\right\}=c\pi.\frac{\text{fin.}\frac{\psi}{2n}}{\text{fin.}\frac{\pi}{2n}}.$$ Ex partibus fecundis fimul fumtis oritur $-\frac{r\psi}{n}=-\frac{\psi}{2}$. Hinc in expreffione fummae, praeter ea membra, quae A. tang. prae fe ferunt, reliqua omnia rite coniuncta contrahuntur in formulam $\frac{c\pi\,\text{fin.}\frac{\psi}{2n}}{\text{fin.}\frac{\pi}{2n}}-\frac{\psi}{2}$.

4) Supponatur fecundo n numerus impar $=2r-1$, tum expreffio fummae modo inuenta (3) aeque ac pro pari n adhibenda eft, ita tamen vt ea continuetur, donec perueniatur ad numerum imparem $r-1^{\text{tum}}$ feu $n-2=2r-3$, praetereaque addatur pars fummae ex parte termini generalis A. t. $\left\{\frac{f\ \text{fin.}\frac{\psi}{n}}{x^2+k\ \text{cof.}\frac{\psi}{n}}\right\}$ oriunda, quae pars feu fumma partialis vltima eft $=\pi c\ \text{fin.}\frac{\psi}{2n}-\frac{\psi}{2n}$

$$+\text{A. tang.}\left\{\frac{\text{fin.}\left(2\pi c\ \text{fin.}\frac{\psi}{2n}\right)}{e^{2\pi c\ \text{cof.}\frac{\psi}{2n}}-\text{cof.}\left(2\pi c\ \text{fin.}\frac{\psi}{2n}\right)}\right\}.$$ Iam fummarum partialium partes primae praebent:

$$2c\pi\,\text{fin.}\frac{\psi}{2n}\left(\text{fin.}\frac{\pi}{2n}+\text{fin.}\frac{3\pi}{2n}\ldots+\text{fin.}\frac{(2r-3)\pi}{2n}\right)+\pi c\ \text{fin.}\frac{\psi}{2n}$$

$=2c\pi$

(*) Haec fummatio finuum ex ferie in EVLERI Introd. T. I. Cap. XIV. §. 259. pag. 218. fummata fequitur. Eadem deduci poteft ex theoremate, quod extat demonftratum apud KAESTNERVM (*Geometrifche Abhandlungen 1fte Samml. Götting. 1791.* pag. 402. nr. 31.) Summatio ita enunciari poteft: Si femicircumferentia diuidatur in partes $=\beta$, numero pari, et finus r multiplorum Arcus β fecundum numeros impares 1, 3, 5 . . addantur, erit fumma = finui verfo arcus $2r\beta$ diuifo per 2 fin. β.

$$= 2c\pi \text{ fin. } \frac{\psi}{2n} \cdot \left\{ \frac{1 - \text{cof.} \frac{(r-1)\pi}{n}}{2 \text{ fin. } \frac{\pi}{2n}} \right\} + c\pi \text{ fin. } \frac{\psi}{2n}$$

$$= c\pi \text{ fin. } \frac{\psi}{2n} \left\{ \frac{1 - \text{cof. } \frac{(n-1)\pi}{2n} + 1}{\text{fin. } \frac{\pi}{2n}} \right\} = c\pi \cdot \frac{\text{fin. } \frac{\psi}{2n}}{\text{fin. } \frac{\pi}{2n}}.$$ Partes fecundae fimul fumtae conficiunt $-\frac{\psi}{n}(r-1) - \frac{\psi}{2n} = -\frac{\psi}{2}$. Vtrumque cum prius inuento pro n pari (3) confentit.

5) Ex his ita demonftratis, fiue n fuerit par, fiue impar, haec obtinetur

Summatio.

Summa feriei infinitae:

$$\text{A. t. } \frac{a}{1^{2n}+b} + \text{A. t. } \frac{a}{2^{2n}+b} + \text{A. t. } \frac{a}{3^{2n}+b} + \ldots + \text{A. t. } \frac{a}{x^{2n}+b} + \text{etc.} = S$$

hac formula exprimitur, fumto angulo $\psi = \text{A. t. } \frac{a}{b}$, $c = (a^2+b^2)^{\frac{1}{4n}}$:

$$S = \frac{c\pi \text{ fin. } \frac{\psi}{2n}}{\text{fin. } \frac{\pi}{2n}} - \frac{\psi}{2}$$

$$+ \text{A. t.} \left\{ \frac{\text{fin.}\left(2c\pi \text{ cof. } \left(\frac{\pi-\psi}{2n}\right)\right)}{e^{2c\pi \text{ fin } \left(\frac{\pi-\psi}{2n}\right)} - \text{cof.}\left(2c\pi \text{ cof.}\left(\frac{\pi-\psi}{2n}\right)\right)} \right\} - \text{A. t.} \left\{ \frac{\text{fin.}\left(2c\pi \text{ cof. } \left(\frac{\pi+\psi}{2n}\right)\right)}{e^{2c\pi \text{ fin. } \left(\frac{\pi+\psi}{2n}\right)} - \text{cof.}\left(2c\pi \text{ cof.}\left(\frac{\pi+\psi}{2n}\right)\right)} \right\}$$

$$+ \text{A. t.} \left\{ \frac{\text{fin.}\left(2c\pi \text{ cof. } \left(\frac{3\pi-\psi}{2n}\right)\right)}{e^{2c\pi \text{ fin. } \left(\frac{3\pi-\psi}{2n}\right)} - \text{cof.}\left(2c\pi \text{ cof.}\left(\frac{3\pi-\psi}{2n}\right)\right)} \right\} - \text{A. t.} \left\{ \frac{\text{fin.}\left(2c\pi \text{ cof. } \left(\frac{3\pi+\psi}{2n}\right)\right)}{e^{2c\pi \text{ fin. } \left(\frac{3\pi+\psi}{2n}\right)} - \text{cof.}\left(2c\pi \text{ cof.}\left(\frac{3\pi+\psi}{2n}\right)\right)} \right\}$$

$$+ \text{etc.} \qquad - \text{etc.}$$

+ A. t.

$$+\text{A. t.}\left\{\frac{\text{fin.}\left(2c\pi\ \text{cof.}\left(\frac{(2k+1)\pi-\psi}{2n}\right)\right)}{e^{2c\pi\,\text{fin.}\left(\frac{(2k+1)\pi-\psi}{2n}\right)}-\text{cof.}\left(2c\pi\,\text{cof.}\left(\frac{(2k+1)\pi-\psi}{2n}\right)\right)}\right\}$$

$$-\text{A. t.}\left\{\frac{\text{fin.}\left(2c\pi\ \text{cof.}\left(\frac{(2k+1)\pi+\psi}{2n}\right)\right)}{e^{2c\pi\,\text{fin.}\left(\frac{(2k+1)\pi+\psi}{2n}\right)}-\text{cof.}\left(2c\pi\,\text{cof.}\left(\frac{(2k+1)\pi+\psi}{2n}\right)\right)}\right\}$$

$+$ etc. $\qquad\qquad$ $-$ etc.

vbi pro $2k+1$ accipiuntur omnes numeri impares ab 1 vsque ad numerum imparem vel proxime minorem numero n vel huic aequalem: posteriori autem casu vltimae bigae Arcuum prior solus accipitur,

$$=\text{A. t.}\left\{\frac{\text{fin.}\left(2c\pi\ \text{fin.}\ \frac{\psi}{2n}\right)}{e^{2c\pi\ \text{cof.}\ \frac{\psi}{2n}}-\text{cof.}\left(2c\pi\ \text{fin.}\ \frac{\psi}{2n}\right)}\right\},$$ quippe cui alter cum figno $-$ aequalis eft,

ob $\text{cof.}\left(\frac{n\pi-\psi}{2n}\right)=-\text{cof.}\left(\frac{n\pi+\psi}{2n}\right)=\text{fin.}\frac{\psi}{2n}$, et $\text{fin.}\left(\frac{n\pi-\psi}{2n}\right)=\text{fin.}\left(\frac{n\pi+\psi}{2n}\right)$

$=\text{cof.}\ \frac{\psi}{2n}$. Vtroque igitur cafu expreffio fummae continuatur, vsque dum Arcus tangentium lege praefcripta procedentium numero n adfint.

Corollarium 1.

§. CL. Formulae fummatoriae praecedentis termini fubtractiui aliter exprimi poffunt.

1) Confiderentur pro n pari fequentes binae feries Arcuum:

a) $\frac{\pi+\psi}{2n}$; $\frac{3\pi+\psi}{2n}$; $\frac{5\pi+\psi}{2n}$; ... $\frac{(n-1)\pi+\psi}{2n}$

b) $\frac{(n-1)\pi-\psi}{2n}$; $\frac{(n-3)\pi-\psi}{2n}$; $\frac{(n-5)\pi-\psi}{2n}$; ... $\frac{\pi-\psi}{2n}$

earum termini initiales et quicunque fibi inuicem refpondentes fimul fumti quadranti $\frac{\pi}{2}$ aequantur, hinc cofinus et finus vnius Arcus aequales funt finui ac cofinui alterius. Iam concipiantur Arcus fubtractiui in expreffione fummae (§ CXLIX.) inuerfo ordine fcripti, vltimus primo loco, penultimus fecundo, et fic porro: tum fummatio fub alia forma exhibetur, quam ex praecedente ftatim deriuare licet. dum in hac

loco:

loco: cof. $\left(\frac{\pi+\psi}{2n}\right)$; cof. $\left(\frac{3\pi+\psi}{2n}\right)$; cof. $\left(\frac{5\pi+\psi}{2n}\right)$; ...

et fin. $\left(\frac{\pi+\psi}{2n}\right)$; fin. $\left(\frac{3\pi+\psi}{2n}\right)$; fin. $\left(\frac{5\pi+\psi}{2n}\right)$; ...

ponantur fuo ordine:

fin. $\left(\frac{\pi-\psi}{2n}\right)$; fin. $\left(\frac{3\pi-\psi}{2n}\right)$; fin. $\left(\frac{5\pi-\psi}{2n}\right)$; ...

et cof. $\left(\frac{\pi-\psi}{2n}\right)$; cof. $\left(\frac{3\pi-\psi}{2n}\right)$; cof. $\left(\frac{5\pi-\psi}{2n}\right)$; ...

2) Simili ratione pro impari n Arcus:

$$\frac{\pi+\psi}{2n}; \quad \frac{3\pi+\psi}{2n}; \ldots \frac{(n-4)\pi+\psi}{2n}; \quad \frac{(n-2)\pi+\psi}{2n}$$

cum Arcubus:

$$\frac{(n-1)\pi-\psi}{2n}; \quad \frac{(n-3)\pi-\psi}{2n}; \ldots \frac{4\pi-\psi}{2n}; \quad \frac{2\pi-\psi}{2n}$$

fcilicet primus cum primo, fecundus cum fecundo, et fic porro, quadrantem aequant. Hinc iterum expreffio fummae (§. CXLIX.), Arcuum fubtractiuorum ordinem inuertendo, ita transformari poteft, vt

loco: cof. fin. $\left(\frac{\pi+\psi}{2n}\right)$; cof. fin. $\left(\frac{3\pi+\psi}{2n}\right)$; cof. fin. $\left(\frac{5\pi+\psi}{2n}\right)$; ...

ponatur: fin. cof. $\left(\frac{2\pi-\psi}{2n}\right)$; fin. cof. $\left(\frac{4\pi-\psi}{2n}\right)$; fin. cof. $\left(\frac{6\pi-\psi}{2n}\right)$; ...

Expreffiones fummae hac ratione tam pro n pari quam impari transformatas in extenfo apponere fuperfluum nimisque longum eft.

Corollarium 2.

§. CLI. Si b negatiuum valorem habeat, vel fummanda fit feries

$$A.\ t. \frac{a}{1^{2n}-b} + A\ t. \frac{a}{2^{2n}-b} + A.\ t. \frac{a}{3^{2n}-b} + \ldots + A.\ t. \frac{a}{x^{2n}-b} + \text{etc.}$$

tum loco A. tang. $\frac{a}{-b}$ fumi debet π — A. t. $\frac{a}{b}$, feu in expreffione fummae §. CXLIX. inuenta vel ex §. CL. transformata pro Arcu ψ ponendus vbique eft $\pi-\psi$.

Corollarium 3.

§. CLII. Sit $b=0$, erit $\psi=\frac{\pi}{2}$, $c=a^{\frac{1}{2n}}$. Inde haec obtinetur Summatio:

$$A.\ tang. \frac{c^{2n}}{1^{2n}} + A.\ t. \frac{c^{2n}}{2^{2n}} + A.\ t. \frac{c^{2n}}{3^{2n}} + \ldots + A.\ t. \frac{c^{2n}}{x^{2n}} + \text{etc.} = -\frac{\pi}{4} + \frac{c\pi}{2\ \text{cof.}\ \frac{\pi}{4n}}$$

+ A t.

$$+\text{A.t.}\left\{\frac{\text{ſin.}\left(2c\pi\ \text{coſ.}\ \frac{\pi}{4n}\right)}{e^{2c\pi\ \text{ſin.}\ \frac{\pi}{4n}}-\text{coſ.}\left(2c\pi\ \text{coſ.}\ \frac{\pi}{4n}\right)}\right\}-\text{A.t.}\left\{\frac{\text{ſin.}\left(2c\pi\ \text{coſ.}\ \frac{3\pi}{4n}\right)}{e^{2c\pi\ \text{ſin.}\ \frac{3\pi}{4n}}-\text{coſ.}\left(2c\pi\ \text{coſ.}\ \frac{3\pi}{4n}\right)}\right\}$$

$$+\text{A.t.}\left\{\frac{\text{ſin.}\left(2c\pi\ \text{coſ.}\ \frac{5\pi}{4n}\right)}{e^{2c\pi\ \text{ſin.}\ \frac{5\pi}{4n}}-\text{coſ.}\left(2c\pi\ \text{coſ.}\ \frac{5\pi}{4n}\right)}\right\}-\text{A.t.}\left\{\frac{\text{ſin.}\left(2c\pi\ \text{coſ.}\ \frac{7\pi}{4n}\right)}{e^{2c\pi\ \text{ſin.}\ \frac{7\pi}{4n}}-\text{coſ.}\left(2c\pi\ \text{coſ.}\ \frac{7\pi}{4n}\right)}\right\}$$

$$+\text{etc.} \qquad\qquad -\text{etc.}$$

$$+\text{A.t.}\left\{\frac{\text{ſin.}\left(2c\pi\ \text{coſ.}\ \frac{(4l-3)\pi}{4n}\right)}{e^{2c\pi\ \text{ſin.}\ \frac{(4l-3)\pi}{4n}}-\text{coſ.}\left(2c\pi\ \text{coſ.}\ \frac{(4l-3)\pi}{4n}\right)}\right\}-\text{A.t.}\left\{\frac{\text{ſin.}\left(2c\pi\ \text{coſ.}\ \frac{(4l-1)\pi}{4n}\right)}{e^{2c\pi\ \text{ſin.}\ \frac{(4l-1)\pi}{4n}}-\text{coſ.}\left(2c\pi\ \text{coſ.}\ \frac{(4l-1)\pi}{4n}\right)}\right\}$$

$$+\text{etc.} \qquad\qquad -\text{etc.}$$

qui arcus continuantur, donec numerum n compleant, ſiue pro n pari fiat $l=\frac{n}{2}$, pro n impari $l=\frac{n+1}{2}$; at poſteriori caſu vltimae bigae Arcuum primus tantum ſumitur $=$

$$\text{A. tang.}\left\{\frac{\text{ſin.}\left(2c\pi\ \text{ſin.}\ \frac{\pi}{4n}\right)}{e^{2c\pi\ \text{coſ.}\ \frac{\pi}{4n}}-\text{coſ.}\left(2c\pi\ \text{ſin.}\ \frac{\pi}{4n}\right)}\right\}$$

Corollarium 4.

§. CLIII. Haec formula ſimili ratione ac generalior §. CL. transformari poteſt; Arcus nimirum $\frac{(2n-1)\pi}{4n}$; $\frac{(2n-3)\pi}{4n}$; $\frac{(2n-5)\pi}{4n}$; . . . ſunt complementa ad quadrantem Arcuum: $\frac{\pi}{4n}$; $\frac{3\pi}{4n}$; $\frac{5\pi}{4n}$; etc. Hinc ſi expreſſionis ſummae (§. CLII.) vltimus terminus adiungitur primo, penultimus ſecundo, et ſic porro: tum pro ſumma haec quoque prodit formula: $-\frac{\pi}{4}+\frac{c\pi}{2\ \text{coſ.}\ \frac{\pi}{4n}}$

+A. t.

$$+\text{A.t.}\left\{\frac{\sin.\left(2c\pi\cos.\frac{\pi}{4n}\right)}{e^{2c\pi\sin.\frac{\pi}{4n}}-\cos.\left(2c\pi\cos.\frac{\pi}{4n}\right)}\right\}\mp\text{A.t.}\left\{\frac{\sin\left(2c\pi\sin.\frac{\pi}{4n}\right)}{e^{2c\pi\sin.\frac{\pi}{4n}}-\cos.\left(2c\pi\sin.\frac{\pi}{4n}\right)}\right\}$$

$$-\text{A.t.}\left\{\frac{\sin.\left(2c\pi\cos.\frac{3\pi}{4n}\right)}{e^{2c\pi\sin.\frac{3\pi}{4n}}-\cos.\left(2c\pi\cos.\frac{3\pi}{4n}\right)}\right\}\pm\text{A.t.}\left\{\frac{\sin.\left(2c\pi\sin.\frac{3\pi}{4n}\right)}{e^{2c\pi\cos.\frac{3\pi}{4n}}-\cos.\left(2c\pi\sin.\frac{3\pi}{4n}\right)}\right\}$$

$$+\text{A.t.}\left\{\frac{\sin.\left(2c\pi\cos.\frac{5\pi}{4n}\right)}{e^{2c\pi\sin.\frac{5\pi}{4n}}-\cos.\left(2c\pi\cos.\frac{5\pi}{4n}\right)}\right\}\mp\text{A t.}\left\{\frac{\sin.\left(2c\pi\sin.\frac{5\pi}{4n}\right)}{e^{2c\pi\cos.\frac{5\pi}{4n}}-\cos.\left(2c\pi\sin.\frac{5\pi}{4n}\right)}\right\}$$

$$-\text{etc.}\qquad\qquad\pm\text{etc.}$$

vbi ſigna ſuperiora valent pro pari n, inferiora pro impari. Arcus continuantur, vsque dum numerus eorum ad n compleatur.

Corollarium 5.

§. CLIV. 1) Pro $n=1$ et $=2$ prodeunt ſummationes ſupra (§. XCIII. CXVIII.) inuentae. Pro $n=3$ eſt:

$$\text{A. tang.}\frac{a}{1^6+b}+\text{A.t.}\frac{a}{2^6+b}+\text{A.t.}\frac{a}{3^6+b}+\text{etc.}+\text{A.t.}\frac{a}{x^6+b}+\ldots$$

$$=-\frac{\psi}{2}+2c\pi\sin.\frac{\psi}{6}+\text{A.t.}\left\{\frac{\sin.\left(2c\pi\cos.\left(\frac{\pi-\psi}{6}\right)\right)}{e^{2c\pi\sin.\left(\frac{\pi-\psi}{6}\right)}-\cos.\left(2c\pi\cos.\left(\frac{\pi-\psi}{6}\right)\right)}\right\}$$

$$-\text{A.t.}\left\{\frac{\sin.\left(2c\pi\sin.\left(\frac{2\pi-\psi}{6}\right)\right)}{e^{2c\pi\cos.\left(\frac{2\pi-\psi}{6}\right)}-\cos.\left(2c\pi\sin.\left(\frac{2\pi-\psi}{6}\right)\right)}\right\}+\text{A.t.}\left\{\frac{\sin.\left(2c\pi\sin.\frac{\psi}{6}\right)}{e^{2c\pi\cos.\frac{\psi}{6}}-\cos.\left(2c\pi\sin.\frac{\psi}{6}\right)}\right\}$$

ex. ſtente $\psi=\text{A. t.}\frac{a}{b}$, et $c=(a^2+b^2)^{\frac{1}{12}}$.

 2) Sit

2) Sit $b=0$, erit $A.\,t.\,\frac{c^6}{1^6}+A.\,t.\,\frac{c^6}{2^6}+A.\,t.\,\frac{c^6}{3^6}+\ldots+A.\,t.\,\frac{c^6}{x^6}+$ etc.

$$=-\frac{\pi}{4}+\frac{c\pi}{2\,\text{cof.}\,\frac{\pi}{12}}+A.\,t.\left\{\frac{\text{fin.}\left(2c\pi\,\text{cof.}\,\frac{\pi}{12}\right)}{e^{2c\pi\,\text{fin.}\,\frac{\pi}{12}}-\text{cof.}\left(2c\pi\,\text{cof.}\,\frac{\pi}{12}\right)}\right\}$$

$$+A.\,t.\left\{\frac{\text{fin.}\left(2c\pi\,\text{fin.}\,\frac{\pi}{12}\right)}{e^{2c\pi\,\text{cof.}\,\frac{\pi}{12}}-\text{cof.}\left(2c\pi\,\text{fin.}\,\frac{\pi}{12}\right)}\right\}-A.\,t.\left\{\frac{\text{fin.}\left(2c\pi\,\text{cof.}\,\frac{\pi}{4}\right)}{e^{2c\pi\,\text{fin.}\,\frac{\pi}{4}}-\text{cof.}\left(2c\pi\,\text{cof.}\,\frac{\pi}{4}\right)}\right\}.$$

Eſt autem fin. $\frac{\pi}{4}=\text{cof.}\,\frac{\pi}{4}=\frac{1}{\surd 2}$; fin. $\frac{\pi}{12}=\text{fin.}\,\frac{30^0}{2}=\surd\left(\frac{1-\text{cof.}\,30^0}{2}\right)=\surd\left(\frac{1-\frac{1}{2}\surd 3}{2}\right)=\frac{1}{2}\surd(2-\surd 3)=\frac{1}{2\surd 2}(\surd 3-1)$; et cof. $\frac{\pi}{12}=\frac{1}{2\surd 2}(\surd 3+1)$.

3) Pro $n=4$, habetur: $A.\,t.\,\frac{a}{1^8+b}+A.\,t.\,\frac{a}{2^8+b}+A.\,t.\,\frac{a}{3^8+b}+\ldots+A.\,t.\,\frac{a}{x^8+b}+$ etc.

$$=-\frac{\psi}{2}+\frac{c\pi\,\text{fin.}\,\frac{\psi}{8}}{\text{fin.}\,\frac{\pi}{8}}+A.\,t.\left\{\frac{\text{fin.}\left(2c\pi\,\text{cof.}\left(\frac{\pi-\psi}{8}\right)\right)}{e^{2c\pi\,\text{fin.}\left(\frac{\pi-\psi}{8}\right)}-\text{cof.}\left(2c\pi\,\text{cof.}\left(\frac{\pi-\psi}{8}\right)\right)}\right\}$$

$$-A.\,t.\left\{\frac{\text{fin.}\left(2c\pi\,\text{fin.}\left(\frac{\pi-\psi}{8}\right)\right)}{e^{2c\pi\,\text{cof.}\left(\frac{\pi-\psi}{8}\right)}-\text{cof.}\left(2c\pi\,\text{fin.}\left(\frac{\pi-\psi}{8}\right)\right)}\right\}+A.\,t.\left\{\frac{\text{fin.}\left(2c\pi\,\text{cof.}\left(\frac{3\pi-\psi}{8}\right)\right)}{e^{2c\pi\,\text{fin.}\left(\frac{3\pi-1\psi}{8}\right)}-\text{cof.}\left(2c\pi\,\text{cof.}\left(\frac{3\pi-\psi}{8}\right)\right)}\right\}$$

$$-A.\,t.\left\{\frac{\text{fin.}\left(2c\pi\,\text{fin.}\left(\frac{3\pi-\psi}{8}\right)\right)}{e^{2c\pi\,\text{cof.}\left(\frac{3\pi-\psi}{8}\right)}-\text{cof.}\left(2c\pi\,\text{fin.}\left(\frac{3\pi-\psi}{8}\right)\right)}\right\},$$ poſito $c=(a^2+b^2)^{\frac{1}{16}}$, $\psi=A.\,t.\,\frac{a}{b}$.

4) Inde eſt pro $b=0$,

$A.\,\text{tang.}\,\frac{c^8}{1^8}+A.\,t.\,\frac{c^8}{2^8}+A.\,t.\,\frac{c^8}{3^8}+\ldots+A.\,t.\,\frac{c^8}{x^8}+$ etc.

$$= -\frac{\pi}{4} + \frac{c\pi}{2\cos.\frac{\pi}{16}} + A.t.\left\{\frac{\sin.\left(2c\pi\cos.\frac{\pi}{16}\right)}{e^{2c\pi\sin.\frac{\pi}{16}} - \cos.\left(2c\pi\cos.\frac{\pi}{16}\right)}\right\}$$

$$-A.t.\left\{\frac{\sin.\left(2c\pi\sin.\frac{\pi}{16}\right)}{e^{2c\pi\cos.\frac{\pi}{16}} - \cos.\left(2c\pi\sin.\frac{\pi}{16}\right)}\right\} - A.t.\left\{\frac{\sin.\left(2c\pi\cos.\frac{3\pi}{16}\right)}{e^{2c\pi\sin.\frac{3\pi}{16}} - \cos.\left(2c\pi\cos.\frac{3\pi}{16}\right)}\right\}$$

$$+A.t.\left\{\frac{\sin.\left(2c\pi\sin.\frac{3\pi}{16}\right)}{e^{2c\pi\cos.\frac{3\pi}{16}} - \cos.\left(2c\pi\sin.\frac{3\pi}{16}\right)}\right\}.$$ Habetur vero $\sin.\frac{\pi}{16} = \sin.\frac{1}{2}\left(\frac{\pi}{8}\right) =$

$$\sqrt{\left(\frac{8^{\frac{1}{4}} - (1+\sqrt{2})^{\frac{1}{2}}}{2.8^{\frac{1}{4}}}\right)};\ \cos.\frac{\pi}{16} = \sqrt{\left(\frac{8^{\frac{1}{4}} + (1+\sqrt{2})^{\frac{1}{2}}}{2.8^{\frac{1}{4}}}\right)};\ \sin.\frac{3\pi}{16} = \sin.\left(\frac{\pi}{4} - \frac{\pi}{16}\right)$$

$$= \frac{\cos.\frac{\pi}{16} - \sin.\frac{\pi}{16}}{\sqrt{2}};\quad \cos.\frac{3\pi}{16} = \frac{\cos.\frac{\pi}{16} + \sin.\frac{\pi}{16}}{\sqrt{2}}.$$

Corollarium 6.

§. CLV. Summatio seriei:

$$A.t.\frac{a}{1^{2n}+b} + A.t.\frac{a}{3^{2n}+b} + A.t.\frac{a}{5^{2n}+b} + \ldots + A.t.\frac{a}{(2x-1)^{2n}+b} + \text{etc.}$$

ex prioribus deriuari potest. Est nimirum $S.A.t.\frac{a}{(2x-1)^2+b} = \frac{\alpha\pi}{2}$

$-A.t.\left(\frac{\sin.\alpha\pi}{e^{\beta\pi}+\cos.\alpha\pi}\right)$. Hinc in expressionibus pro $S.A.t.\frac{a}{x^{2n}+b}$ inuentis (§. CXLIX. CL.) omittenda est pars $-\frac{\psi}{2}$, pro f supponendum $\frac{f}{2}$, et Arcus omnes cum signo opposito accipiendi sunt, ac denique in denominatoribus tangentium signum $-$ cum $+$ permutandum. Quibus ita obseruatis prodit

$$\text{t.}\frac{a}{1^{2n}+b} + A.t.\frac{a}{3^{2n}+b} + A.t.\frac{a}{5^{2n}+b} + \ldots + A.t.\frac{a}{(2x-1)^{2n}+b} + \ldots$$

$$= \frac{c\pi \sin. \frac{\psi}{2n}}{2 \sin. \frac{\pi}{2n}}$$

$$- A.t. \left\{ \frac{\sin.\left(c\pi \cos.\left(\frac{\pi-\psi}{2n}\right)\right)}{e^{c\pi \sin\left(\frac{\pi-\psi}{2n}\right)} + \cos.\left(c\pi \cos.\left(\frac{\pi-\psi}{2n}\right)\right)} \right\} + A.t. \left\{ \frac{\sin.\left(c\pi \cos.\left(\frac{\pi+\psi}{2n}\right)\right)}{e^{c\pi \sin.\left(\frac{\pi+\psi}{2n}\right)} + \cos.\left(c\pi \cos.\left(\frac{\pi+\psi}{2n}\right)\right)} \right\}$$

$$- A.t. \left\{ \frac{\sin.\left(c\pi \cos.\left(\frac{3\pi-\psi}{2n}\right)\right)}{e^{c\pi \sin\left(\frac{3\pi-\psi}{2n}\right)} + \cos.\left(c\pi \cos.\left(\frac{3\pi-\psi}{2n}\right)\right)} \right\} + A.t. \left\{ \frac{\sin.\left(c\pi \cos.\left(\frac{3\pi+\psi}{2n}\right)\right)}{e^{c\pi \sin.\left(\frac{3\pi+\psi}{2n}\right)} + \cos.\left(c\pi \cos.\left(\frac{3\pi+\psi}{2n}\right)\right)} \right\}$$

$$- \text{etc.} \qquad + \text{etc.}$$

cuius expressionis termini eodem modo continuantur, ac §. CXLIX. i. e. vsque dum n Arcus habeantur. Quomodo haec expressio transformari queat, ex §. CL. colligitur.

Corollarium 7.

§. CLVI. Eadem ratione summare licet seriem cum *signis alternantibus:*

$$A. \text{tang.} \frac{a}{1+b} - A.t. \frac{a}{2^{2n}+b} + A.t. \frac{a}{3^{2n}+b} - A.t. \frac{a}{4^{2n}+b} + \text{etc.}$$

Cum nimirum sit (§. XCIX.) $S \pm A.t. \frac{a}{x^2+b} = \frac{1}{2} A.t. \frac{a}{b} - A.t. \left\{ \frac{2e^{\beta\pi} \sin. \alpha\pi}{e^{2\beta\pi} - 1} \right\}$, iam ex summatione §. CXL. colligi potest, esse summam illius seriei seu $S \pm A.t. \frac{a}{x^{2n}+b} = \frac{\psi}{2}$

$$- A.t. \left\{ \frac{2e^{c\pi \sin.\left(\frac{\pi-\psi}{2n}\right)} . \sin.\left(c\pi \cos.\left(\frac{\pi-\psi}{2n}\right)\right)}{e^{2c\pi \sin.\left(\frac{\pi-\psi}{2n}\right)} - 1} \right\} + A.t. \left\{ \frac{2e^{c\pi \sin.\left(\frac{\pi+\psi}{2n}\right)} . \sin.\left(c\pi \cos.\left(\frac{\pi+\psi}{2n}\right)\right)}{e^{2c\pi \sin.\left(\frac{\pi+\psi}{2n}\right)} - 1} \right\}$$

$$- A.t. \left\{ \frac{2e^{c\pi \sin.\left(\frac{3\pi-\psi}{2n}\right)} . \sin.\left(c\pi \cos.\left(\frac{3\pi-\psi}{2n}\right)\right)}{e^{2c\pi \sin.\left(\frac{3\pi-\psi}{2n}\right)} - 1} \right\} + A.t. \left\{ \frac{2e^{c\pi \sin.\left(\frac{3\pi+\psi}{2n}\right)} . \sin.\left(c\pi \cos.\left(\frac{3\pi+\psi}{2n}\right)\right)}{e^{2c\pi \sin.\left(\frac{3\pi+\psi}{2n}\right)} - 1} \right\}$$

$$- \text{etc.} \qquad + \text{etc.}$$

qui Arcus continuantur, donec eorum multitudo numerum n compleat.

Corol-

Corollarium 8.

§. CLVII. Simili modo series infinita:

$$\text{A. tang}\,\frac{a}{1^{2n}+b}+\text{A. t.}\,\frac{a}{(2m-1)^{2n}+b}+\text{A. t.}\,\frac{a}{(2m+1)^{2n}+b}+\text{A. t.}\,\frac{a}{(4m-1)^{2n}+b}$$
$$+\text{A. t.}\,\frac{a}{(4m+1)^{2n}+b}+\text{etc.}$$

summatur. Est nimirum pro $n=1$ seu seriei $\text{A. tang.}\,\frac{a}{1+b}+\text{A. t.}\,\frac{a}{(2m-1)^2+b}$

$$+\text{A. t.}\,\frac{a}{(2m+1)^2+b}+\text{etc. summa}=\frac{\alpha\pi}{m}+\text{A. t.}\left\{\frac{2e^{\frac{\beta\pi}{m}}\,\text{cos.}\,\frac{\pi}{m}\,\text{sin.}\,\frac{\alpha\pi}{m}-\text{sin.}\,\frac{2\alpha\pi}{m}}{e^{\frac{2\beta\pi}{m}}-2e^{\frac{\beta\pi}{m}}\,\text{cos.}\,\frac{\pi}{m}\,\text{cos.}\,\frac{\alpha\pi}{m}+\text{cos.}\,\frac{2\alpha\pi}{m}}\right\}.$$

Hinc erit summa illius seriei pro quouis n (dum breuitatis gratia duo tantum Arcus, instar omnium, in summae formula exprimuntur),

$$=\frac{c\pi}{m}\,\frac{\text{sin.}\,\frac{\psi}{2n}}{\text{sin.}\,\frac{\pi}{2n}}+\ldots\ldots-\ldots\ldots$$

$$+\text{A. t.}\left\{\frac{2e^{\frac{\beta\pi}{m}}\,\text{cos.}\,\frac{\pi}{m}\,\text{sin.}\,\frac{\alpha\pi}{m}-\text{sin.}\,\frac{2\alpha\pi}{m}}{e^{\frac{2\beta\pi}{m}}-2e^{\frac{\beta\pi}{m}}\,\text{cos.}\,\frac{\pi}{m}\,\text{cos.}\,\frac{\alpha\pi}{m}+\text{cos.}\,\frac{2\alpha\pi}{m}}\right\}$$

$$-\text{A. t.}\left\{\frac{2e^{\frac{\beta^{I}\pi}{m}}\,\text{cos.}\,\frac{\pi}{m}\,\text{sin.}\,\frac{\alpha^{I}\pi}{m}-\text{sin.}\,\frac{2\alpha^{I}\pi}{m}}{e^{\frac{2\beta^{I}\pi}{m}}-2e^{\frac{\beta^{I}\pi}{m}}\,\text{cos.}\,\frac{\pi}{m}\,\text{cos.}\,\frac{\alpha^{I}\pi}{m}+\text{cos.}\,\frac{2\alpha^{I}\pi}{m}}\right\}$$

$+$ etc. $\qquad - $ etc.

sumto $\alpha=\text{sin.}\left(\frac{(2\lambda+1)\pi-\psi}{2n}\right)$, $\beta=\text{cos.}\left(\frac{(2\lambda+1)\pi-\psi}{2n}\right)$;

$\alpha^{I}=\text{sin.}\left(\frac{(2\lambda+1)\pi+\psi}{2n}\right)$, $\beta^{I}=\text{cos.}\left(\frac{(2\lambda+1)\pi+\psi}{2n}\right)$; et numero impari

$2\lambda+1$

$2\lambda+1$ extenso ab 1 vsque ad $n-1$ vel n, prouti n fuerit par vel impar: posteriori autem casu vltimae Arcuum bigae altero omisso, ita vt vtroque casu n Arcus in summae expressione occurrant.

Problema XX.

§. CLVIII. Summare seriem infinitam:

$$\text{A. tang.}\frac{a}{1^{2n-1}}-\text{A. t.}\frac{a}{3^{2n-1}}+\text{A. t.}\frac{a}{5^{2n-1}}-\text{etc.}\pm\text{A. t.}\frac{a}{(2x-1)^{2n-1}}\mp\text{etc.}$$

Solutio.

Ad dispescendum terminum generalem in Arcus simplices resoluenda est ex §. CXXXIX. aequatio: $z^{2n-1}-a=0$, vel $z^{2n-1}+a=0$, prouti n fuerit impar vel par. Hinc posito $a=f^{2n-1}$, radices harum aequationum erunt:

$$z=\pm f$$

. . . . . .

$$z=\mp f\left(\text{cos.}\frac{(2\lambda+1)\pi}{2n-1}+\sqrt{-1}\,.\,\text{sin.}\left(\frac{2\lambda+1}{2n-1}\right)\pi\right)$$

$$z=\mp f\left(\text{cos.}\frac{(2\lambda+1)\pi}{2n-1}-\sqrt{-1}\,.\,\text{sin.}\left(\frac{2\lambda+1}{2n-1}\right)\pi\right)$$

. . . .

vbi pro $2\lambda+1$ omnes numeri impares ab 1 vsque ad $2n-3$ sumuntur: prima radix spectari potest ceu ex $2\lambda+1=2n-1$ deriuata et simpliciter sumta. Ex prima radice oritur pars summae seu summa partialis $=\text{A. t. }e^{\mp\frac{\pi f}{2}}-\frac{\pi}{4}=\pm\left(\text{A. t. }e^{\frac{\pi f}{2}}-\frac{\pi}{4}\right)$.

Ex reliquarum radicum binis coniunctis prodit summa partialis (§. CXLIV. 2.)

$$=\text{A. t.}\left\{\frac{e^{\mp\pi f\,\text{cos.}\frac{(2\lambda+1)\pi}{2n-1}}-1}{2e^{\mp\frac{\pi f}{2}\text{cos.}\frac{(2\lambda+1)\pi}{2n-1}}\,.\,\text{cos.}\left(\frac{\pi f}{2}\,.\,\text{sin.}\left(\frac{2\lambda+1}{2n-1}\right)\pi\right)}\right\}$$

$$=\mp\text{A. t.}\left\{\frac{e^{\pi f\,\text{cos.}\frac{(2\lambda+1)\pi}{2n-1}}-1}{2e^{\frac{\pi f}{2}\text{cos.}\frac{(2\lambda+1)\pi}{2n-1}}\,.\,\text{cos.}\left(\frac{\pi f}{2}\,\text{sin.}\,.\left(\frac{2\lambda+1}{2n-1}\right)\pi\right)}\right\}.$$

Iam

Iam vero Arcus $\frac{(2n-3)\pi}{2n-1}$; $\frac{(2n-5)\pi}{2n-1}$; $\frac{(2n-7)\pi}{2n-1}$; ....

cum Arcubus $\frac{2\pi}{2n-1}$; $\frac{4\pi}{2n-1}$; $\frac{6\pi}{2n-1}$; ....

coniunctim efficiunt π, hinc sinus aequales, cosinus oppositos habent. Quare, si summarum partialium vltima secundae adiungitur, penultima tertiae, et sic porro, haec obtinetur

Summatio.

$$\text{A. tang.}\frac{f^{2n-1}}{1^{2n-1}} - \text{A. t.}\frac{f^{2n-1}}{3^{2n-1}} + \text{A. t.}\frac{f^{2n-1}}{5^{2n-1}} - \text{etc.} \pm \text{A. t.}\frac{f^{2n-1}}{(2x-1)^{2n-1}} \mp \text{etc.}$$

$$= \pm\left(\text{A. tang.}\left\{e^{\frac{\pi f}{2}}\right\} - \frac{\pi}{4}\right) \mp \text{A. t.}\left\{\frac{e^{\pi f \operatorname{cos}\frac{\pi}{2n-1}} - 1}{2e^{\frac{\pi f}{2}\operatorname{cos}\frac{\pi}{2n-1}} \cdot \operatorname{cos}\left(\frac{\pi f}{2}\cdot\operatorname{sin}\frac{\pi}{2n-1}\right)}\right\}$$

$$\pm \text{A. t.}\left\{\frac{e^{\pi f \operatorname{cos}\frac{2\pi}{2n-1}} - 1}{2e^{\frac{\pi f}{2}\operatorname{cos}\frac{2\pi}{2n-1}} \cdot \operatorname{cos}\left(\frac{\pi f}{2}\cdot\operatorname{sin}\frac{2\pi}{2n-1}\right)}\right\}$$

$$\mp \text{A. t.}\left\{\frac{e^{\pi f \operatorname{cos}\frac{3\pi}{2n-1}} - 1}{2e^{\frac{\pi f}{2}\operatorname{cos}\frac{3\pi}{2n-1}} \cdot \operatorname{cos}\left(\frac{\pi f}{2}\cdot\operatorname{sin}\frac{3\pi}{2n-1}\right)}\right\}$$

$$\pm \text{etc.}$$

vbi signa inferiora pro n pari, superiora pro impari n valent. Arcus in expressione summae continuantur, donec perueniatur ad $\frac{(n-1)\pi}{2n-1}$, seu donec adsint n Arcus, incluso

$$\text{A. tang.}\left\{e^{\frac{\pi f}{2}}\right\} - \frac{\pi}{4} = \text{A. t.}\frac{e^{\frac{\pi f}{2}} - 1}{e^{\frac{\pi f}{2}} + 1} = \tfrac{1}{2}\text{A. t.}\left\{\frac{e^{\pi f} - 1}{2e^{\frac{\pi f}{2}}}\right\},$$ quae expressio ostendit

dit, Arcum primum eadem lege progredi $\left(\text{ob cos. } \frac{0\pi}{2n-1} = 1, \text{ sin. } \frac{0\pi}{2n-1} = 0\right)$, ac ceteros, illum vero dimidiatum accipi debere, quoniam ipse factor, ex quo oritur, simplex, nec quadraticus, sumitur.

Corollarium I.

§. CLIX. 1) Casus n = 1 et = 2 supra §. CX et CXXVII. expositi sunt. Pro n = 3 est: $A.t.\frac{f^5}{1^5} - A.t.\frac{f^5}{3^5} + A.t.\frac{f^5}{5^5} - A.t.\frac{f^5}{7^5} + \text{etc.}$

$$= -\frac{\pi}{4} + A.t.\, e^{\frac{\pi f}{2}} - A.t.\left\{\frac{e^{\pi f \cos.\frac{\pi}{5}} - 1}{2e^{\frac{\pi f}{2}\cos.\frac{\pi}{5}} . \cos.\left(\frac{\pi f}{2}\sin.\frac{\pi}{5}\right)}\right\}$$

$$+ A.t.\left\{\frac{e^{\pi f \cos.\frac{2\pi}{5}} - 1}{2e^{\frac{\pi f}{2}\cos.\frac{2\pi}{5}} . \cos.\left(\frac{\pi f}{2}\sin.\frac{2\pi}{5}\right)}\right\},$$ vbi reperitur $\cos.\frac{\pi}{5} = \cos.\ 36^\circ = \frac{1}{4}(\sqrt{5}+1)$, $\sin.\frac{\pi}{5} = \frac{1}{4}\sqrt{(10-2\sqrt{5})}$; $\cos.\frac{2\pi}{5} = \sin.\ 18 = \frac{1}{4}(\sqrt{5}-1)$; $\sin.\frac{2\pi}{5} = \frac{1}{4}\sqrt{(10+2\sqrt{5})}$.

2) Pro n = 4 prodit

$$A.\text{tang}.\frac{f^7}{1^7} - A.t\frac{f^7}{3^7} + A.t.\frac{f^7}{5^7} - \text{etc.}$$

$$= \frac{\pi}{4} - A.t.\, e^{\frac{\pi f}{2}} + A.t.\left\{\frac{e^{\pi f \cos.\frac{\pi}{7}} - 1}{2e^{\frac{\pi f}{2}\cos.\frac{\pi}{7}} . \cos.\left(\frac{\pi f}{2}\sin.\frac{\pi}{7}\right)}\right\}$$

$$- A.t.\left\{\frac{e^{\pi f \cos.\frac{2\pi}{7}} - 1}{2e^{\frac{\pi f}{2}\cos.\frac{2\pi}{7}} . \cos.\left(\frac{\pi f}{2}\sin.\frac{2\pi}{7}\right)}\right\} + A.t.\left\{\frac{e^{\pi f \cos.\frac{3\pi}{7}} - 1}{2e^{\frac{\pi f}{2}\cos.\frac{3\pi}{7}} . \cos.\left(\frac{\pi f}{2}\sin.\frac{3\pi}{7}\right)}\right\}.$$

Corol-

Corollarium 2.

§. CLX. Simili ratione summatur series infinita:

$$A.\,t.\frac{a}{1^{2n-1}} - A.\,t.\frac{a}{(2m-1)^{2n-1}} + A\,t.\frac{a}{(2m+1)^{2n-1}} - A.\,t.\frac{a}{(4m-1)^{2n-1}} + A.\,t.\frac{a}{(4m+1)^{2n-1}} - \text{etc.}$$

Est nimirum summa ex §. CXII. $= S = \frac{(2n-1)\pi(m-1)}{2m} - A.t.\left\{\frac{\text{fin.}\frac{\pi}{m}}{e^{\pm\frac{\pi f}{m}} - \text{cof.}\frac{\pi}{m}}\right\} - \ldots$

$$-A.\,t.\left\{\frac{\text{fin.}\frac{\pi}{m}}{e^{\mp\frac{\pi f}{m}\left(\text{cof.}\frac{\pi(2\lambda+1)}{2n-1} + \sqrt{-1}\,.\,\text{fin.}\frac{\pi(2\lambda+1)}{2n-1}\right)} - \text{cof.}\frac{\pi}{m}}\right\}$$

$$-A.\,t.\left\{\frac{\text{fin.}\frac{\pi}{m}}{e^{\mp\frac{\pi f}{m}\left(\text{cof.}\frac{(2\lambda+1)\pi}{2n-1} - \sqrt{-1}\,.\,\text{fin.}\frac{(2\lambda+1)\pi}{2n-1}\right)} - \text{cof.}\frac{\pi}{m}}\right\}$$

— etc., quae fublatis quantitatibus imaginariis abit in:

$$S = \frac{(2n-1)(m-1)\pi}{2m} - A.\,t.\left\{\frac{\text{fin.}\frac{\pi}{m}}{e^{\pm\frac{\pi f}{m}} - \text{cof.}\frac{\pi}{m}}\right\} - \ldots$$

$$-A\,t.\left\{\frac{2\,\text{fin.}\frac{\pi}{m}\,.\,e^{\mp\frac{\pi s}{m}}\,\text{cof.}\frac{\pi t}{m} - \text{fin.}\frac{2\pi}{m}}{e^{\mp\frac{2\pi s}{m}} - 2\,\text{cof.}\frac{\pi}{m}\,.\,e^{\mp\frac{\pi s}{m}}\,.\,\text{cof.}\frac{\pi t}{m} + \text{cof.}\frac{2\pi}{m}}\right\} - \ldots$$

fumto $f = a^{\frac{1}{2n-1}}$; $s = f\,\text{cof.}\frac{(2\lambda+1)\pi}{2n-1}$, $t = f\,\text{fin.}\frac{(2\lambda+1)\pi}{2n-1}$; et extenfo numero

impari

impari $2\lambda+1$ ab 1 vsque ad $2n-3$; signis praeterea superioribus pro impari n, inferioribus pro pari assumtis. Arcus in summae formula primus exhiberi etiam potest tanquam dimidium eius arcus, qui vltimo loco prodiret ex arcu generali, sumendo numerum imparem $2\lambda+1=2n-1$.

Problema XXI.

§. CLXI. Summare seriem infinitam:

$$\text{A. tang.}\,\frac{a+b}{1+c+d}+\text{A. t.}\,\frac{2^{2n}a+b}{2^{4n}+2^{2n}c+d}+\text{A. t.}\,\frac{3^{2n}a+b}{3^{4n}+3^{2n}c+d}-+\ldots$$

$$+\text{A. t.}\,\frac{ax^{2n}+b}{x^{4n}+cx^{2n}+d}+\text{etc.}$$

Solutio.

1) Posito $x^{2n}=u$, terminus generalis $=\text{A. t.}\,\frac{ax^{2n}+b}{x^{4n}+cx^{2n}+d}=\text{A. t.}\,\frac{au+b}{u^2+cu+d}$ in binos Arcus dispesci potest. Quem in finem ex praecepto §. CXXXIV. 3. resoluenda est aequatio:

$$u^2+cu+d+(au+b)\sqrt{-1}=0;$$

vnde fit $u=\frac{-(c+a\sqrt{-1})\pm\sqrt{(c^2-4d-a^2+2(ac-2b)\sqrt{-1})}}{2}$. Radix quantitatis imaginariae in hac expressione occurrentis ad formam $\mathfrak{A}+\mathfrak{B}\sqrt{-1}$ reuocatur, definiendo $\mathfrak{A}$, $\mathfrak{B}$ ex aequationibus: $\mathfrak{A}^2-\mathfrak{B}^2=c^2-4d-a^2$; $\mathfrak{A}\mathfrak{B}=ac-2b$. Tum fit radicum u vna $=\frac{-(c-\mathfrak{A})-(a-\mathfrak{B})\sqrt{-1}}{2}$, altera

$$=\frac{-(c+\mathfrak{A})-(a+\mathfrak{B})\sqrt{-1}}{2}.$$ Quare prodeunt Arcus simplices $\text{A. tang.}\,\frac{a+\mathfrak{B}}{2u+c+\mathfrak{A}}+\text{A. t.}\,\frac{a-\mathfrak{B}}{2u+c-\mathfrak{A}}=\text{A. t.}\,\frac{au+b}{u^2+cu+d}$. Quantitates $\mathfrak{A}$, $\mathfrak{B}$ ex aequationibus commemoratis ita etiam exprimi possunt:

$$\mathfrak{A}=\left(\frac{c^2-4d-a^2}{\text{cos.}\,\zeta}\right)^{\frac{1}{2}}.\,\text{cos.}\,\frac{\zeta}{2};\ \mathfrak{B}=\left(\frac{c^2-4d-a^2}{\text{cos.}\,\zeta}\right)^{\frac{1}{2}}.\,\text{sin.}\,\frac{\zeta}{2},$$ sumto $\zeta=\text{A. tang.}\,\frac{2(ac-2b)}{c^2-4d-a^2}$.

2) Quibus praemissis terminus generalis seriei in problemate propositae in binos Arcus resoluitur: $\text{A. t.}\,\frac{a+\mathfrak{B}}{2x^{2n}+c+\mathfrak{A}}$ et $\text{A. tang.}\,\frac{a-\mathfrak{B}}{2x^{2n}+c-\mathfrak{A}}$. Quare series summanda

manda in duas difpefcitur, quarum vtraque ex Probl. XIX. §. CXLIX. fummabilis eft.

Scholion 1.

§. CLXII. Refolutio A. tang. $\frac{au+b}{u^2+cu+d}$ in binos Arcus fimplices fequenti etiam ratione tentari poteft. Pofito A. t. $\frac{au+b}{u^2+cu+d}$ = A. t. $\frac{f}{u+g}$ + A. t. $\frac{F}{u+G}$ =

A. tang. $\frac{(f+F)u+Gf+Fg}{u^2+(g+G)u+gG-Ff}$, quatuor oriuntur aequationes:

$$1)\ f+F=a;\quad 2)\ Gf+Fg=b;\quad 3)\ g+G=c;\quad 4)\ gG-Ff=d.$$

Quae aequationes fi more eliminationis vfitato tractentur, prodit tandem aequatio completa quarti gradus, difficilis refolutu. His difficultatibus, quas refolutio ex regula §. CXXXIV. 3. euitat, hac quoque ratione occurritur: Ob $f+F=a$ et $g+G=c$, ex (1) et (2); ponatur $F=\frac{a+s}{2}$, $f=\frac{a-s}{2}$; $G=\frac{c+r}{2}$, $g=\frac{c-r}{2}$; tum aequationes (2) et (4) in has abeunt:

$$\frac{(c+r)(a-s)}{4}+\frac{(c-r)(a+s)}{4}=b;\quad \frac{c^2-r^2}{4}-\frac{(a^2-s^2)}{4}=d;\ \text{id eft}$$

$$c^2-4d-a^2=r^2-s^2,\ \text{et}\ ca-rs=2b.$$

Hinc oritur aequatio biquadratica ad quadraticam reducibilis, fitque

$$r^2=\frac{c^2-4d-a^2}{2}+\surd\left(\left(\frac{c^2-4d-a^2}{2}\right)^2+(ca-2b)^2\right);$$ vnde s, et F, f, G, g prodeunt. Quantitates r et s cum $\mathfrak{A}$, $\mathfrak{B}$, §. CLXI. conueniunt.

Scholion 2.

§. CLXIII. Eadem adhibita refolutione fummabilis eft feries Problem. XXI. (§. CLXI.), dum figna *alternantur;* nec minus feries, cuius terminus generalis x^{tus} eft

$= \text{A. t.} \frac{a(2x-1)^{2n}+b}{(2x-1)^{4n}+c(2x-1)^{2n}+d}$; porro etiam feries haec:

$$\text{A. tang.} \frac{a+b}{1+c+d}+\text{A. t.} \frac{a(2m-1)^{2n}+b}{(2m-1)^{4n}+c(2m-1)^{2n}+d}$$

$$+\text{A. t.} \frac{a(2m+1)^{4n}+b}{(2m+1)^{4n}+c(2m+1)^{2n}+d}+\ldots$$ (cf. §. CLVII.)

 Ex

Ex iisdem principiis ſummare licet ſeries, quarum termini generales x^{ti} ſunt:

$$\text{A. tang.}\frac{ax^{4n}+bx^{2n}+c}{x^{6n}+\beta x^{4n}+\gamma x^{2n}+\delta},\ \text{A. tang.}\frac{ax^{6n}+bx^{4n}+cx^{2n}+d}{x^{8n}+\beta x^{6n}+\gamma x^{4n}+\delta x^{2n}+\varepsilon},$$ cum aliis ſeriebus, quae ad harum ſimilitudinem efformatae ſunt. Quae tamen omnia cum ex praeceptis generalibus ſupra ſtabilitis, atque ex ſummationibus hactenus ſatis ample euolutis repeti queant, fuſius iſta perſequi ſuperfluum videtur.

NOVA

NOVA DISQVISITIO
DE INTEGRATIONE AEQVATIONIS DIFFERENTIO-DIFFERENTIALIS:

$$x^2(a+bx^n)d^2y+x(c+ex^n)dydx+(f+gx^n)ydx^2=Xdx^2.$$

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$$x^2(a+bx^n)d^2y+x(c+ex^n)dydx+(f+gx^n)ydx^2=Xdx^2.$$

"Cum ad aequationes differentiales, quae generaliter integrari nequeunt methodis adhuc vſitatis, peruenitur, non parum augmenti analyſis accipere cenſenda eſt, ſi caſus ſaltem particulares aſſignentur, quibus integratio locum inueniat. Dum enim integratio caſuum ab integratione generalis aequationis non pendet, eo magis erit abſcondita atque inuentu difficilis, quo minus per generaliores integrandi methodos perfici poterit."

L. EVLERVS Comment. Petrop. T. X. p. 40.

§. I.

Aequationis, quam hoc loco euoluendam ſuſcipio, accuratius conſiderandae occaſionem mihi praebuit Diſquiſitio, quam eram aggreſſus, de ſummis ſerierum hypergeometricarum: quippe ſummatio ſerierum hypergeometricarum ſecundi ordinis ad integrationem aequationis differentialis iſta forma praeditae reduci poteſt. (*) Quae tamen ipſa aequatio cum ab aliis Analyſtis iam paſſim tractata fuerit, ceperam primo conſilium, ea, quae alibi extabant inuenta ac demonſtrata, tanquam Lemmata in vſus meos adhibendi. Quum vero cauſſae propius accederem, vidi, haud pauca, ſi non plurima, in hac quaeſtione enodanda adhuc ſupereſſe, eamque nec ſatis ſimpliciter, nec ſatis generaliter tractatam fuiſſe. Quare hunc laborem denuo ſuſcipere, pro ſcopo illo quem mihi propoſueram, neceſſe erat: idemque nec aliis ſuperfluum viſum iri ſpero.

§. II.

Aequatio commemorata (**), praeterea quam quod ſummatio ſerierum hypergeometricarum ſecundi ordinis ab eius integratione pendeat (§. I.), alios inſuper variosque habet

(*) De notione huiusmodi ſerierum, earum, quas voco, *ordinibus* ac ſummis amplius exponetur *Volumine altero.*

(**) Cum forma generalior, pro qua X denotat quamuis functionem τοῦ x, ad caſum ſimpliciorem $X=0$ reduci queat (§. LXX.), de hoc maxime caſu in ſequentibus ſermo erit.

habet in Analyfi vfus. Quare plerique auctores, qui Calculum integralem fufius pertractarunt, illam quoque aequationem ad examen reuocarunt. Plurimum vero ac iterata vice in ea integranda occupatus erat L. EVLERVS: *primo* quidem in Commentatione de aequationibus differentialibus certis tantum cafibus integrationem admittentibus (*), quae tota in confideranda noftra aequatione verfatur; *deinde* in Inftitt. Calc. Integr. Vol. II, cuius Capp. VII et VIII (pp. 182-253.) *eidem* peculiariter deftinata funt, quam *tandem* poft editum illud Opus denuo in fingulari fchediasmate contemplatus eft (**).

Ea, quae EVLERVS praeceperat, alii deinceps auctores, BOUGAINVILLE (***), P. P. LE SEUR et JACQUIER (†), COUSIN (††), etc. maxime vel ad ductum Commentationis primo laudatae, vel ex Inftitt. Calc. Integr. propfuerunt (†††). Cum vero recentior feu tertia EVLERI tractatio nouas exhibeat meditationes, prioribus fupplementi inftar adiungendas: quo exactius conftet, quid de noftra aequatione inuentum iam fit ac traditum, breuiter illius fummam commemorabo.

Quoniam aequatio, nifi ad feries infinitas confugeris, vniuerfaliter integrari nequit, cardo quaeftionis in eo verfatur, vt determinentur cafus, quibus integrationem admittat.

Iam *primo* offert fe cafus integrabilis, dum affumta ferie indefinita coëfficientes aequationis ita determinentur, vt feries alicubi abrumpatur (a). Qua ratione (vel immediate, vel mediate, praeuia fcilicet transmutatione aequationis per fubftitutionem) prodeunt aequationes conditionales binae fequentes:

f + λ

(*) Commentar. Petropol. Tom. X. pag. 40-55.

(**) Nou. Comm. Petrop. T. XVII. Petropoli MDCCLXXIII. Confideratio aequationis differentio-differentialis: $(a+bx)ddz + (c+ex)\frac{dxdz}{x} + (f+gx)\frac{zdx^2}{xx} = 0$ (pp. 125-154.).

(***) *Calculi Infinitefimalis Pars II.* feu Calculus integralis, expofitus opere Bipartito D. *Bougainville*, ex edit. Parif. anni MDCCLIV et MDCCLVI. in Latinum conuerfo a C. S. S. I. MDCCLXIV Vindobonae 4. (Part. 2. Sect. 2. Cap. 9. pp. 205-229.)

(†) *Elemens de Calcul Integral*, Seconde Partie, par les P. P. *le Seur* et *Jacquier*, à Parme MDCCLXVIII. 4. (Cap. VIII. pp. 410-440.)

(††) *Lecons de Calcul differentiel et de Calcul integral*, par M. *Coufin*, Paris MDCCLXXVII. 8. (II. Partie Chap. VIII. pp. 497-508.) In editione huius operis altera, quae nuperrime prodiit, infcripta: *Traité de Calcul Différential et Calcul Intégral*, par J. A. J. *Coufin*, de l'Inftitut National des Sciences et des Arts, à Paris, l'an. 4^e. — 1796, aequatio differentialis eadem omnino ratione, ac in editione prima, tractatur (II. Partie pp. 68-76.).

(†††) Meditationum Celeberr. *Lorgna* circa eandem aequationem infra (§. LXIX.) mentio fiet.

(a) De methodo, per feries integrandi, cf. inprimis Ill. *Kaeftneri* Analyf. infinit. §. 419 fq. Problemate §. 462. p. 404. aequationis noftrae cafus particularis refoluitur: de quo infra amplius exponetur (§. XLVIII.).

$f + \lambda(\lambda - 1)a + \lambda c = 0$
$g + (\lambda + in)(\lambda + in - 1)b + (\lambda + in)e = 0;$

vbi λ arbitrarie, et i numero cuiuis *integro*, fiue *affirmatiuo*, fiue *negatiuo* aequalis fumi poteft. Huncce maxime cafum integrabilem, quem aequationes iftae coniunctim fiftunt, fatis ample euoluit EVLERVS in Inftitt Calc. Integr. pro valoribus affirmatiuis numeri i, eundemque ad negatiuos etiam patere, obferuauit l. c. pag. 258.

Praeter hunc *cafum primum*, qui ab Auctoribus modo laudatis tanquam *vnicus* profertur, in recentiori Commentatione EVLERVS *nouem* infuper *cafus fingulares* exhibuit, quibus aequatio itidem eft integrabilis: *fingulares* inquam, quippe quiuis eorum certam ac determinatam relationem inter coëfficientes aequationis fupponit, cum contra cafus primus ob quantitates λ et i indeterminatas latiorem ambitum habeat (*). Methodus, qua ifti cafus reperti funt, huc fere redit. Quaeritur multiplicator formae $Pdy + Q dx$, per quem aequatio differentio-differentialis propofita prima vice integrari feu ad primum gradum deprimi queat. Cuius autem factoris inueftigatio rurfus perducit ad aequationem differentialem fecundi gradus: quae et ipfa cum directe et generaliter integrari nequeat, affumitur eius integrale certae formae algebraicae, quo quidem in aequatione actu fubftituto, rite peractis calculis inueniuntur debitae relationes inter coëfficientes aequationes primitiuae. Sic tandem prodeunt cafus fequentes (**):

1) $e = \frac{b(2a+c)}{a}$; $g = \frac{b(c+f)}{a}$

2) $e = \frac{bc}{a}$; $g = \frac{bf}{a}$

3) $e = \frac{b(3a+2c)}{2a}$; $g = \frac{bc(a+c)}{4aa}$

4) $e = \frac{b(a+2c)}{2a}$; $g = \frac{bc(c-a)}{4aa}$

5) $e = \frac{b(3a+2c)}{2a}$; $f = \frac{(2c-a)(2c-3a)}{16a}$

6) $e = \frac{b(a+2c)}{2a}$; $f = \frac{(2c-a)(2c-3a)}{16a}$

7) $f = \frac{(e-2b)(2bc-ae)}{4bb}$; $g = \frac{e(e-2b)}{4b}$

8) $f = \frac{(2e-a)(2c-3a)}{16a}$; $g = \frac{(b-2e)(3b-2e)}{16b}$

9) $f = \frac{c(c-2a)}{4a}$; $g = \frac{-c(2ab-2ae+bc)}{4aa}$ (***)

quibus

(*) Quatuor horum nouem cafuum ad cafum primum redire, hincque non pro nouis effe habendos, infra demonftrabitur (§. LXVIII).

(**) l. c. p. 151.

quibus aequationes differentiales primi gradus assignantur, ad quas aequatio secundi gradus reducta est. Qua autem ratione istae aequationes, et ipsae difficiles integratu, vlterius sint tractandae, vt solutio completa obtineatur, haud ostendit EVLERVS. At pro casibus 3. 4. 5. 6, in auxilium adsumto alio multiplicatore magis particulari, qui duas partes, in quas aequatio diuidi potest, seorsim integrabiles reddat, aequationes differentiales primi gradus *separatas* exhibuit: qui igitur quatuor casus singulares reuera pro integratis habendi sunt.

§. III.

Quo eorum, quae deinceps proponenda sunt, sensus et nexus clarius percipiantur, è re esse videtur, de methodo, in vsum adhibenda, ac de iis, quae illius auxilio inuenire mihi contigit, pauca praefari.

Cum aequatio proposita certis tantum casibus integrationem admittat, ea commode ita tractari posse videtur, vt *primo* inuestigentur casus in suo genere *simplicissimi*, quibus integrale sponte innotescat. *Deinde* aequatio in alias forma similes transformanda, ad easue reducenda est: vnde ex casibus simplicissimis alii magis compositi iidemque integrabiles oriuntur. Quo vero casuum integrabilium tractatio vna serie procedat,

I) *Capite primo* de *transformationibus* et *reductionibus* aequationis propositae quaerendum est. *Transformationes* nituntur *vna* substitutione: $y = x^p (a + b x^n)^q . v$, cuius ope tres aequationes transformatae, propositae similes, reperiuntur. *Reductio* fundamentalis simplici omnino ratione obtinetur, dum aequationis differentialis, propositae quoad formam similis, differentiale r^{tum} sumitur, et aequatio inde nata ipsi propositae aequipollens statuitur. Ex hac reductione, primo quidem ad casum $f = 0$, $n = 1$ restricta, ope transformationum commemoratarum aliae reductiones sponte consequuntur, quae rite combinatae perducunt tandem ad vnam reductionem generaliorem.

II) His praemissis Cap. II. *Casus integrabiles ipsi euoluuntur.* Fundamenti loco ponuntur duo casus simplicissimi:

a) *Primus* per se manifestus est, pro $f = 0$, $g = 0$. Ex quo, ope transformationum et reductionum Cap. I., obtinetur *casus generalior primus*, idem quem ab aliis auctoribus ex assumta serie indefinita deriuari supra (§. II.) dictum est. Quanquam hic casus satis iam notus sit, eundem tamen denuo considerare haud superfluum duxi. Methodo hic adhibita prodit expressio finita pro y, ab vsitata quoad

(***) l. c. p. 150. ponitur $g = \frac{-c(4ab - 2ae + bc)}{4aa}$, idemque iterata vice occurrit p. 153; vbi vero pro 4 scribendum esse 2, accuratiore calculi examine compertum habeo.

quoad formam omnino diuersa, quaeque cum simplicior esse videtur, tum loco integralis *particularis* statim *completum* exhibet. Praeterea ad varias obseruationes nouas deductus sum, quae non tantum ad ipsum casum primum spectant, verum etiam generalem eamque completam aequationis differentialis per series saltem infinitas integrationem illustrant. Inde factum est, vt haec nostra tractatio paullo prolixior euaserit, quo ea, quae post aliorum studia desiderari adhuc videbantur, supplere, atque singula distincte satis et accurate euoluere liceret.

b) *Alter* casus in suo genere simplicissimus oritur, ponendo $c = \frac{a}{2}$, $e = b$, $f = 0$, $n = 1$; cuius integratio haud difficulter eruitur. Ex hoc porro casu, adhibitis reductionibus et transformationibus Cap. I., *tres casus generaliores* deriuantur, qui, respiciendo ad *primum* (a), *secundus*, *tertius* et *quartus* vocentur, hi nimirum:

$$2)\ e = b\left(\frac{c}{a} + (\tfrac{1}{2} + r + \varrho)n\right);\ f = \frac{a}{4}\left(\left(\frac{c}{a} - 1\right)^2 - n^2\left(\tfrac{1}{2} - r + \varrho\right)^2\right)$$

$$3)\ e = b\left(\frac{c}{a} + (\tfrac{1}{2} + r + \varrho)n\right);\ g = \frac{b}{4}\left(\left(\frac{e}{b} - 1\right)^2 - n^2\left(\tfrac{1}{2} - r + \varrho\right)^2\right)$$

$$4)\ f = \frac{a}{4}\left(\left(\frac{c}{a} - 1\right)^2 - n^2\left(\tfrac{1}{2} - r - \varrho\right)^2;\ g = \frac{b}{4}\left(\left(\frac{e}{b} - 1\right)^2 - n^2\left(\tfrac{1}{2} - r + \varrho\right)^2\right)\right.$$

quibus non tantum aequationem integrabilem esse demonstrari potest, verum etiam integralia completa per expressiones finitas satis simplices deinceps euoluentur. Qui tres casus, praeter casus singulares ab EVLERO prolatos, ceu exempla particularia, innumeras insuper comprehendunt aequationes differentiales secundi gradus, ab Analystis hactenus nondum integratas: quippe pro r et ϱ sumi possunt numeri quiuis *integri* tam affirmatiui quam negatiui. Itaque iis, quae de nostra aequatione ab aliis iam auctoribus tradita fuisse supra (§. II.) retuli, quaeque praeter exempla EVLERI singularia primum tantum casum concernunt, euolutione reliquorum trium casuum generalium augmentum haud mediocre accedit. Ceterum ipsae *formae* integralium in sequentibus exprimendae quodammodo memorabiles esse videntur, cum in aliis integrationibus similes formae nondum fuerint adhibitae. Ita pro aequatione *Riccatiana*, quae dudum exhausta videbatur, quanquam haud nouos casus integrabiles, nouam tamen eamque concinniorem integralis expressionem ad istas formas accommodatam exhibere licuit (*).

CAPVT

(*) Dum in hac scriptione versarer, incidit in manus liber, profundioris analyseos specimina exhibens, inscriptus: *Memoires analytiques par le Comte R. de C. à Milan.* MDCCLXXVI. 4. Prima commentatio agit de ipsa nostra aequatione. Auctor praemittit quatuor theoremata generaliora, aequationes *lineares* secundi ordinis concernentia. Deinde ad aequationem propositam specialem transiens,

CAPVT I.

DE TRANSFORMATIONIBVS ET REDVCTIONIBVS AEQVATIONIS DIFFERENTIALIS PROPOSITAE.

Problema I.

§. IV. Aequationem: $0 = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)y\,dx^2$ in aliam ſimilis formae transformare, pro qua exponens n ſit $= 1$.

Solutio.

1) Aequatio propoſita, cuius differentiale conſtans eſt dx, ſequenti modo exhibeatur:

$$0 = x^2(a+bx^n)\frac{d\left(\frac{dy}{dx}\right)}{dx} + x(c+ex^n)\frac{dy}{dx} + (f+gx^n)y;$$

tum

iens, caſum primum integrabilem ex ſerie indefinita petendum breuiter commemorat, prouocans ad ea, quae amplius iam expoſita ſint in Inſtitt. Calc. Integr. *Euleri*. Tum quinque caſus integrabiles ſingulares ſeparatim tractantur: via quidem indirecta, dum ſcilicet integralia per formulas tranſcendentes finitas expreſſa aſſumuntur, ex iisque demum aequationis coëfficientes determinantur. Qui caſus cum caſibus 3. 4. 5. 6. 8 in recentiore *Euleri* commentatione (II) demonſtratis omnino conueniunt. Porro probl. 4. 5. 6. variae transformationes aequationis exponuntur; ex iisque tandem concluditur, innumeros dari caſus integrabiles. In quo igitur auctor vlterius progreſſus eſt, quam *Eulerus*. Quinam vero ſint caſus iſti integrabiles, et quaenam iis integralia reſpondeant, nec ille oſtendit. Ipſe potius in limine commentationis ait: acquieſcere ſe, "*de faire ſentir la neceſſité, d'examiner encore plus a profond cette importante et ſinguliere equation.*" Subſtitutiones, quibus vtitur, quaeque ad *Euleri* praecepta (Cap. IX. l. c.) efformatae videntur, ita ſunt comparatae, vt, niſi noua adhibeantur artificia, vix pateat, quo pacto eae pro lubitu continuandae, earumque ope integralia ſint definienda. Hinc ſorte factum eſt, quod praeter caſus quinque praedictos haud inſuper alius caſus eiusque noui integrationem exhibuerit. Aſſerit porro, transformationes in infinitum variare, earumque numerum haud agnoſcere limites, indeque etiam multitudinem caſuum integrabilium eſſe prorſus inexhauſtam: verumtamen plurimas transformationes tam complicatas euadere, vt difficile foret, regulas faciles condere, quibus dignoſcere liceret, num data quaepiam aequatio ſit integrabilis (pag. III. XXIV.). Ex quo immenſus labor in abſoluenda hac, de qua agimus, diſquiſitione, i. e. complete enumerandis caſibus integrabilibus, iisque rite integrandis, poſtulari, quin id, quod in hac quaeſtione maxime deſiderandum erat, vix effici poſſe videtur. Quid enim iuuat, noſſe, aequationem innumeris caſibus eſſe integrabilem, niſi ſimul certa lex conſtaret, quae ſingulos hosce caſus contineat, et ad quam quaeuis aequatio propoſita, num ea ſit integrabilis, examinari queat: niſi porro ipſa integralia actu euoluere liceret? Verum enim vero rem ſecus ſe habere, ac ex iſtis aſſertis colligi poterat et auctor opinatus eſſe videtur, ipſa noſtra tractatio abunde teſtabitur. Adhiberi nimirum poſſunt ſubſtitutiones et reductiones ſimpliciores, vel potius vna ſubſtitutio, vnaque reductio ſufficiunt. Sic omnes caſus integrabiles, praeter primum, ad tres generales, eosque facile dignoſcendos, redeunt: et integralia formulis ſenſu analytico omnino ſimplicibus exprimuntur. — Quae hactenus dicta commemoranda videbantur, partim quoniam libri in hoc genere egregii et parum noti mentio erat facienda; partim, vt intelligatur, poſt hunc etiam laborem nouum conamen, via planiore ſuſceptum, ac ad abſoluendam quaeſtionem imperfectam omnino relictam, eamque prima ſpecie quaſi infinitam, tendens, nequaquam fuiſſe ſuperfluum.

tum differentiali secundo ad formam primi reducto, illius differentialis constantis haud amplius ratio est habenda, verum aliud quodpiam pro constanti sumere licet.

2) Quare ponatur $x^n = \chi$, eritque $x = \chi^{\frac{1}{n}}$, $dx = \frac{1}{n}\chi^{\frac{1}{n}-1} d\chi$, $\frac{dy}{dx} = \frac{n\chi^{1-\frac{1}{n}} dy}{d\chi}$; hinc sumto $d\chi$ pro differentiali constanti (1), prodit $d\left(\frac{dy}{dx}\right) = n\chi^{1-\frac{1}{n}}\frac{d^2y}{d\chi} + n\left(1-\frac{1}{n}\right)\chi^{-\frac{1}{n}} dy$. Proinde ex aequatione proposita sequitur:

$$0 = \chi^2(a+b\chi)d^2y + \left(\frac{n-1}{n}\right)\chi(a+b\chi)dyd\chi + (c+e\chi)\chi\frac{dyd\chi}{n} + (f+g\chi)\frac{yd\chi^2}{n^2}.$$

3) Posito igitur $c^I = \frac{a(n-1)+c}{n}$, $e^I = \frac{b(n-1)+e}{n}$, $f^I = \frac{f}{n^2}$, $g^I = \frac{g}{n^2}$;

Solutio problematis ita exprimi potest:

Aequatio $0 = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+g^n)ydx^2$
abit in hanc: $0 = \chi^2(a+b\chi)d^2y + \chi(c^I+e^I\chi)dyd\chi + (f^I+g^I\chi)yd\chi^2$,
sumto $\chi = x^n$, et $d\chi$ pro differentiali constanti, quod in priori aequatione erat dx.

Corollarium.

§. V. 1) Sicuti secunda aequatio (§. IV. 3.) ex prima deriuata est, ita simili ratione vice versa haec ex illa reperitur, seu aequatio, pro qua est exponens $n = 1$, transmutatur in aliam, pro qua n quemuis valorem recipere potest. Coëfficientes nimirum ex his formulis prodeunt: $c = nc^I - a(n-1)$; $e = ne^I - (n-1)b$; $f = n^2 f^I$; $g = n^2 g^I$.

2) Hinc apparet, pro nostra aequatione sufficere considerationem casus $n = 1$, (*) eaque quae ad hunc casum inuenta fuerint, cum quoad transformationes aequationis, tum quoad ipsam integrationem, transferri deinde posse ad aequationis formam generaliorem. Ita quidem tractatio sequens omnino simplicior concinniorque fit, et calculorum ambages haud parum contrahuntur.

PRO-

(*) Sic etiam *Eulerus* N. C. P. Tom. XVII. casum $n = 1$ considerauit; nec vero ea, quae pro hac forma particulari demonstrata sunt, ad formam generalem extendit.

Problema II.

§. VI. Aequationem: $0 = x^2(a+bx)d^2y + x(c+ex)dydx + (f+gx)ydx^2$ ope substitutionis $y = x^p(a+bx)^q . v$ in aliam similis formae transmutare.

Solutio.

1) Logarithmice differentiando ex substitutione assumta sequitur:

$\frac{dy}{y} = \frac{pdx}{x} + \frac{bqdx}{a+bx} + \frac{dv}{v}$. Repetita differentiatio praebet:

$\frac{d^2y}{y} - \frac{dy^2}{y^2} = -\frac{pdx^2}{x^2} - \frac{b^2qdx^2}{(a+bx)^2} + \frac{d^2v}{v} - \frac{dv^2}{v^2}$; hincque fit, addendo vtrinque $\frac{dy^2}{y^2}$,

$$\frac{d^2y}{y} = (p^2-p)\frac{dx^2}{x^2} + \frac{2bpqdx^2}{x(a+bx)} + \frac{b^2q(q-1)dx^2}{(a+bx)^2} + 2\left(\frac{pdx}{x} + \frac{bqdx}{a+bx}\right)\frac{dv}{v} + \frac{d^2v}{v}.$$

2) Quibus valoribus suppositis, aequatio problematis abit in hanc:

$$0 = x^2(a+bx)\frac{d^2v}{vdx^2} + \left(\begin{matrix}2(px(a+bx)+bqx^2)\\ +x(c+ex)\end{matrix}\right)\frac{dv}{vdx} + p(p-1)(a+bx)$$
$$+ 2bpqx + \frac{b^2q(q-1)x^2}{a+bx} + p(c+ex) + \frac{bqx(c+ex)}{a+bx} + f+gx$$

sive: $0 = x^2(a+bx)d^2v + x(c+2pa+(e+2pb+2qb)x)dvdx$
$+ (f+pc+p(p-1)a+(g+pe+p(p-1)b+2pqb)x).vdx^2$
$+ \frac{bqx}{a+bx}(c+(e+bq-b)x)vdx^2.$

3) Iam facile apparet, aequationem transformatam propositae quoad formam similem reddi, si fractio, quae vltimum membrum implicat, tollatur. Quod quidem duplici ratione obtinetur, sumendo 1) $q = 0$; vel 2) $\frac{e+b(q-1)}{c} = \frac{b}{a}$, id est

$q = 1 + \frac{c}{a} - \frac{e}{b}$.

4) Ex *prima* positione fluit

Transformatio prima.

Aequatio $0 = x^2(a+bx)d^2y + x(c+ex)dydx + (f+gx)ydx^2$ abit in hanc:
$0 = x^2(a+bx)d^2v + x(c+2pa+(e+2pb)x)dvdx + (f+pc+p(p-1)a$
$+(g+pe+p(p-1)b)x)vdx^2$

adhibita substitutione $y = x^p . v$.

5) Ex *altera* positione fit coëfficiens τȣ vdx^2 in aequatione transformata (2)
$= f+pc+p(p-1)a+(g+pe+p(p-1)b+2pqb+bq\frac{c}{a})x$;

in qua expressione coëfficiens τȣ x est

$=g$

$$=g+pe+p(p-1)b+b\left(1+\frac{c}{a}-\frac{e}{b}\right)\left(2p+\frac{c}{a}\right)$$

$$=g+pe+p(p-1)b+pb\left(1+\frac{c}{a}-\frac{e}{b}\right)+b\left(1+\frac{c}{a}-\frac{e}{b}\right)\left(p+\frac{c}{a}\right)$$

$$=g+pb\left(p+\frac{c}{a}\right)+b\left(1+\frac{c}{a}-\frac{e}{b}\right)\left(p+\frac{c}{a}\right)$$

$$=g+b\left(p+1+\frac{c}{a}-\frac{e}{b}\right)\left(p+\frac{c}{a}\right).$$ Inde haec prodit

Transformatio fecunda.

Aequatio $o=x^2(a+bx)d^2y+x(c+ex)dydx+(f+gx)ydx^2$ abit in hanc:

$$o=x^2(a+bx)d^2v+x\left(c+2pa+\left(2\left(p+1+\frac{c}{a}\right)b-e\right)x\right)dvdx$$
$$+\left(f+pc+p(p-1)a+\left(g+b\left(p+1+\frac{c}{a}-\frac{e}{b}\right)\left(p+\frac{c}{a}\right)\right)x\right)vdx^2$$

ope fubftitutionis $y=x^p(a+bx)^{1+\frac{c}{a}-\frac{e}{b}}.v.$

6) Qui bini modi, (4) et (5) exhibiti, aequationem transformatam datae fimilem reddendi, quanquam primo obtutu foli locum habere videantur; *tertius* tamen modus, isque minus obuius, fupereft. Statuatur nimirum $f+pc+p(p-1)a=o$, tum fingula aequationis transformatae (2) membra per x diuidi poffunt. Quo facto, ac pofito infuper $a+bx=-bX$, prodit aequatio:

$$o=X(a+bX)d^2v-\left(c+2pa-(e+2pb+2qb)\left(\frac{a+bX}{b}\right)\right)dvdX$$
$$+(g+pe+p(p-1)b+2pqb)vdX^2-\frac{q}{X}\left(c-(e+bq-b)\left(\frac{a+bX}{b}\right)\right)vdX^2,$$

fiue, per X multiplicando:

$$o=X^2(a+bX)d^2v+X\left(-c+a\left(\frac{e}{b}+2q\right)+(e+2pb+2qb)X\right)dvdX$$
$$+\left(aq\left(\frac{e}{b}+q-1-\frac{c}{a}\right)+(g+pe+p(p-1)b+2pqb+qe+q(q-1)b).X\right)vdX^2.$$

Coëfficiens τȣ X in vltimo membro huius aequationis eft $=g+(p+q)e+(p+q)^2b-(p+q)b=g+(p+q)(e+b(p+q-1))$. Exinde fequens oritur

Transformatio tertia.

Aequatio $o=x^2(a+bx)d^2y+x(c+ex)dydx+(f+gx)ydx^2$ abit in hanc:

$$o=X^2(a+bx)d^2v+X\left(-c+a\left(\frac{e}{b}+2q\right)+(e+2(p+q)b)X\right)dvdX$$
$$+\left(aq\left(\frac{e}{b}-\frac{c}{a}+q-1\right)+(g+(p+q)(b(p+q-1)+e))X\right)vdx^2$$

pofito

posito $y = x^p (a + bx)^q . v$, et $a + bx = - bX$. Quantitas p definitur ex aequatione quadratica $o = f + pc + p(p - 1)a$. In priori aequatione dx, in altera dX pro differentiali constanti habetur.

Corollarium 1.

§. VII. Cum in transformationibus prima et secunda quantitas p, in tertia q arbitrio nostro relinquatur, quantitatem arbitrariam ita semper determinare licebit, vt membri vltimi aequationis transformatae alteruter terminus euanescat. Hinc apparet, in aequatione nostra semper poni posse $f = o$; omnemque igitur inuestigationem vertendam esse ad aequationem hanc:

$$o = x^2(a + bx) d^2y x + (c + ex) dy dx + gxy dx^2$$

$$\text{siue } o = x\ (a + bx) d^2y + (c + ex) dy dx + gy dx^2;$$

quippe, quae pro hac forma reperiuntur, ad formam generalem sponte traduci possunt. Hoc procedendi modo calculi admodum contrahuntur.

Corollarium 2.

§. VIII. Haec ipsa aequatio: $o = x(a + bx) d^2y + (c + ex) dy dx + gy dx^2$ ope transformationis 1 et 2 (§. VI.) triplici ratione in formam similem transmutari potest. Posito nimirum in transformatione (1) $p = 1 - \frac{c}{a}$, obtinetur:

$$1)\ o = x(a + bx) d^2 v + \left(2a - c + \left(e + 2b - \frac{2bc}{a}\right)x\right) dv dx$$
$$+ \left(g + \left(1 - \frac{c}{a}\right)\left(\frac{e}{b} - \frac{c}{a}\right)b\right) v dx^2,$$

vbi est $y = x^{1 - \frac{c}{a}} . v$.

Ex transformatione (2) sequitur, posito primum $p = 1 - \frac{c}{a}$, deinde $p = o$:

$$2)\ o = x(a + bx) d^2v + (2a - c + (4b - e)x) dv dx + (g + 2b - e) v dx^2;$$

existente $y = x^{1 - \frac{c}{a}} (a + bx)^{1 + \frac{c}{a} - \frac{e}{b}} . v$;

$$3)\ o = x(a + bx) d^2 v + x\left(c + \left(2b + \frac{2bc}{a} - e\right)x\right) dv dx$$
$$+ \left(g + \frac{bc}{a}\left(1 + \frac{c}{a} - \frac{e}{b}\right)\right) v dx^2;$$

existente $y = (a + bx)^{1 + \frac{c}{a} - \frac{e}{b}} . v$.

Corol-

Corollarium 3.

§. IX. Posito $a+bx=-bX$, aequatio:

$$o=x(a+bx)d^2y+(c+ex)dydx+gydx^2 \quad \text{abit in hanc:}$$

$$o=X(a+bX)d^2y+\left(-c+\frac{ae}{b}+eX\right)dydX+gydX^2;$$

quae ipsa rursus triplici ratione, ex §pho praecedente, transformari potest.

Corollarium 4.

§. X. Transformationes tres §. VI. inuentas ad aequationem

$$o=x^2(a+bx^n)d^2y+x(c+ex^n)dydx+(f+gx^n)ydx^2$$

extendi posse, ex §. V. facile intelligitur. Ita oriuntur aequationes transformatae sequentes:

Aequatio transformata 1.

$$o=x^2(a+bx^n)d^2v+x(c+2pa+(e+2pb)x^n)dvdx+$$
$$(f+pc+p(p-1)a+(g+pe+p(p-1)b)x^n)vdx^2;$$

posito $y=x^p.v.$

Aequatio transformata 2.

$$o=x^2(a+bx^n)d^2v+x\left(c+2pa+\left(2\left(p+n+\frac{c}{a}\right)b-e\right)x^n\right)dvdx$$
$$+\left(f+pc+p(p-1)a+\left(g+b\left(p+n+\frac{c}{a}-\frac{e}{b}\right)\left(p+n+\frac{c}{a}-1\right)\right)x^n\right)vdx^2;$$

posito $y=x^p(a+bx^n)^{1+\frac{c}{na}-\frac{e}{nb}}.v.$

Aequatio transformata 3.

$$o=X^2(a+bX^n)d^2v+X\left(-c+a\left(\frac{e}{b}+2nq-n+1\right)+(e+2b(p+q)n)X^n\right)dvdX$$
$$+\left(anq\left(\frac{e}{b}-\frac{c}{a}+n(q-1)\right)+(g+n(p+q)(e+b(n(p+q)-1)))X^n\right)vdX^2;$$

posito $y=x^{np}(a+bx^n)^q.v$; $a+bx^n=-bX^n$; $f+pnc+pn(pn-1)a=o$; et sumto dX pro differentiali constanti.

Scholion.

§ XI. Transformationes prima et secunda ex iis etiam deriuari possunt, quae extant apud EVLERVM (Inst. Calc. Integr. Vol. II. Probl. 125. pag. 254. 256.), quaeque hunc in vsum amplius euoluit Auctor supra (§. III. *Not.* *) laudatus. *Tertiam* vero transformationem, et ipsam deinceps vtiliter adhibendam, vterque praetermisit. Ceterum

rum EVLERVS de vſu transformationum in integranda aequatione noſtra ſic ſtatuit (l. c. pag. 272): "*ope huiusmodi transformationum vix vnquam nouos caſus integrabiles erui poſſe;*" quam ſententiam minus veram eſſe, ex Capite ſequenti ſponte elucescet, quo quippe omnino noui caſus integrabiles eruentur.

Problema III.

§. XII. Aequationem $o = x(a+bx)d^2y + (c+ex)dydx + gydx^2$ (§. VII.) ad aliam, ſimilis formae, reducere.

Solutio.

1) Conſideretur aequatio ſimilis propoſitae:

$$o = x(a+bx)d^2z + (c^I + e^Ix)dzdx + g^Izdx^2,$$

eiusque differentiale quoduis r^{tum} ſumatur; erit:

$$d^r((ax+bx^2)d^2z) = (ax+bx^2)d^{r+2}z + r(a+2bx)d^{r+1}z.dx + \frac{r(r-1)}{1.2}2bd^rz.dx^2;$$

$$d^r((c^I+e^Ix)dzdx) = (c^I+e^Ix)d^{r+1}z.dx + re^Id^rz.dx^2$$

$$d^r(g^Izdx^2) = g^Id^rz.dx^2.$$

Hinc obtinetur:

$$o = x(a+bx)d^{r+2}z + (ra+c^I+(2rb+e^I)x)d^{r+1}z.dx + (r(r-1)b+re^I+g^I)d^rz.dx^2.$$

2) Quae iam aequatio vt cum propoſita conueniat, ponatur $\frac{d^rz}{dx^r} = y$; $ra+c^I = c$; $2rb+e^I = e$; $r(r-1)b+re^I+g^I = g$; et erit $c^I = c - ra$; $e^I = e - 2rb$; $g^I = g - re^I - r(r-1)b = g - re + r(r+1)b$. Inde haec prodit

Reductio fundamentalis.

Aequatio $o = x(a+bx)d^2y + (c+ex)dydx + gydx^2$ reducitur ad hanc:

$$o = x(a+bx)d^2z + (c - ra + (e-2rb)x)dzdx + (g + r(r+1)b - re)zdx^2;$$

denotante r quemuis numerum *integrum*, et poſito $y = \frac{d^rz}{dx^r}$.

Corollarium.

§. XIII. Ex hac reductione, quae, reſpectu ad ſequentes habito, *prima* vocatur, tres nouae deriuari poſſunt, dum binae aequationes differentiales (XII. 2.), quarum prior *reducenda*, altera *reducta* appelletur, tribus modis §. VIII. exhibitis *transformentur*.

1) Quodſi nimirum aequatio propoſita, priusquam ea reducatur, ex (§. VIII. 1.) transformetur, ex transformata orietur aequatio reducta haec (§. XII.):

o = x

$$0 = x(a+bx)d^2z + (2a - c - ra + (e + 2b - \frac{2bc}{a} - 2rb)x)dzdx$$
$$+ (g + (1 - \frac{c}{a})(\frac{e}{b} - \frac{c}{a})b + r(r+1)b - re - 2rb + \frac{2rbc}{a})zdx^2$$

existente $y = x^{1-\frac{c}{a}}.v$, et $v = \frac{d^rz}{dx^r}$. Quae aequatio rursus ex (§. VIII. 1.) transformetur, tum prodibit transformatae membri secundi coëfficiens primus $= 2a - (2a - c - ra) = c + ra$; coëfficiens alter $= e + 2b - \frac{2bc}{a} - 2rb + 2b - \frac{2b}{a}(2a - c - ra) = e$; membri tertii coëfficiens

$$= g + (1 - \frac{c}{a})(\frac{e}{b} - \frac{c}{a})b + r(r-1)b - re + \frac{2rbc}{a}$$
$$+ (-1 + \frac{c}{a} + r)(\frac{e}{b} + 2 - \frac{2c}{a} - 2r - 2 + \frac{c}{a} + r)b$$
$$= g + (1 - \frac{c}{a})(\frac{e}{b} - \frac{c}{a})b + r(r-1)b - re + \frac{2rbc}{a}$$
$$- (r-1)rb + (r-1)(\frac{e}{b} - \frac{c}{a})b + \frac{c}{a}(\frac{e}{b} - \frac{c}{a} - r)b$$

$= g$, sublatis partibus se mutuo destruentibus. Huius aequationis postremae quantitas incognita vocetur Z, eritque $z = x^{-1+\frac{c}{a}+r}.Z$. Inde haec oritur

Reductio secunda.

Aequatio $0 = x(a+bx)d^2y + (c+ex)dydx + gydx^2$ reducitur ad hanc:
$$0 = x(a+bx)d^2Z + (c + ra + ex)dZdx + gZdx^2;$$
existente $y = x^{1-\frac{c}{a}} \frac{d^r\left(x^{r-1+\frac{c}{a}}.Z\right)}{dx^r}$.

2) Simili ratione adplicetur transformatio (§. VIII. 2.) ad reductionem (§. XII.); tum erit reducta primae transformatae:
$$0 = x(a+bx)d^2z + (2a - c - ra + (4b - e - 2rb)x)dzdx$$
$$+ (g + 2b - e + r(r+1)b - (4b-e)r)zdx^2;$$
et $y = x^{1-\frac{c}{a}}(a+bx)^{1+\frac{c}{a}-\frac{e}{b}} d^rz : dx^r$; qua reducta similiter transformata, fit membri secundi coëfficiens primus $= 2a - (2a - c - ra) = c + ra$; coëfficiens alter $= 4b - (4b - e - 2rb) = e + 2rb$; membri tertii coëfficiens $=$
$g + 2b$

$$g+2b-e+r(r+1)b-r(4b-e)+2b-(4b-e-2rb)=g+r(r-1)b+re.$$

Est porro $z = x^{1-2+\frac{c}{a}+r}\,(a+bx)^{1+2-\frac{c}{a}-r-4+\frac{e}{b}+2r}\,.\,Z$
$= x^{\frac{c}{a}+r-1}\,(a+bx)^{\frac{e}{b}-\frac{c}{a}+r-1}\,.\,Z.$ Inde nascitur haec

Reductio tertia.

Aequatio $0 = x(a+bx)d^2y + (c+ex)dydx + gydx^2$ redit ad hanc:

$$0 = x(a+bx)d^2Z + (c+ra+(e+2rb)x)dZdx + (g+r(r-1)b+re)Zdx^2;$$

posito $y = x^{1-\frac{c}{a}}(a+bx)^{1+\frac{c}{a}-\frac{e}{b}}\,.\,\dfrac{d^r\left(x^{\frac{c}{a}+r-1}(a+bx)^{\frac{e}{b}-\frac{c}{a}+r-1}.Z\right)}{dx^r}.$

3) Ope transformationis tertiae (§. VIII. 3.) fit reducta primae transformatae:

$$0 = x(a+bx)d^2z + x\left(c-ra+\left(2b+\frac{2bc}{a}-e-2rb\right)x\right)dzdx$$
$$+\left(g+\frac{bc}{a}\left(1+\frac{c}{a}-\frac{e}{b}\right)+r(r+1)b-r\left(2b+\frac{2bc}{a}-e\right)\right)zdx^2;$$

et pro transformata huius reductae, membri secundi coëfficiens primus $= c - ra$; coefficiens alter $= 2b + \frac{2bc}{a} - 2br - \left(2b + \frac{2bc}{a} - e - 2rb\right) = e$; membri tertii coëfficiens $= g + \frac{bc}{a}\left(1 + \frac{c}{a} - \frac{e}{b}\right) + r(r+1)b - r\left(2b + \frac{2bc}{a} - e\right) + b\left(\frac{c}{a} - r\right)\left(1 + \frac{c}{a} - r - 2 - \frac{2c}{a} + \frac{e}{b} + 2r\right) = g.$ Inde prodit

Reductio quarta.

Aequatio $0 = x(a+bx)d^2y + (c+ex)dydx + gydx^2$ redit ad hanc:

$$0 = x(a+bx)d^2Z + (c-ra+ex)dZdx + gZdx^2;$$

posito $y = (a+bx)^{1+\frac{c}{a}-\frac{e}{b}}\,\dfrac{d^r\left((a+bx)^{\frac{e}{b}-\frac{c}{a}+r-1}.Z\right)}{dx^r}.$

4) Aequationes reductae tertia et quarta (2. 3.) ex prima (§. XII.) et secunda (1) oriuntur, dum numero integro r valor negatiuus tribuitur.

Problema IV.

§. XIV. Ex reductionibus praecedentibus (§. XII. XIII.) nouas generaliores deducere.

Solu-

Solutio.

Harum reductionum binae inter se coniungi possunt, ita quidem, vt aequatio proposita primum ex vna formula reducatur, et reducta deinceps rursus ex altera formula. Sic nouae oriuntur reductiones, eaeque generaliores, quippe quae duos iam numeros integros arbitrarios r et ϱ inuoluunt. Pro reductione nimirum primo adhibita loco numeri integri r ponatur ϱ, pro altera autem litera r seruetur.

1) Qua ratione si reductiones secunda et prima (§. XIII. 1. XII.) combinentur, illaque primum, deinde haec aequationi propositae adplicetur: tum haec oritur

Reductio prima.

Aequatio $0 = x(a+bx)d^2y + (c+ex)dydx + gydx^2$ reducitur ad hanc:

$$0 = x(a+bx)d^2z + (c+(\varrho-r)a+(e-2rb)x)dzdx + (g-re+r(r+1)b)zdx^2;$$

existente $y = x^{1-\frac{c}{a}} \dfrac{d^\varrho\left(x^{\varrho-1+\frac{c}{a}} d^r z\right)}{dx^{\varrho+r}}$.

2) Ex simili combinatione reductionum quartae et primae (§. XIII. 3. XII.) prodit sequens

Reductio secunda.

Prior aequatio (1) redit ad hanc:

$$0 = x(a+bx)d^2z + (c-(\varrho+r)a+(e-2rb)x)dzdx + (g-re+r(r+1)b)zdx^2;$$

posito $y = (a+bx)^{1+\frac{c}{a}-\frac{e}{b}} \dfrac{d^\varrho\left((a+bx)^{\frac{e}{b}-\frac{c}{a}+\varrho-1} d^r z\right)}{dx^{\varrho+r}}$.

3) Combinentur porro reductiones secunda et tertia (§. XIII. 1. 2.), et obtinebitur

Reductio tertia,

pro qua est aequatio reducta:

$$0 = x(a+bx)d^2z + (c+(\varrho+r)a+(e+2rb)x)dzdx + (g+re+r(r-1)b)zdx^2;$$

substituendo

$$y = x^{1-\frac{c}{a}} \frac{d^\varrho\left((a+bx)^{1+\frac{c}{a}+\varrho-\frac{e}{b}} d^r\left(x^{\frac{c}{a}+\varrho+r-1}(a+bx)^{\frac{e}{b}-\frac{c}{a}-\varrho+r-1}.z\right)\right)}{dx^{\varrho+r}}.$$

4) Tandem ex combinatione reductionum quartae et tertiae (§. XIII. 3. 2.) consequitur

Reductio quarta.

Aequatio $0 = x(a+bx)d^2y + (c+ex)dydx + gydx^2$ reducitur ad hanc:

 $0 = x$

$$o = x(a+bx)d^2z + (c+(r-\varrho)a+(e+2rb)x)dzdx + (g+re+r(r-1)b)zdx^2;$$

posito

$$y = (a+bx)^{1+\frac{c}{a}-\frac{e}{b}} \frac{d^{\varrho}\left(x^{1-\frac{c}{a}+\varrho} d^{r}\left(x^{\frac{c}{a}-\varrho+r-1}(a+bx)^{\frac{e}{b}-\frac{c}{a}+\varrho+r-1} z\right)\right)}{dx^{\varrho+r}}.$$

Corollarium.

§. XV. 1) Aequationes quatuor reductas, praecedenti §pho exhibitas, hac *vna* comprehendere licet: $o = x(a+bx)d^2z + (c+(\varrho-r)a+(e-2rb)x)dxdz + (g-re+r(r+1)b)zdx^2$; modo obseruetur, numeros integros r et ϱ tam *affirmatiue* quam *negatiue* accipi posse. Alterutro eorum euanescente prodeunt reductiones §§. (XII) et (XIII). Sic igitur reductiones hactenus demonstratae ad *vnam* omnes redeunt, eamque ob valores τῶν r et ϱ indeterminatos late patentem.

2) Quodsi aequatio reducta complete integrabilis est, exinde simul innotescet integrale completum alterius aequationis; quippe expressiones pro y inuentae (§. XIV.) duas constantes arbitrarias inuoluent, ex valoribus τȣ̃ z ortas.

Scholion.

1) Enumeratio quatuor combinationum, (§. XIV.) licet incompleta, (sex enim locum habere combinationes constat), nostro tamen fini sufficit; quippe reductiones 1 et 3, 2 et 4 (§§. XII. XIII.) inuicem coniunctas haud quicquam noui praebere, statim intelligetur. Nec minus manifestum est, iteratas combinationes reductionum §. XIV. inuentarum haud nouas suppeditare reductiones.

2) Quanquam reductiones hactenus expositae ad aequationem $o = x(a+bx)d^2y + (c+ex)dydx + gydx^2$ spectent, eaedem tamen etiam ad aequationem generaliorem: $o = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)ydx^2$ transferri possunt; haec enim semper ad formam priorem reuocabilis est (§§. IV. VII.). Cum vero reductiones iam demonstratae ad eruendos casus integrabiles sufficiant, reductiones aequationis generalioris amplius euoluere minus necesse esse videtur. Transeamus potius ad id, cuius caussa maxime haec praemissa sunt, scilicet ad ipsam aequationis nostrae differentialis integrationem, quae quando et quomodo peragi queat, disquirendum est.

CAPVT II.

INVESTIGATIO CASVVM INTEGRABILIVM, ET EVOLVTIO INTEGRALIVM ILLIS RESPONDENTIVM.

ARTICVLVS PRIMVS.

Euolutio casus integrabilis primi;
vna cum obferuationibus nouis circa integrationem aequationis differentialis propofitae generalem eamque completam per feries faltem infinitas.

PROBLEMA V.

§. XVII. Integrare aequationem differentialem; $o = x(a+bx)d^2y + (c+ex)dydx$.

Solutio.

Ex $\frac{d^2y}{dy} = -\frac{(c+ex)dx}{x(a+bx)} = -\frac{cdx}{ax} + \frac{\left(\frac{cb}{a}-e\right)dx}{a+bx}$, fequitur per integrationem logarithmicam, ob dx conftans, $\log.\frac{dy}{dx} = \log. M - \frac{c}{a}\log. x + \frac{1}{b}\left(\frac{cb}{a}-e\right)\log.(a+bx)$, denotante M conftantem arbitrariam. Hinc porro fit $\frac{dy}{dx} = Mx^{-\frac{c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}}$, et rurfus fumendo integralia, $y = N + M\int x^{-\frac{c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}}dx$; quae formula ob binas conftantes N et M praebet integrale completum.

Corollarium.

§. XVIII. Aequatio $o = x(a+bx)d^2y + (c+ex)dxdx + gydx^2$ ope reductionis (§. XIII. 3.) ad hanc reducitur:
$o = x(a+bx)d^2z + (c+ra+(e+2rb)x)dzdx + (g+re+r(r-1)b)zdx^2$.
Pofito iam $g = -re - r(r-1)b$, aequatio reducta ex problemate praecedenti integrabilis erit; eft nimirum $z = N + M\int x^{-\frac{c}{a}-r}(a+bx)^{\frac{c}{a}-\frac{e}{b}-r}dx$. Hinc etiam aequationem priorem: $o = x(a+bx)d^2y + (c+ex)dydx - (re+r(r-1)b)ydx^2$ integrare licet, cuius integrale ex (§. XIII. 3.) fic exprimitur:

$$y = x^{1-\frac{c}{a}}(a+bx)^{1+\frac{c}{a}-\frac{e}{b}}\frac{d^r\left(x^{\frac{c}{a}+r-1}(a+bx)^{\frac{e}{b}-\frac{c}{a}+r-1}z\right)}{dx^r}.$$

 Pro

Pro z valor modo exhibitus ponitur, et pro r numerus quisque integer affirmatiuus sumi potest.

Problema VI.

§. XIX. Inuenire integrale completum aequationis:

$$o = x^2 (a + b x) d^2 y + x (c + e x) d y d x + (f + g x) y d x^2;$$

dum inter eius coëfficientes hae relationes locum habeant:

$$f = - p c - p (p - 1) a, \quad g = - e (p \pm r) - b (p \pm r) (p \pm r - 1);$$

denotante litera p quantitatem arbitrariam, et r numerum quemuis integrum.

Solutio.

Aequatio proposita ope transformationum (§. VI.) semper in aliam transmutari potest, cuius membri vltimi primus coëfficiens euanescat (§. VII.). Quo facto integratio praecedentis §phi adhibenda est. Iam pro ambiguitate signorum τῶ r (qui absolute semper numerum affirmatiuum significet) duo casus discernendi sunt, quorum primus ope transformationis primae (§. VI. 4.), alter per secundam (§. VI. 5.) tractetur.

1) Pro signo τῶ r affirmatiuo adhibeatur transformatio 1 (§. VI. 4.), et aequatio nostra mutabitur in hanc:

$$o = x^2 (a + b x) d^2 v + x (c + 2 p a + (e + 2 p b) x) d v d x$$
$$+ (f + p c + p (p - 1) a + (g + p e + p (p - 1) b) x) v d x^2;$$

posito $y = x^p . v$. Quae iam aequatio differentialis, vt ex §. VIII. integrabilis fiat, statuendum est primo: $f = - p c - p (p - 1) a$; deinde $g + p e + p (p - 1) b = - r (r - 1) b - r (e + 2 p b)$, siue $g = - e (p + r) - b (p (p - 1) + r (r - 1) + 2 p r) = - e (p + r) - b (p + r) (p + r - 1)$. Tum erit

$$v = x^{1 - \frac{c}{a} - 2p} (a + b x)^{1 + \frac{c}{a} - \frac{e}{b}} \frac{d^r \left(x^{\frac{c}{a} + 2p + r - 1} (a + b x)^{\frac{e}{b} - \frac{c}{a} + r - 1} . z \right)}{d x^r};$$

vnde sponte consequitur pro integrali aequationis propositae: $y = x^p v =$

$$x^{1 - \frac{c}{a} - p} (a + b x)^{1 + \frac{c}{a} - \frac{e}{b}} d^r \left(x^{\frac{c}{a} + 2p + r - 1} (a + b x)^{\frac{e}{b} - \frac{c}{a} + r - 1} . z \right) : d x^r.$$

Est autem z (§. XVIII.) $= N + M \int x^{- \frac{c}{a} - 2p - r} (a + b x)^{\frac{c}{a} - \frac{e}{b} - r} d x$.

2) Pro signo τῶ r negatiuo, ope transformationis 2 (§. VI. 5.), posito iterum $f = - p c - p (p - 1) a$, aequatio nostra abit in hanc:

$$0 = x(a+bx)d^2v + \left(c+2pa+\left(2\left(p+1+\frac{e}{a}\right)b-e\right)x\right)dv\,dx$$
$$+\left(g+b\left(p+1+\frac{c}{a}-\frac{e}{b}\right)\left(p+\frac{c}{a}\right)\right)v\,dx^2.$$

Quae vt ad formam integrabilem (§. XVIII.) reuocetur, ponendum est:

$$g+b\left(p+1+\frac{c}{a}-\frac{e}{b}\right)\left(p+\frac{c}{a}\right) = -r(r-1)b - r\left(2\left(p+1+\frac{c}{a}\right)b-e\right).$$

Loco quantitatis p introducatur alia P, vt fit $p+P = 1-\frac{c}{a}$; tum prima aequatio conditionalis pro f praebebit similem: $f = -Pc - P(P-1)a$; ex altera pro g prodit

$$g = -b\left(2-P-\frac{e}{b}\right)(1-P) - r(r-1)b - r(2(2-P)b-e)$$
$$= -b((2-P)(1-P)+r(r-1)+2r(2-P))+e(1-P+r)$$
$$= -b((P-1)^2+r^2-2r(P-1)-(P-1)+r)+e(1-P+r)$$
$$= -e(P-r-1)-b(P-1-r)(P-2-r).$$

Iam pro integrali aequationis transformatae obtinetur (§. XVIII.) $v =$

$$x^{1-\frac{c}{a}-2p}(a+bx)^{1+\frac{c}{a}+2p-2\left(p+1+\frac{c}{a}\right)+\frac{e}{b}} \cdot \frac{d^r\left(x^{\frac{c}{a}+2p+r-1}(a+bx)^{2\left(p+1+\frac{c}{a}\right)-\frac{e}{b}-\frac{c}{a}-2p+r-1}\cdot z\right)}{dx^r},$$

hincque fit

$$y = x^P(a+bx)^{1+\frac{c}{a}-\frac{e}{b}}\cdot v = x^{1-\frac{c}{a}-p}\, d^r\left(x^{\frac{c}{a}+2p+r-1}(a+bx)^{1+\frac{c}{a}-\frac{e}{b}+r}\cdot z\right);$$

vbi est $z = N + M\int x^{-\frac{c}{a}-2p-r}(a+bx)^{-2-\frac{c}{a}+\frac{e}{b}-r}$.

Ad seruandam analogiam cum integratione praecedente (1) scribatur r loco r+1, tum aequationes binae conditionales:

$$f = -Pc - P(P-1)a;\quad g = -e(P-r)-b(P-r)(P-r-1)$$

prioribus (1) omnino similes sunt, nisi quod in secunda numerus integer r signo negatiuo affectus sit. Quibus igitur aequationibus inter coëfficientes locum habentibus, integrale completum aequationis propositae sic exprimere licet:

$$y = x^P\,\frac{d^{r-1}\left(x^{-\frac{c}{a}-2P+r}(a+bx)^{\frac{c}{a}-\frac{e}{b}+r}\cdot z\right)}{dx^{r-1}};$$

existente $z = N + M\int x^{\frac{c}{a}+2P-r-1}(a+bx)^{\frac{e}{b}-\frac{c}{a}-r-1}\,dx.$ *Corol-*

Corollarium 1.

§. XX. Ex praecedenti problemate facile colligi poteſt, quando aequatio generalior:

$$0 = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)ydx^2$$

integrabilis futura ſit. Haec enim, poſito $x^n = \chi$, abit in ſequentem (§. IV.):

$$0 = \chi^2(a+b\chi)d^2y + \chi\left(\frac{a(n-1)+c}{n} + \left(\frac{b(r-1)+e}{n}\right)\chi\right)dyd\chi + \left(\frac{f}{n^2} + \frac{g\chi}{n^2}\right)yd\chi^2.$$

Quae aequatio transformata, indeque etiam prior, ex §. XIX. integrabilis erit duplici caſu.

1) *Primus* locum habet, ſi fuerit $\frac{f}{n^2} = -p\left(\frac{a(n-1)+c}{n}\right) - p(p-1)a$, ſiue $f = -anp(n-1+n(p-1)) - npc = -anp(np-1) - cnp$; porro $\frac{g}{n^2} = -(p+r)\left(\frac{b(n-1)+e}{n}\right) - (p+r)(p+r-1)b$, ſiue $g = -bn(p+r)(n(p+r)-1) - en(p+r)$. Integrale ſic exprimitur:

$$y = \chi^{\frac{1}{n} - \frac{c}{na} - p}(a+b\chi)^{1 + \frac{c}{na} - \frac{e}{nb}} \frac{d^r\left(\chi^{-\frac{1}{n} + \frac{c}{na} + 2p + r}(a+b\chi)^{\frac{e}{nb} - \frac{c}{na} + r - 1}.z\right)}{d\chi^r};$$

vbi eſt $z = N + M\int \chi^{-1 + \frac{1}{n} - \frac{c}{na} - 2p - r}(a+b\chi)^{\frac{c}{na} - \frac{e}{nb} - r}.d\chi$.

2) Pro *altero* caſu, ſiue ſigno τȣ r negatiuo ex §. XIX. 2, ponendum eſt: $f = -anP(nP-1) - cnP$; et $g = -bn(P-r)(n(P-r)-1) - en(P-r)$.

Tum prodit integrale hoc:

$$y = \chi^P.\frac{d^{r-1}\left(\chi^{\frac{1}{n} - \frac{c}{na} - 2P + r - 1}(a+b\chi)^{\frac{c}{na} - \frac{e}{nb} + r}.z\right)}{d\chi^{r-1}};$$

exiſtente $z = N + M\int \chi^{-\frac{1}{n} + \frac{c}{na} + 2P - r}(a+b\chi)^{\frac{e}{nb} - \frac{c}{na} - r - 1}$.

Corollarium 2.

§. XXI. Scripto (§. XX. 1.) p pro np, et (§. XX. 2.) p pro nP, ex hactenus demonſtratis haec tandem ſequitur

Integratio completa
Caſus primi.

Dum binos coëfficientes f, g hiſce aequationibus exprimere liceat:

$$f = -ap(p-1) - cp; \qquad g = -b(p \pm nr)(p \pm nr - 1) - e(p \pm nr);$$

deno-

denotante p quantitatem arbitrariam, et r numerum quemuis *integrum:* integrale completum aequationis differentialis

$$o = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)ydx^2$$

formulis sequentibus exhibetur:

1) Pro signo τȣ r affirmatiuo:

$$y=\chi^{\frac{1}{n}\left(1-\frac{c}{a}-p\right)}(a+b\chi)^{1+\frac{c}{na}-\frac{e}{nb}}\frac{d^r\left(\chi^{\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+r}(a+b\chi)^{\frac{e}{nb}-\frac{c}{na}+r-1}.z\right)}{d\chi^r};$$

existente $z = N + M\int\chi^{-\frac{1}{n}\left(\frac{c}{a}+2p-1\right)-r-1}(a+b\chi)^{\frac{c}{na}-\frac{e}{nb}-r}.d\chi$.

2) Pro signo negatiuo:

$$y=\chi^{\frac{p}{n}}.\frac{d^{r-1}\left(\chi^{-\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+r-1}(a+b\chi)^{\frac{c}{na}-\frac{e}{nb}+r}.z\right)}{d\chi^{r-1}};$$

posito $z = N + M\int\chi^{\frac{1}{n}\left(\frac{c}{a}+2p-1\right)-r}(a+b\chi)^{\frac{e}{nb}-\frac{c}{na}-r-1}.d\chi$.

Vtrinque $\chi = x^n$, et in formulis quidem integralibus $d\chi$, in ipsa contra aequatione integrata dx pro differentiali constanti habetur. N et M sunt Constantes arbitrariae.

Corollarium 3.

§. XXII. Casus, quo b vel a euanescit, seorsim tractandus est. Tum quidem integratio fundamentalis (§. XVII.) rite instituta quantitates exponentiales inuoluet. Hinc etiam ceterae integrationes exinde deriuatae alterantur. At vero hoc regressu haud opus est, quippe ipsae formulae generales (§. XXI.) positioni b vel a = o adaptari possunt, modo notetur, esse pro quantitate ω infinite parua seu euanescente, $(1+\omega u)^{\frac{1}{\omega}} = E^u$, vbi E denotat basin logarithmorum hyperbolicorum (*). Inde habetur pro b = o,

$$\left(1+\frac{b}{a}\chi\right)^{\frac{e}{nb}} = \left(1+\frac{nb}{e}.\frac{e\chi}{an}\right)^{\frac{e}{nb}} = E^{\frac{e\chi}{an}}.$$ Quare prodit

1) pro

(*) *Euleri* Introductio in Anal. inf. Tom. I. pag. 92. Basis logarithmorum hyperb. communiter litera e insignitur: quae vero cum hoc loco coëfficientem denotet, pro basi illa literam maiorem E adhibui.

1) pro ſigno τ8 r affirmatiuo:

$$y = \chi^{\frac{1}{n}\left(1-\frac{c}{a}-p\right)} . E^{-\frac{e\chi}{an}} \frac{d^r\left(\chi^{\frac{1}{n}\left(\frac{c+2p-1}{a}\right)+r} . E^{\frac{e\chi}{an}} . z\right)}{d\chi^r};$$

exiſtente $z = N + M\int \chi^{-\frac{1}{n}\left(\frac{c+2p-1}{a}\right)-r-1} . E^{-\frac{e\chi}{an}} d\chi$.

2) pro ſigno negatiuo:

$$y = \chi^{\frac{p}{n}} \frac{d^{r-1}\left(\chi^{-\frac{1}{n}\left(\frac{c+2p-1}{a}\right)+r-1} . E^{-\frac{e\chi}{an}} . z\right)}{d\chi^{r-1}};$$

poſito $z = N + M\int \chi^{\frac{1}{n}\left(\frac{c+2p-1}{a}\right)-r} . E^{\frac{e\chi}{an}} . d\chi$.

Sic igitur integranda eſt aequatio differentialis;

$$0 = ax^2d^2y + x(c + ex^n)dydx + (f + gx^n)ydx^2 = 0;$$

poſito ſimul $f = -ap(p-1) - cp$, et $g = -e(p \pm nr)$, ſiue haſce aequationes in vnam contrahendo, $f = a\left(\frac{g}{e} + nr\right)\left(\frac{c}{a} - 1 - \frac{g}{e} - nr\right)$. Simili ratione caſus $a = 0$ tractandus eſt. Quique etiam ad priorem $b = 0$ reduci poteſt: diuidendo enim per x^n, aequatio pro $a = 0$ hanc formam induit: $0 = bx^2d^2y + x(e + cx^{-n})dydx + (g + fx^{-n})ydx^2$, quam ex formulis modo exhibitis integrare licet.

Scholion.

§. XXIII. Integratio §pho XXI. exhibita ſiſtit caſum ſatis iam notum; quemque, vti ſupra dictum (§. II.), pro affirmatiuo valore τ8 r ample expoſuit, pro negatiuo breuiter ſaltem attigit EVLERVS. Methodus, qua hic caſus a ceteris etiam auctoribus ſupra laudatis demonſtrari ſolet, nititur aſſumtione ſeriei indefinitae, cuius exponentes et coëfficientes determinantur: vnde etiam ipſum integrale per ſeriem, ſcilicet abrumpentem, exprimitur. Noua methodus, hoc loco adhibita, eam quoque commendationem habere videtur, quod expreſſiones inde deriuatae formam nactae ſint analytice ſimpliciorem, eaedemque ſtatim exhibeant integrale *completum*. Nec vero ſuperfluum eſſe videtur, oſtendere, quo pacto ex hiſce formulis integralia *particularia* concinnius expreſſa, ſimulque ipſas ſeries vſitatas deducere liceat. Quod quidem in ſequenti problemate, eiusque corollariis, perficietur.

PROBLE-

Problema VII.

§. XXIV. Aequationis differentialis §. XXI. *complete* integratae, exhibere integrale *particulare.*

Solutio.

Cum in formulis integralibus §. XXI. inuentis quantitates N et M sint constantes arbitrariae, earum vnam M = o ponere licet; vnde consequuntur integralia particularia. Est nimirum z = N, hinc prodit

1) pro signo numeri integri r affirmatiuo:

$$y = N\chi^{\frac{1}{n}\left(1-\frac{c}{a}-p\right)}(a+b\chi)^{1+\frac{c}{na}-\frac{e}{nb}} \cdot \frac{d^r\left(\chi^{\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+r}(a+b\chi)^{\frac{e}{nb}-\frac{c}{na}+r-1}\right)}{d\chi^r},$$

2) pro signo negatiuo:

$$y = N\chi^{\frac{p}{n}}\frac{d^{r-1}\left(\chi^{-\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+r-1}(a+b\chi)^{\frac{c}{na}-\frac{e}{nb}+r}\right)}{d\chi^{r-1}}$$

Vtrinque est $\chi = x^n$, et N constans arbitraria.

Corollarium 1.

§. XXV. 1) Superest, vt ex hisce formulis satis concinnis series vsitatas, quibus integralia exprimuntur, deducamus. Quem in finem differentialia altiora productorum ex potestatibus τȣ χ in potestates τȣ $a+b\chi$, euoluenda sunt. Consideremus *primo* casum (§. XXIV. 1.) signi τȣ r affirmatiui. Iam constat, esse $d^r(uv) = u\,d^r v + r\,du\,d^{r-1}v + \frac{r(r-1)}{1 \cdot 2}d^2u\,d^{r-2}v + \text{etc.}$; porro $d^\mu \chi^\nu = \nu(\nu-1)\ldots(\nu-\mu+1)\chi^{\nu-\mu}\cdot d\chi^\mu$, et $d^\mu(a+b\chi)^\nu = \nu(\nu-1)\ldots(\nu-\mu+1)(a+b\chi)^{\nu-\mu}\cdot b^\mu d\chi^\mu$.

2) Quibus praemissis erit $\dfrac{d^r\left((a+b\chi)^{\frac{e}{nb}-\frac{c}{na}+r-1}\chi^{\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+r}\right)}{d\chi^r}$

X $= (a+b\chi)$

$$\pm(a+b\chi)^{\frac{e}{nb}-\frac{c}{na}+r-1}\cdot\frac{\left(\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+r\right)\left(\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+r-1\right)\ldots}{\left(\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+1\right)\chi^{\frac{1}{n}\left(\frac{c}{a}+2p-1\right)}}$$

$$+r\left(\frac{e}{nb}-\frac{c}{na}+r-1\right).b(a+b\chi)^{\frac{e}{nb}-\frac{c}{na}+r-2}\frac{\left(\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+r\right)\ldots}{\left(\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+2\right)\chi^{\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+1}}$$

$$+\frac{r(r-1)}{1.2}\left(\frac{e}{nb}-\frac{c}{na}+r-1\right)\left(\frac{e}{nb}-\frac{c}{na}+r-2\right)b^2(a+b\chi)^{\frac{e}{nb}-\frac{c}{na}+r-3}\cdot\frac{\left(\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+r\right)\ldots}{\left(\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+3\right)\chi^{\frac{1}{n}\left(\frac{c}{a}+2p-1\right)+2}}$$

$$+\text{etc.}\qquad\qquad+\text{etc.}$$

Coëfficientem membri huius expressionis primi ad Constantem N trahendo, ac restituendo x pro $\chi^{\frac{1}{n}}$, prodit inde:

$$y=Nx^p(a+bx^n)^r\left\{1+r\frac{\left(\frac{e}{b}-\frac{c}{a}+n(r-1)\right)}{\frac{c}{a}+2p-1+n}\cdot\frac{bx^n}{a+bx^n}+\frac{r(r-1)}{1.2}\frac{\left(\frac{e}{b}-\frac{c}{a}+n(r-1)\right)\left(\frac{e}{b}-\frac{c}{a}+n(r-2)\right)}{\left(\frac{c}{a}+2p-1+n\right)\left(\frac{c}{a}+2p-1+2n\right)}\cdot\left\{\frac{bx^n}{a+bx^n}\right\}^2+\text{etc.}\right\}$$

Cuius seriei, termino $r+1^{to}$ desinentis, lex progressus euidens est.

3) Haec series, secundum potestates τῆ $\frac{bx^n}{a+bx^n}=s$ procedens, in aliam transformari potest, secundum potestates ipsius x progredientem: dum potestates r^{ta}, $r-1^{ta}$, $r-2^{ta}$ . . . τῆ $a+bx^n$ consueta ratione euoluantur. Breuitatis gratia designatis in serie

ferie pro y (2) quantitatibus, quae ducuntur in rs, $\frac{r(r-1)}{1.2}s^2$, etc. per $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, . . . ; erit

$$\frac{y}{Nx^p} = a^r + r\left[a^{r-1}bx^n + r\mathfrak{B}\right] + \frac{r(r-1)}{1.2}\left[a^{r-2}b^2x^{2n} + r(r-1)\mathfrak{B} + \frac{r(r-1)}{1.2}\mathfrak{C}\right] + \frac{r(r-1)(r-2)}{1.2.3}\left[a^{r-3}b^3x^{3n} + \frac{r(r-1)(r-2)}{1.2}\mathfrak{B} + \frac{r(r-1)(r-2)}{1.2}\mathfrak{C} + \frac{r(r-1)(r-2)}{1.2.3}\mathfrak{D}\right] + \ldots$$

vbi coëfficientium feries verticales in fummas redigi poffunt; quippe pro termino quouis $m+1^{to}$ eft fumma coëfficientium partialium =

$$\frac{r(r-1)\ldots(r-m+1)}{1.2\ldots m}\left(1+m\mathfrak{B}+\frac{m(m-1)}{1.2}\mathfrak{C}+\frac{m(m-1)(m-2)}{1.2.3}\mathfrak{D}+\ldots\right)$$

$$=\frac{r(r-1)\ldots(r-m+1)}{1.2\ldots m}\cdot\frac{\left(\frac{e}{nb}+\frac{2p-1}{n}+r\right)\left(\frac{e}{nb}+\frac{2p-1}{n}+r+1\right)\ldots\left(\frac{e}{nb}+\frac{2p-1}{n}+r+m-1\right)}{\left(\frac{c}{na}+\frac{2p-1}{n}+1\right)\left(\frac{c}{na}+\frac{2p-1}{n}+2\right)\ldots\left(\frac{c}{na}+\frac{2p-1}{n}+m\right)}.$$

Hinc prodit: $y =$

$$Na^rx^p\left(1+r\frac{(e+(2p-1+rn)b)}{c+(2p-1+n)a}\cdot x^n+\frac{r(r-1)}{1.2}\cdot\frac{(e+(2p-1+rn)b)(e+(2p-1+(r+1)n)b)}{(c+(2p-1+n)a)(c+(2p-1+2n)a)}x^{2n}+\ldots\right);$$

fiue $y = Ax^p + Bx^{p+n} + Cx^{p+2n} +$ etc., dum coëfficientes fequenti ratione procedant, primo earum ad arbitrium fumto:

$$(c+(2p-1+n)a)B = r(e+(2p-1+rn)b)A$$
$$2(c+(2p-1+2n)a)C = (r-1)(e+(2p-1+(r+1)n)b)B$$
$$3(c+(2p-1+3n)a)D = (r-2)(e+(2p-1+(r+2)n)b)C$$

etc. etc.

Lex, quam EVLERVS inuenit (l. c. p. 222.), aliter expreffa eft, vti ex his aequationibus apparet:

$$((p+n)(p+n-1)a+(p+n)c+f)B = -(p(p-1)b+pe+g)A$$
$$((p+2n)(p+2n-1)a+(p+2n)c+f)C = -((p+n)(p+n-1)b+(p+n)e+g)B$$
$$((p+3n)(p+3n-1)a+(p+3n)c+f)D = -((p+2n)(p+2n-1)b+(p+2n)e+g)C$$

etc. etc.

Exinde autem, fuppofitis loco f, g valoribus: $f = -ap(p-1)-cp$, $g = -b(p+nr)(p+nr-1)-e(p+nr)$, ipfae noftrae aequationes prodeunt.

Corollarium 2.

§. XXVI. Quod porro *alterum* casum, signi τῦ r negatiui, attinet: formula integralis pro hoc casu (§. XXIV. 2.) ex formula pro casu priore (§. XXIV. 1.) prodit, dum in hac ponatur pro r, r − 1; pro n, − n; pro e, e − 2nb; eaque deinde multiplicetur per $\chi^{\frac{1}{n}\left(1-\frac{c}{a}\right)} \cdot (a+b\chi)^{1+\frac{c}{na}-\frac{e}{nb}}$. Hinc peculiari euolutione integralis in seriem pro hoc casu haud opus est: sed ex iis iam, quae §pho praecedente demonstrata sunt, consequitur haec series:

$$y = (a+bx^n)^{1+\frac{c}{na}-\frac{e}{nb}} \left(Ax^{1-\frac{c}{a}-p} + Bx^{1-\frac{c}{a}-p+n} + Cx^{1-\frac{c}{a}-p+2n} \right.$$

[illegible] $\left. + \text{etc.} \right)$;

vbi lex coëfficientium A, B, C, D . . . hisce aequationibus exprimitur:

$$(c+(2p-1-n)a)B = (r-1)(e+(2p-1-(r+1)n)b)A$$
$$2(c+(2p-1-2n)a)C = (r-2)(e-(2p-1-(r+2)n)b)B$$
$$3(c+(2p-1-3n)a)D = (r-3)(e-(2p-1-(r+3)n)b)C$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

quae etiam sic exhiberi possunt:

$$((p-n)(p-n-1)a+(p-n)c+f)B = -((p-n)(p-n-1)b+(p-n)e+g)A$$
$$((p-2n)(p-2n-1)a+(p-2n)c+f)C = -((p-2n)(p-2n-1)b+(p-2n)e+g)B$$
$$((p-3n)(p-3n-1)a+(p-3n)c+f)D = -((p-3n)(p-3n-1)b+(p-3n)e+g)C$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

Est nimirum $f = -ap(p-1) - cp$; $g = -b(p-nr)(p-nr-1) - e(p-nr)$.

Corollarium 3.

§. XXVII. 1) Conditiones integrabilitatis pro casu hactenus (§. XXI–XXVI.) pertractato, hoc etiam modo enuntiari possunt. Ex aequationibus: $f = -ap(p-1) - cp$, $g = -b\pi(\pi-1) - e\pi$, definiantur valores quantitatum p et π; tum $\frac{\pi - p}{n}$ debet aequari numero integro ± r, siue affirmatiuo siue negatiuo. Notandum hoc loco est, vtramque istarum aequationum binas habere radices: quippe, posito $\pi + \pi^I = 1 - \frac{e}{b}$, $p + p^I = 1 - \frac{c}{a}$, satisfaciunt non minus valores π^I, p^I, ac π, p.

2) Respiciendo ad hanc duplicitatem radicum, integralia ipsa alia forma exhiberi possunt. Primo quidem pro signo τῦ r affirmatiuo ponatur $\frac{\pi - p^I}{n} = r^I$, cum sit $\frac{\pi - p}{n} = r$; porro $\frac{p - p^I}{n} = k$; tum erit $nk = \frac{c}{a} + 2p - 1$; $\frac{e}{b} - 1 + 2p = -\pi - \pi^I + 2p$

$\pi^I + 2p = -nr + p - \pi^I = -nr + n(k - r^I)$; $\frac{c}{a} - \frac{e}{b} = \pi + \pi^I - p - p^I = n(r + r^I)$. Hinc, posito insuper $\frac{b}{a} = \beta$, prodit integrale ex §. XXI. 1:

$$y = \chi^{\frac{p^I}{n}} (1 + \beta\chi)^{1 + r + k} \frac{d^r \left(\chi^{k + r} (1 + \beta\chi)^{-r^I - 1} . z \right)}{d\chi^r},$$

existente $\chi = x^n$ et $z = N + M \int \chi^{-k - r - 1} (1 + \beta\chi)^{r^I} d\chi$.

Series (§. XXV. 3) hanc praebet expressionem:

$$y = x^p \left(1 + \frac{r(k - r^I)}{k + 1} \beta x^n + \frac{r(r - 1)}{1.2} \frac{(k - r^I)(k - r^I + 1)}{(k + 1)(k + 2)} \beta^2 x^n + \ldots \right)$$

3) Pro signo τȣ r negatiuo, $\frac{\pi - p}{n}$ aequatur numero negatiuo, qui ponatur $= -\varrho$; porro sit $\frac{\pi^I - p^I}{n} = -\varrho^I$, et $\frac{p - p^I}{n}$ rursus $= k$. Tum habetur ex §. XXI. 2. $y = \chi^{\frac{p}{n}} . d^{\varrho - 1} \left(\chi^{-k + \varrho - 1} (1 + \beta\chi)^{-\varrho^I} . z \right) : d\chi^{\varrho - 1}$, sumto $\chi = x^n$, et $z = N + M \int \chi^{k - \varrho} (1 + \beta\chi)^{\varrho^I - 1} d\chi$. Ex serie (§. XXVI.) prodit, omisso factore arbitrario constante:

$$y = x^{p^I} (1 + \beta x^n)^{1 - \varrho - \varrho^I} \left(1 + (\varrho - 1) \frac{(\varrho^I + k - 1)}{k - 1} \beta x^n + \frac{(\varrho - 1)(\varrho - 2)}{1.2} \frac{(\varrho^I + k - 1)(\varrho^I + k - 2)}{(k - 1)(k - 2)} \beta^2 x^{2n} + \ldots \right)$$

4) Obseruatio de duplicitate valorum τῶν p, π, quae deinceps saepius in vsum vocabitur, accuratius euolui meretur, respectu habito ad formulas superiores (§. XXVII. 2, 3.). Permutatis *primum* p et p^I, simulque π et π^I, mutabitur r in r^I, r^I in r, k in $-k$. *Deinde* permutatis tantum inuicem p et p^I, nec simul π et π^I, abit $\frac{\pi - p}{n} = r$ in $\frac{\pi - p^I}{n} = \frac{\pi - p}{n} + \frac{p - p^I}{n} = r + k$; $\frac{\pi^I - p^I}{n} = r^I$ in $\frac{\pi^I - p}{n} = \frac{\pi^I - p^I + p^I - p}{n} = r^I - k$; k in $-k$. *Tertio*, permutando tantum π et π^I, mutabitur $\frac{\pi - p}{n} = r$ in $\frac{\pi^I - p}{n} = r^I - k$; $r^I = \frac{\pi^I - p^I}{n}$ in $\frac{\pi - p^I}{n} = r + k$; k valorem seruabit. Quae mutationes coniunctim hac forma exhibentur:

Loco	ponere	licet	
p	p^I	p^I	p
π	π^I	π	π^I
r	r^I	$r+k$	r^I-k
r^I	r	r^I-k	$r+k$
k	$-k$	$-k$	k

5) Ex quantitatibus assumtis p, r . . . coëfficientes c, e, f, g sic exprimi possunt:

$$\frac{c}{a}=1-p-p^I=1+nk-2p;$$

$$\frac{e}{b}=1-\pi-\pi^I=1+n(k-r-r^I)-2p;$$

$$\frac{f}{a}=pp^I=p(p-nk);$$

$$\frac{g}{b}=\pi\pi^I=(p+nr)(p-nk+nr^I).$$

Scholion I.

§. XXVIII. 1) Casus integrabilis, qui oritur, sumto $\frac{\pi-p}{n}$ (quae differentia, siue affirmatiua sit siue negatiua, in sequentibus semper litera r denotetur) = numero integro *negatiuo*, in priore EVLERI commentatione (Comment. Petrop. T. X.) omissus est: eundemque auctores supra laudati (§. II.), BOUGAINVILLE, LE SEUR et JACQUIER, praetermiserunt. In Institt. Calc. Integr. et Nou. Comment. Petrop. Tom. XVII. iste casus ab EVLERO ita tractatur, vt aequatio differentialis per substitutionem in aliam transformetur, pro quo iam sumi possit r = numero integro *affirmatiuo:* vnde hac existente integrabili, illam quoque integrabilem esse concluditur. Similis solutio apud COUSINIVM (l. c.) reperitur.

2) Quae solutio quanquam rite se habeat, notatione tamen dignum videtur, idque nondum animaduersum video, quod absque praeuia transformatione integrabilitas aequationis pro negatiuo r ex ipsa serie vsitata immediate sequatur: quae quippe series non tantum pro affirmatiuo r abrumpit, verum etiam pro integro *negatiuo* r^I finitam summam habet. Est nimirum series pro y (quae indefinite sumta integrale exprimit, siue r fuerit numerus integer, siue non):

$$x^p\left(1+\frac{r(k-r^1)}{k+1}\beta x^n+\frac{r(r-1)}{1.2}\frac{(k-r^1)(k-r^1+1)}{(k+1)(k+2)}\beta^2x^{2n}+\text{etc.}\right)$$

aequalis seriei huic:

$$x^p(1+\beta x^n)^{1+r+r^I}\left(1-\frac{(r^1+1)(r+k+1)}{k+1}\beta x^n+\frac{(r^1+1)(r^1+2)}{1.2}\frac{(r+k+1)(r+k+2)}{(k+1)(k+2)}\beta^2x^{2n}-\text{etc.}\right)$$

Quae

Quae aequalitas, quanquam aliunde deduci queat, hoc tamen loco breuius sic demonstrari potest. Prior series ex valore

$$y=\chi^{\frac{p^1}{n}}(1+\beta\chi)^{1+r+r^1}\, d^r\left(\chi^{k+r}(1+\beta\chi)^{-r^1-1}\right):d\chi^r$$

per factorem constantem diuiso eliciebatur (§. XXV. XXVII.); iam vero idem valor similiter diuisus suppeditat alteram seriem, dum $\chi^{k+r}(1+\beta\chi)^{-r^1-1}$ ex formula binomiali euoluatur, et differentiale r^{tum} consueta ratione sumatur. Quarum serierum mutuo sibi aequipollentium (non tantum pro integro r, verum etiam, quod sponte exinde sequitur, pro quocunque valore τοῦ r), primam pro affirmatiuo r, alteram pro negatiuo r^1 abrumpi, euidens est. Facile autem intelligitur, quantitates π et π^1, p et p^1, hincque etiam r et r^1 inuicem permutari posse, vbi ex k fit $-$ k: sicque altera series abit in eam ipsam, quae §. XXVII. (3) exhibita est.

3) Nec minus notanda videtur series supra (§. XXV. 2.) inuenta, quaeque sic exprimi potest:

$$y=x^p(1+\beta x^n)^r\left\{1-\frac{r(r^1+1)}{k+1}\,\frac{\beta x^n}{1+\beta x^n}+\frac{r(r-1)}{1.2}\,\frac{(r^1+1)(r^1+2)}{(k+1)(k+2)}\left\{\frac{\beta x^n}{1+\beta x^n}\right\}^2-\ldots\right\}$$

Haec enim series, tam pro affirmatiuo r quam pro negatiuo r^1 abrumpens, ita est comparata, *vt ea simul binos casus comprehendat*, quin reductione vnius ad alterum, vel peculiari pro vtroque integralis expressione opus sit: quod quidem commodum formulae vsitatae haud praestant.

Scholion 2.

§. XXIX. 1) Expressiones pro integrali y hactenus inuentae, quanquam ex suppositione numeri r tanquam integri deriuatae sunt, nihilominus tamen vniuersaliter obtinent: iisque integrale semper per series saltem infinitas exhibetur. Id quidem iam exinde apparet, quod series pro y in aequatione differentiali substituta, hanc identicam reddere debeat: haec autem identitas perinde se habet, siue r fuerit numerus integer, siue non, nec ea, si pro integris r adest, cessare potest pro non integris.

2) Modus, quo series pro integrali y communiter demonstrari solent, supra iam (§. XXIII.) breuiter indicatus est. De quo vt exactius constet, addendum est, ab EVLERO, ceterisque auctoribus ipsum secutis, pro integratione aequationis differentialis: $0=x^2(a+bx^n)d^2y+x(c+ex^n)dydx+(f+gx^n)ydx^2$ binas series assumi, vnam *ascendentem*: $y=Ax^p+Bx^{p+n}+Cx^{p+2n}+\ldots$; alteram *descendentem*: $y=\mathfrak{A}x^q+\mathfrak{B}x^{q-n}+\mathfrak{C}x^{q-2n}+\ldots$; quarum coëfficientes, per substitutionem

tionem in aequatione differentiali propoſita, modo vſitato determinantur. Ita quidem duplex obtinetur ſolutio. Verumtamen haud ſuperflua videtur ſequens obſeruatio (*), quae oſtendit, primam iam ſolutionem ſufficere, et alteram ſponte ex illa conſequi. Aequatio nimirum propoſita, diuidendo per x^n, hanc formam recipit:

$$o = x^2 (b + a x^{-n}) d^2 y + x (e + c x^{-n}) dy dx + (g + f x^{-n}) y dx^2.$$

Quodſi igitur pro forma prima inuentum fuerit integrale:

$y = A x^p + B x^{p+n} + C x^{p+2n} + \ldots$, ſtatim inde pro forma altera habebitur alterum integrale: $y = \mathfrak{A} x^q + \mathfrak{B} x^{q-n} + \mathfrak{C} x^{q-2n} + \ldots$, vbi q; $\mathfrak{A}$; $\mathfrak{B}$; $\mathfrak{C}$; ... prodeunt ex p; A; B; C; ... permutatis inuicem a et b; c et e; f et g; $+n$ et $-n$.

3) Binae ſeries pro y modo commemoratae abrumpunt, dum inter coëfficientes aequationis propoſitae certa ſupponitur relatio. Quam relationem EVLERVS (Comm. Petrop. T. X. p. 44.) pro vtraque ſerie, tam aſcendente, quam deſcendente, quaerit, ac concludit, *duplici modo* infinitos caſus aſſignari poſſe, quibus aequatio differentialis noſtra integrabilis exiſtat; poſito nimirum $f = -ap(p-1) - cp$, $g = -b\pi(\pi - 1) - e\pi$, ſeriem primam finitam fore, ſi fuerit $\frac{1 - p - \pi - \frac{e}{b}}{n}$ numerus integer affirmatiuus, alteram, ſi $\frac{-1 + p + \pi + \frac{c}{a}}{n}$ eiusmodi numero aequetur. Eundem in modum LE SEUR et JACQUIER (l. c. p. 424. 425.), et BOUGAINVILLE (l. c. p. 212.) duas *diuerſas* eſſe vias ſtatuerunt, reperiundi innumeros caſus, quibus aequatio finite integrari queat. In recentiori contra diſſertatione (Nou. Comm. Petr. T. XVII. p 131.) EVLERVS, caſum $n = 1$ conſiderans, aſſerit, ex ſerie deſcendente haud nouos ſed eosdem potius caſus integrabiles prodire, ac ex ſerie aſcendente: eademque integralia ordine tantum retrogrado ſcripta obtineri (**). Quae ſibi apparenter contraria vt inuicem concilientur, recurrendum eſt ad obſeruationem (§. XXVII. 1.) commemoratam, quod ſcilicet aequationes pro p et π binas habeant radices: p, p^1; π, π^1; exiſtentibus ſummis $p + p^1 = 1 - \frac{c}{a}$; $\pi + \pi^1 = 1 - \frac{e}{b}$. Hinc erit $\frac{1 - p - \pi - \frac{e}{b}}{n} = \frac{\pi^1 - p}{n}$, et $\frac{-1 + p + \pi + \frac{c}{a}}{n} = \frac{\pi - p^1}{n}$. Vnde

(*) Haec obſeruatio, *quod ſcilicet aequatio ſub duplici ſemper forma exhiberi queat*, deinceps quoque vtilis erit, eaque ad calculorum ambages et moleſtias minuendas multum facit. Quare miror, eam ceteros auctores, interque eos Analyſtam ſupra §. III. (Not. *) laudatum, effugiſſe.

(**) Idem obſeruat *Couſin* l. c. p. 498. ed. alter. P. II. p. 70.

Vnde apparet, binas conditiones superiores sub hac vna comprehendi posse, quod $\frac{\pi - p}{n}$ debeat esse numerus integer affirmatiuus, denotantibus π et p alterutras radices aequationum commemoratarum: quam ipsam conditionem supra inuenimus (§. XXVII. 1.). Itaque clare intelligitur, series ascendentem et descendentem haud diuersos casus integrabiles suppeditare; hosce contra ad vnum casum generalem modo expressum redire.

Scholion 3.

§. XXX. 1) Quanquam series (§. XXV. 3. XXVII. 2.) demonstrata exhibeat integrale tantum *particulare*, exinde tamen colligi etiam potest integrale *completum*. Etenim cum aequatio $f = -ap(p-1) - cp$ duas habeat radices p, p^I, duae obtinentur series, vna ab x^p, altero ab x^{p^I} inchoans; itaque bina habentur integralia particularia, ex quorum per arbitrarias constantes multiplicatorum additione oritur integrale completum. Constat nimirum, si υ et ω denotent integralia particularia *diuersa* aequationis generalioris $o = Pd^2y + Qdxdy + Rydx^2$, fore integrale completum $y = A\upsilon + B\omega$ (*).

2) Ab hac, integrale completum inuestigandi, ratione excipiendus tamen est casus, quo differentia radicum, $p - p^I$, per n est diuisibilis. Tum, ait EVLERVS (**), "sola series, quae incipit a potestate x^μ (nobis x^p) determinari potest; si enim altera a "potestate $x^{\mu - in}$ ($x^{p - kn} = x^{p^I}$) incipiens pro y assumeretur, coëfficiens cu-"iusdam termini repperiretur infinitus, vnde sequentes omnes forent quoque infiniti." Genuina ac sufficiens ratio, quam nec ab EVLERO nec ab aliis satis declaratam video, cur casu substrato altera series seorsim considerata inutilis sit, ita concipi posse videtur. Ob terminos huius seriei infinitos, termini praecedentes finiti omittendi sunt; cumque integrale particulare in factorem constantem arbitrarium ducere liceat, denominator euanescens (***) ex singulis terminis tolli potest, sicque prodibit series terminis finitis constans; vnde intelligitur, terminos infinitos per se haud necessario seriem inutilem reddere, nedum *impossibilem*, quo verbo EVLERVS in inscriptione Probl. 123. (l. c. p. 227.) vtitur. At vero, quod inprimis iam obseruandum est, haec ipsa series praedicto modo variata seriei primae ex valore p ortae omnino aequipollet: quae identitas ex formulis superioribus (§. XXVII. 2.) sine negotio comprobatur. Cum igitur series ex valore p^I deriuata alioquin peculiare integrale ab altero diuersum suppeditet (1), casu contra supposito eadem *seorsim considerata* hunc vsum haud praestat.

3) Cui

(*) *Euler* Institt. Calc. Integr. Vol. II. Cap. IV. §. 837.

(**) l. c. p. 233. §. 976.

(***) qui est $k - k$, §. XXVII. 2, ob mutationem τοῦ p in p^I, hinc τοῦ k in $-k$, cf. §. XXVIII. 2. in fine.

3) Cui igitur incommodo vt medela paretur, EVLERVS hoc praeceptum condidit (*): introducendo logarithmum ipsius x ponendum esse $y = u + \alpha v + v \log. x$, et pro u, v has supponendas series:

$$v = A x^{p} + B x^{p+n} + C x^{p+2n} + \ldots$$
$$u = \mathfrak{A} x^{p^{I}} + \mathfrak{B} x^{p^{I}+n} + \mathfrak{C} x^{p^{I}+2n} + \ldots$$

Quae analysis siue ratiocinatio ad hanc logarithmi substitutionem perducat, auctor haud docuit: eam tantum ceu *artificium*, quo istud incommodum feliciter tollatur, adhibens (l. c. §. 976.)

4) *Idem porro*, ab exemplo particulari occasionem nactus, tanquam "*phaenomenon* "*singulare*" obseruauit: (l. c. §. 980.) "etiamsi integrale completum in genere log. x "inuoluat, (existente scilicet k = numero integro), tamen id a logarithmo liberum prod-"ire certis casibus." Plenam vero huius phaenomeni rationem haud reddidit, nec conditiones euoluit, quibus positis illud locum habeat.

5) Quanquam ex obseruatione (1), vna cum praecepto EVLERI (3), appareat, quo pacto integrale completum semper per series saltem infinitas exhiberi queat: exinde tamen nondum constat, quando et quomodo integrale completum *finite*, et quidem vel algebraice, vel per quantitates transcendentes vsitatas, logarithmos atque Arcus circulares, assignare liceat. Equidem ex supra demonstratis integrale per seriem abrumpentem exprimitur, dum fuerit r = numero integro siue affirmatiuo siue negatiuo (§. XXV. XXVI.); at vero hoc ipsum integrale est tantum particulare, nec quicquam obstat, quominus, dum vna series finita est, altera ex p^{I} orta (1) in infinitum excurrat: quo casu ad integrale sub forma finita exhibendum, si quidem id fieri potest, nouis artificiis opus est.

6) Aduertendo igitur animum ad ea, quae modo exposita sunt (3. 4. 5.), tria adhuc desiderari videntur, quae potissimum expressionem integralis *completi*, siue per series infinitas siue per formulas finitas, concernunt.

a) Primo casus (2), quo ponitur $k = \frac{p - p^{I}}{n}$ = numero integro, quique ceu a regula generali (1) exceptus singulariter tractari solet, accuratius considerandus est, necessitas logarithmum ipsius x introducendi (3) declaranda, huiusque artificii origo explicanda est. Ostendere nimirum licet, etiamsi series ex p^{I} orta seorsim considerata inutilis sit (2), eandem tamen cum altera serie debito modo iunctam integrale completum suppeditare: sicque casum istum ex formulis generalibus resolui, seu ad regulam communem reuocari posse.

b) Deinde

(*) l. c. §§. 973-75. Eodem modo tractat *Cousin* hunc, quem vocat, casum *exceptionis* (l. c. p. 499. 501.).

b) Deinde ratio eſt reddenda, cur et quando, ſuppoſito etiam $k =$ numero integro, integrale tamen a logarithmo liberum prodat (4); vbi apparebit, tum integrale completum ſemper per expreſſionem finitam eamque algebraicam aſſignari poſſe, ſecus ac EVLERVS opinatus fuiſſe videtur.

c) Tandem inueſtigandum eſt, quando et quomodo integrale completum, non tantum particulare, per expreſſiones finitas, easque vel algebraicas vel circulares et logarithmicas, exhibere liceat. Ea enim, quae EVLERVS circa hanc quaeſtionem protulit, partim minus ſufficientia, partim non omnino vera videntur.

Quae igitur deſiderata in ſequentibus explendi tentamen faciam. Ad primum (a) pertinet problema VIII. (§. XXXI.); ad alterum (b) problema IX. (§. XXXIII.); ad tertium (c) idem problema, vna cum problematibus X, XI, XII (§§. XXXVII. XLI. XLII.) eorumque corollariis.

PROBLEMA VIII.

§. XXXI. Aequationis differentialis:

$$o = x^2 (a + bx^n) d^2y + x (c + ex^n) dy dx + (f + gx^n) y dx^2$$

integrale *completum* ſaltem per ſeries infinitas exhibere, dum fuerit $k = \frac{p - p^I}{n}$ aequalis numero *integro*, exiſtentibus p, p^I binis radicibus aequationis, $o = ap(p-1) + cp + f$.

Solutio.

1) Cum radices p, p^I inuicem permutari queant, earum differentiam, indeque quantitatem k affirmatiue ſumere licet. Hinc ſeries ſupra inuenta (§. XXVII. 2. cf. §. XXIX. 1.) ſemper integrale ſaltem particulare aequationis differentialis propoſitae ſuppeditat, hoc nimirum:

$$y = Ax^p \left(1 + \frac{r(k - r^I)}{k+1}\beta x^n + \frac{r(r-1)}{1.2} \frac{(k - r^I)(k - r^I + 1)}{(k+1)(k+2)} \beta^2 x^{2n} \ldots\right)$$

At vero alterum integrale particulare, quod ex illo ſequitur, permutatis inuicem p et p^I, π et π^I, hincque r et r^I, k et $-k$, ad terminos cum denominatoribus euaneſcentibus, ſeu infinitos, perducit (§. XXX. 2.). Quare id maxime agitur, vt oſtendatur, quomodo haec altera feries cum priore iungi poſſit, vt ex hac combinatione prodeat integrale completum.

2) Ponamus primo valorem τȣ k fractione exigua $= \omega$ a numero integro proxime minori $= \mathfrak{k}$ diſcrepare, vt ſit $k = \mathfrak{k} + \omega$: vbi deinceps quantitas ω infinite parua ſeu $= o$ ſumenda eſt. Iam integrale particulare alterum, ex p^I ortum, ita exprimitur:

$$y = A^{I} x^{p^{I}} \left\{ 1 + \frac{r^{I}(k+r)}{k-1} \beta x^{n} + \frac{r^{I}(r^{I}-1)}{1 \cdot 2} \frac{(k+r)(k+r-1)}{(k-1)(k-2)} \beta^{2} x^{2n} \ldots \right.$$
$$+ \frac{r^{I}(r^{I}-1) \ldots (r^{I}-\mathfrak{k}+2)}{1 \cdot 2 \ldots \mathfrak{k}-1} \cdot \frac{(k+r) \ldots (r+2+\omega)}{(k-1) \ldots (1+\omega)} \beta^{\mathfrak{k}-1} x^{(\mathfrak{k}-1)n}$$
$$+ \frac{r^{I}(r^{I}-1) \ldots (r^{I}-\mathfrak{k}+1)}{1 \cdot 2 \ldots \mathfrak{k}} \frac{(k+r) \ldots (r+1+\omega)}{(k-1) \ldots (1+\omega)\omega} \beta^{\mathfrak{k}} x^{\mathfrak{k} n}$$
$$\left. + \frac{r^{I} \ldots (r^{I}-\mathfrak{k})}{1 \cdot 2 \ldots \mathfrak{k}+1} \cdot \frac{(k+r) \ldots (r+\omega)}{(k-1) \ldots (\omega-1)} \beta^{\mathfrak{k}+1} x^{(\mathfrak{k}+1)n} + \ldots \right\}$$

Huius expreſſionis duae partes ſeorſim conſiderandae ſunt. Prima vsque ad terminum $\mathfrak{k}^{tum}$ extenſa, pro $\omega = 0$ ſeu $k =$ numero integro $\mathfrak{k}$, ſponte abit in:

$$A^{I} \left(x^{p-kn} + \frac{r^{I}(k+r)}{k-1} \beta x^{p-kn+n} + \frac{r^{I}(r^{I}-1)}{1 \cdot 2} \frac{(k+r)(k+r-1)}{(k-1)(k-2)} \beta^{2} x^{p-kn+2n} \ldots \right.$$
$$\left. + \frac{r^{I}(r^{I}-1) \ldots (r^{I}-k+2)}{1 \cdot 2 \ldots k-1} \cdot \frac{(k+r) \ldots (r+2)}{(k-1) \ldots 1} \cdot \beta^{k-1} x^{p-n} \right).$$

3) Altera pars ſingularem euolutionem poſtulat. Aſſumta primum pro ω fractione indefinita, erit illa pars $=$

$$A^{I} \frac{r^{I}(r^{I}-1) \ldots (r^{I}-\mathfrak{k}+1)}{1 \cdot 2 \ldots \mathfrak{k}} \cdot \frac{(k+r) \ldots (r+1+\omega)}{(k-1) \ldots (\omega+1)\omega} \cdot x^{p} \beta^{\mathfrak{k}} x^{-\omega n}$$
$$\left(1 + \frac{(k-r^{I}-\omega)}{k-\omega+1} \cdot \frac{(r+\omega)}{1-\omega} \beta x^{n} + \frac{(k-r^{I}-\omega)(k-r^{I}-\omega+1)}{(k-\omega+1)(k-\omega+2)} \cdot \frac{(r+\omega)(r+\omega-1)}{(1-\omega)(2-\omega)} \cdot \beta^{2} x^{2n} \ldots \right)$$

Quo iam conſtet, quemnam valorem haec expreſſio pro euaneſcente ω habitura ſit, coëfficientes τῶν βx^{n}, $\beta^{2} x^{2n}$ ... euoluendi ſunt in ſeries, ſecundum poteſtates τȣ̃ ω progredientes: vbi vero poteſtates ſecunda et vlteriores negligi poſſunt. Poſito igitur pro coëfficiente τȣ̃ $\beta^{\mu} x^{\mu n}$,

$$\frac{(r+\omega)(r+\omega-1) \ldots (r+\omega-\mu+1) \cdot (k-r^{I}-\omega)(k-r^{I}-\omega+1) \ldots (k-r^{I}-\omega+\mu-1)}{(1-\omega)(2-\omega) \ldots (\mu-\omega) \cdot (k-\omega+1)(k-\omega+2) \ldots (k+\mu-\omega)}$$

$= a^{\mu} + b^{\mu}\omega + c^{\mu}\omega^{2} + \ldots$ (vbi litera μ coëfficientibus a, b, c ... ſuperſcripta indices, non exponentes, notat), ſponte ſequitur ex $\omega = 0$: $a^{\mu} =$

$$\frac{r(r-1) \ldots (r-\mu+1) \cdot (k-r^{I})(k-r^{I}+1) \ldots (k-r^{I}+\mu-1)}{1 \cdot 2 \ldots \mu \cdot (k+1)(k+2) \ldots (k+\mu)};$$

porro ſumendo logarithmos ac differentiando habetur:

$$\frac{1}{r+\omega} + \frac{1}{r+\omega-1} + \frac{1}{r+\omega-2} \ldots + \frac{1}{r+\omega-\mu+1} - \frac{1}{k-r^{I}-\omega} - \frac{1}{k-r^{I}-\omega+1} \ldots - \frac{1}{k-r^{I}-\omega+\mu-1}$$
$$+ \frac{1}{1-\omega} + \frac{1}{2-\omega} + \frac{1}{3-\omega} \ldots + \frac{1}{\mu-\omega} + \frac{1}{k-\omega+1} + \frac{1}{k-\omega+2} \ldots + \frac{1}{k-\omega+\mu}$$

$=$

$$= \frac{b^{\mu} + 2c^{\mu}\omega + \ldots}{a^{\mu} + b^{\mu}\omega + c^{\mu}\omega^2 + \ldots}$$; hinc, pro $\omega = 0$, est

$$\frac{b^{\mu}}{a^{\mu}} = \frac{1}{r} + \frac{1}{r-1} + \ldots + \frac{1}{r-\mu+1} - \frac{1}{k-r^1+1} - \frac{1}{k-r^1+2} - \ldots - \frac{1}{k-r^1+\mu-1}$$

$$+ \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{\mu} + \frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{k+\mu};$$

quod sit $= c^{\mu}$, vt fiat $b^{\mu} = c^{\mu} a^{\mu}$. Est insuper pro euanescente ω, $x^{-\omega n} = (E^{-n\log. x})^{\omega} = 1 - n\omega \log. x + \ldots$, seu $\frac{x^{-\omega n}}{\omega} = \frac{1}{\omega} - n \log. x$, vbi log. x logarithmum hyperbolicum ipsius x, pro basi E, denotat (cf. XXII.). Exinde pars integralis (2), quae erat determinanda, prodit $=$

$$A^1 . \frac{r^1(r^1-1)\ldots(r^1-\kappa+1).(\kappa+r)\ldots(r+1)}{1.2 \quad \ldots \quad \kappa \quad .(\kappa-1)\ldots \quad 1} . \beta^k \left(\frac{1}{\omega} - n \log. x\right) x^p$$

$$\left\{\begin{matrix} 1 + \frac{r(\kappa-r^1)}{\kappa+1}\beta x^n + \frac{r(r-1)(\kappa-r^1)(\kappa-r^1+1)}{1.2 \quad (\kappa+1)(\kappa+2)}\beta^2 x^{2n} + \ldots \\ + \quad b^1\omega\beta x^n \quad + \quad b^{11}\omega\beta^2 x^{2n} \quad + \ldots \end{matrix}\right\}$$

Qua iam parti priori antea (2) exhibitae addita, conficitur integrale particulare alterum ex p^1 ortum.

4) Hoc porro integrale cum prius inuento (1) combinando, obtinetur tandem *integrale completum hoc:*

$$y = -\frac{A^1 r^1(r^1-1)\ldots(r^1-\kappa+1).(\kappa+r)\ldots(r+1)}{1.2 \quad \ldots \quad \kappa \quad .(\kappa-1)\ldots \quad 1} . \beta^k n \log. x$$

$$\left(x^p + \frac{r(\kappa-r^1)}{\kappa+1}\beta x^{p+n} + \frac{r(r-1)}{1.2}\frac{(\kappa-r^1)(\kappa-r^1+1)}{(\kappa+1)\ (\kappa+2)}\beta^2 x^{p+2n} + \ldots\right)$$

$$+ A^1\left(x^{p-kn} + \frac{r^1(\kappa+r)}{\kappa-1}\beta x^{p-kn+n} + \frac{r^1(r^1-1)}{1.2}\frac{(\kappa+r)(\kappa+r-1)}{(\kappa-1)(\kappa-2)}\beta^2 x^{p-kn+2n} \ldots\right.$$

$$\left. + \frac{r^1(r^1-1)\ldots(r^1-\kappa+2)}{1.2 \quad \ldots \quad \kappa-1} . \frac{(\kappa+r)\ldots(r+2)}{(\kappa-1)\ldots \quad 1}\beta^{k-1} x^{p-n}\right)$$

$$+ \mathfrak{H} x^p + \mathfrak{H}^1\beta x^{p+n} + \mathfrak{H}^{11}\beta^2 x^{p+2n} + \mathfrak{H}^{111}\beta^3 x^{p+3n} + \ldots;$$

vbi ponitur $A + A^1 \frac{r^1(r^1-1)\ldots(r^1-\kappa+1).(\kappa+r)\ldots(r+1)}{1.2 \quad \ldots \quad \kappa \quad .(\kappa-1)\ldots \quad 1} . \frac{\beta^k}{\omega} = \mathfrak{H}$, quippe quae summa tanquam constans arbitraria spectari potest, ob factorem A itidem arbitrarium: vnde formula integralis ab infinito prorsus liberatur; pro reliquis enim coëfficientibus $\mathfrak{H}^1$, $\mathfrak{H}^{11}$, $\mathfrak{H}^{111}$ ... hae prodeunt aequationes: $\frac{r(\kappa-r^1)}{\kappa+1}$

$$\frac{r(\kappa - r^I)}{\kappa+1} \cdot \mathfrak{H} + \frac{r^I(r^I-1)\ldots(r^I-\kappa+1)}{1.2 \ldots \kappa} \cdot \frac{(\kappa+r)\ldots(r+1)}{(\kappa-1)\ldots 1} \cdot \beta^k b^r A^I = \mathfrak{H}^I$$

$$\frac{r(r-1)}{1.2} \cdot \frac{(\kappa-r^I)(\kappa-r^I+1)}{(\kappa+1)\;(\kappa+2)} \cdot \mathfrak{H} + \frac{r^I(r^I-1)\ldots(r^I-\kappa+1)}{1.2 \ldots \kappa} \cdot \frac{(\kappa+r)\ldots(r+1)}{(\kappa-1)\ldots 1} \cdot \beta^k b^{II} A^I = \mathfrak{H}^{II}$$

etc. etc.

Quae, scripto $\mathfrak{A}^I$ pro $A^I \cdot \frac{r^I(r^I-1)\ldots(r^I-\kappa+1)}{1.2 \ldots \kappa} \cdot \frac{(\kappa+r)\ldots(r+1)}{(\kappa-1)\ldots 1} \cdot \beta^k$, sic breuius exhiberi possunt:

$$\mathfrak{H}^I = \frac{r(\kappa-r^I)}{\kappa+1} \cdot (\mathfrak{H} + c^I \mathfrak{A}^I)$$

$$\mathfrak{H}^{II} = \frac{r(r-1)}{1.2} \cdot \frac{(\kappa-r^I)(\kappa-r^I+1)}{(\kappa+1)\;(\kappa+2)} \cdot (\mathfrak{H} + c^{II} \mathfrak{A}^I)$$

$$\mathfrak{H}^{III} = \frac{r(r-1)(r-2)}{1.2.3} \cdot \frac{(k-r^I)(k-r^I+1)(k-r^I+2)}{(k+1)(k+2)(k+3)} \cdot (\mathfrak{H} + c^{III} \mathfrak{A}^I)$$

. . . .

vbi lex progressus satis manifesta est. De quantitatibus litera c insignitis ex formula pro c^μ (3) constat: est nimirum

$$c^I = \frac{1}{r} + \frac{1}{1} - \frac{1}{k-r^I} + \frac{1}{k+1};$$

$$c^{II} = \frac{1}{r} + \frac{1}{r-1} + \frac{1}{1} + \frac{1}{2} - \frac{1}{k-r^I} - \frac{1}{k-r^I+1} + \frac{1}{k+1} + \frac{1}{k+2};$$

$$c^{III} = \frac{1}{r} + \frac{1}{r-1} + \frac{1}{r-2} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{k-r^I} - \frac{1}{k-r^I+1} - \frac{1}{k-r^I+2} + \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3}$$

. . . .

siue

$$c^I = \frac{r+1}{1.r} - \frac{(r^I+1)}{(k+1)(k-r^I)};$$

$$c^{II} = (r+1)\left(\frac{1}{1.r} + \frac{1}{2(r-1)}\right) - (r^I+1)\left(\frac{1}{(k+1)(k-r^I)} + \frac{1}{(k+2)(k-r^I+1)}\right)$$

$$c^{III} = (r+1)\left(\frac{1}{1.r} + \frac{1}{2(r-1)} + \frac{1}{3(r-1)}\right) - (r^I+1)\left(\frac{1}{(k+1)(k-r^I)} + \frac{1}{(k+2)(k-r^I+1)} + \frac{1}{(k+3)(k-r^I+2)}\right)$$

. . . .

5) Formulis *independentibus*, quibus coëfficientes $\mathfrak{H}$, $\mathfrak{H}^I$, $\mathfrak{H}^{II}$ . . . singuli ex primo eoque arbitrario, $\mathfrak{H}$, et Constante A^I vel $\mathfrak{A}^I$ deducuntur, haud superfluum est addere *legem recursus*, qua quiuis ex proxime antecedente definitur, quaeque ex istis formulis sponte fluit. Est nimirum $(k+\mu+1)(\mu+1)\mathfrak{H}^{\mu+1} - (k-r^I+\mu)(r-\mu)\mathfrak{H}^\mu = \mathfrak{A}^I$.

$$= \mathfrak{A}^{I} \cdot \frac{r(r-1)\ldots(r-\mu)}{1.2\ \ldots\ \mu} \cdot \frac{(k-r^{I})\ldots(k-r^{I}+\mu)}{(k+1)\ \ldots\ (k+\mu)} \cdot \left(\frac{1}{r-\mu} + \frac{1}{\mu+1} - \frac{1}{k-r^{I}+\mu} + \frac{1}{k+\mu+1}\right)$$

$$= \mathfrak{A}^{I} \cdot \frac{r(r-1)\ldots(r-\mu+1)}{1.2\ \ldots\ \mu} \cdot \frac{(k-r^{I})\ldots(k-r^{I}+\mu-1)}{(k+1)\ldots\ (k+\mu)} \left((k+2\mu+2)\frac{(r-\mu)\cdot(k-r^{I}+\mu)}{\mu+1 \cdot k+\mu+1} + k - r - r^{I} + 2\mu\right).$$

Corollarium.

§. XXXII. 1) Sic igitur via directa ad idem praeceptum peruenimus, quod pro hoc casu assumi solet (§. XXX. 3.): simulque integrale completum ita expressum est, vt formula generalis legi satis manifestae subdita pro quouis exemplo particulari statim in vsum conuerti queat, qualem formulam EVLERVS haud euoluit, verum ad singula exempla (§§. 977. 78. 79. l. c.) substitutionem (§. XXX. 3.) applicat, sicque demum coëfficientes sérierum determinat.

2) Ex formula nostra generali simul diiudicare licet, de quo §. XXX. 6. b. quaerebatur, cur et quando integrale, quod in genere log. x inuoluit, tamen a logarithmo liberum prodeat. Cum nimirum pars integralis (§. XXXI. 4.), quae log. x continet, factorem habeat $r^{I}(r^{I}-1)\ldots(r^{I}-k+1)\,.\,(k+r)(k+r-1)\ldots(r+1)$, sponte sequitur, illam cum factore euanescere, si fuerit vel 1) r^{I} numerus integer affirmatiuus minor simili numero k, vel 2) r negatiuus, cuius oppositum $<$ k siue $=$ k. Ob p, p^{I} et π, π^{I} inuicem permutabiles, vbi k oppositum valorem recipit, prior positio ita exprimi potest, vt sumatur k $=$ numero negatiuo, et r $=$ affirmatiuo $< -$ k.

3) Quanquam ita appareat, sub conditionibus modo commemoratis integrale a log. x liberari, ex formula tamen superiore (4) nondum constat, tum etiam integrale semper finite exprimi posse. Quod pro vno casu $r < -k$ demonstrare sufficit, quippe alter casus, quo $-r <$ vel $= k$, sponte ad illum redit. Ille igitur casus peculiarem euolutionem meretur, cui sequens problema destinatum est: quo simul ex parte ad quaestionem (§. XXX. 6. c) respondebitur.

PROBLEMA IX.

§. XXXIII. Posito $\frac{\pi-p}{n} = r =$ numero integro affirmatiuo, $\frac{p-p^{I}}{n} = k =$ integro negatiuo, et $r < -k$: integrale completum aequationis differentialis:

$$o = x^{2}(a+bx^{n})d^{2}y + x(c+ex^{n})dy\,dx + (f+gx^{n})y\,dx^{2}$$

per expressionem *finitam* et *algebraicam* exhibere (*).

Solu-

(*) De significatu literarum π, p, p^{I} cf. §. XXVII. 1. 2.

Solutio.

1) Sit $k = -\mathfrak{k}$, erit

$$y = x^p\left(1 + \frac{r(\mathfrak{k}+r^I)}{\mathfrak{k}-1}\beta x^n + \frac{r(r-1)}{1.2}\,\frac{(\mathfrak{k}+r^I)(\mathfrak{k}+r^I-1)}{(\mathfrak{k}-1)(\mathfrak{k}-2)}\beta^2 x^{2n} + \ldots\right) = U,$$

quae feries praebet integrale particulare finitum, cum ea, ob r numerum integrum affirmatiuum minorem numero $\mathfrak{k}$, abrumpat.

2) Eft porro $\frac{\pi - p^I}{n} = r + k = r - \mathfrak{k} =$ numero integro negatiuo. Hinc formula altera fupra (§. XXVII. 3.) demonftrata applicari poteft: dum, permutatis inuicem p et p^I, manentibus vero π et π^I, illic ponatur $\mathfrak{k} - r$ loco ϱ; $\mathfrak{k}$ loco k; $-r^I - \mathfrak{k}$ loco ϱ^I, ob $\frac{\pi^I - p}{n} = r^I - k = r^I + \mathfrak{k}$. Ita prodit

$$y = x^p(1+\beta x^n)^{r+r^I+1}\left(1 - \frac{(\mathfrak{k}-r-1)(r^I+1)}{\mathfrak{k}-1}\beta x^n + \frac{(\mathfrak{k}-r-1)(\mathfrak{k}-r-2)}{1.2}\,\frac{(r^I+1)(r^I+2)}{(\mathfrak{k}-1)(\mathfrak{k}-2)}\beta^2 x^{2n} - \ldots\right) = U^I$$

= alteri integrali particulari, itidem finito.

3) Exinde confequitur integrale completum: $y = AU + A^I U^I$, vbi A et A^I denotant binas Conftantes arbitrarias.

Corollarium.

§. XXXIV. 1) Cafus, quo eft $r =$ numero integro negatiuo, $k =$ integro affirmatiuo, et $-r$ fiue $\varrho <$ vel $= k$, ad cafum praecedentis problematis reduci poteft (§. XXXII. 3.), permutando tantum inuicem p et p^I. Eft nimirum $\frac{\pi - p^I}{n} = k + r = k - \varrho =$ numero integro affirmatiuo, vel $= 0$; porro $\frac{p^I - p}{n} = -k =$ negatiuo, cuius oppofitum $k > \frac{\pi - p^I}{n}$. Hinc ex folutione praecedente (vel etiam ex binis formulis §. XXVII. 2. 3, dum in priore r et r^I, k et $-k$ permutantur) obtinetur integrale completum:

$$y = Ax^{p^I}\left(1 + \frac{r^I(k-\varrho)}{k-1}\beta x^n + \frac{r^I(r^I-1)}{1.2}\,\frac{(k-\varrho)(k-\varrho-1)}{(k-1)(k-2)}\beta^2 x^{2n} + \ldots\right)$$
$$+ A^I x^{p^I}(1+\beta x^n)^{r^I-\varrho+1}\left(1 + \frac{(\varrho-1)(k-r^I-1)}{k-1}\beta x^n + \frac{(\varrho-1)(\varrho-2)}{1.2}\,\frac{(k-r^I-1)(k-r^I-2)}{(k-1)(k-2)}\beta^2 x^{2n} + \ldots\right)$$

quod ob vtramque feriem abrumpentem eft finitum.

2) Cum

2) Cum r et r^I inuicem permutari queant, vbi k oppositum valorem recipit, hactenus demonstrata sponte applicari possunt, si fuerint r^I et k vel 1) numeri integri affirmatiui, et $r^I < k$, vel 2) integri negatiui, et $-r^I <$ seu $= -k$.

Corollarium.

§. XXXV. 1) Aequatio: $0 = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)ydx^2$ diuidendo per x^n ad hanc reducitur:

$0 = x^2(b+ax^{-n})d^2y + x(e+cx^{-n})dydx + (g+fx^{-n})ydx^2$ (cf. §. XXIX. 2.), vnde p et π, p^I et π^I, $+n$ et $-n$ inuicem permutantur, sicque $\frac{\pi-p}{n}$ et $\frac{\pi^I-p^I}{n}$ siue r et r^I valores seruant, at $\frac{p-p^I}{n}$ siue k abit in $\frac{\pi^I-\pi}{n} = r^I - k - r$. Hinc ex binis casibus (§. XXXIII. XXXIV. 1.) nouos deducere licet.

2) Ex *priori* conditione ($r < -k$) manet $r =$ numero integro affirmatiuo, at loco $-k$ fit $r+k-r^I =$ simili numero maiori vel r. Hinc $k-r^I$ debet esse numerus integer affirmatiuus. Existentibus igitur r et $k-r^I$ binis numeris integris affirmatiuis, habetur integrale completum algebraice et finite expressum hoc:

$$y = Ax^{\pi}\left(1 + \frac{r(k+r)}{r+k-r^I-1}\cdot\frac{1}{\beta x^n} + \frac{r(r-1)}{1.2}\,\frac{(k+r)(k+r-1)}{(r+k-r^I-1)(r+k-r^I-2)}\cdot\frac{1}{\beta^2 x^{2n}} + \ldots\right)$$

$$+ A^I x^{\pi}\left(1+\frac{1}{\beta x^n}\right)^{r+r^I+1}\left(1 - \frac{(k-r^I-1)(r^I+1)}{k-r^I+r-1}\cdot\frac{1}{\beta x^n} + \frac{(k-r^I-1)(k-r^I-2)}{1.2}\,\frac{(r^I+1)(r^I+2)}{(k-r^I+r-1)(k-r^I+r-2)}\cdot\frac{1}{\beta^2 x^{2n}} - \ldots\right),$$

3) Simili ratione *altera* conditio ($-r \leqq k$) abit in hanc, quod fit $r =$ numero integro negatiuo, et $-r \leqq r^I - r - k$, siue $0 \leqq r^I - k$. Tunque, existentibus nimirum r et $k - r^I$ numeris integris negatiuis, quorum posterior etiam $= 0$ esse potest, integrale completum sic exprimitur:

$$y = Ax^{\pi^I}\left(1 + \frac{r^I(r^I-k)}{r^I-k+\rho-1}\cdot\frac{1}{\beta x^n} + \frac{r^I(r^I-1)}{1.2}\cdot\frac{(r^I-k)(r^I-k-1)}{(r^I-k+\rho-1)(r^I-k+\rho-2)}\cdot\frac{1}{\beta^2 x^{2n}} + \ldots\right)$$

$$+ A^I x^{\pi}\left(1+\frac{1}{\beta x^n}\right)^{r^I-\rho+1}\left(1 + \frac{(\rho-1)(\rho-k-1)}{(\rho+r^I-k-1)}\cdot\frac{1}{\beta x^n} + \frac{(\rho-1)(\rho-2)}{1.2}\,\frac{(\rho-k-1)(\rho-k-2)}{(\rho+r^I-k-1)(\rho+r^I-k-2)}\cdot\frac{1}{\beta^2 x^{2n}} + \ldots\right).$$

 Exem-

Exempla:

§. XXXVI. 1) *Sit proposita aequatio differentialis:*

$$0 = x^2(1 + bx^2)d^2y - x(5 - ex^2)dydx + (5 - ex^2)ydx^2;$$

erit pro p, $5 = -p(p-1) + 5p$; vnde $p = 1$, $p^{I} = 5$; pro π, $-e = -b\pi(\pi - 1) - e\pi$, siue $e(\pi - 1) = -b\pi(\pi - 1)$, hinc $\pi = 1$, $\pi^{I} = -\frac{e}{b}$. Quare habetur $r = 0$, $k = -2$, $r^{I} = \frac{-\frac{e}{b} - 5}{2}$. Proinde ex §. XXXIII, prodit integrale completum:

$$y = Ax + A^{I}x(1 + bx^2)^{\frac{-e-3b}{2b}}\left(1 + \frac{(e+3b)}{2}x^2\right).$$

2) *Pro aequatione:*

$$0 = x^2(1 + bx^2)d^2y - x(5 - ex^2)dydx + (5 - 3(e + 2b)x^2)ydx^2,$$

est rursus $p = 1$, $p^{I} = 5$, $k = -2$; at $-6b - 3e = -b\pi(\pi - 1) - e\pi$, siue $0 = -b(\pi^2 - \pi - 6) - e(\pi - 3) = -(\pi - 3)(b(\pi + 2) + e)$, vnde $\pi = 3$, $\pi^{I} = -\frac{e}{b} - 2$; porro $r = 1$, $r^{I} = \frac{-\frac{e}{b} - 7}{2}$. Hinc prodit integrale completum (§. XXXIII.): $y = Ax\left(1 - \frac{(e+3b)}{2}x^2\right) + A^{I}x(1 + bx^2)^{\frac{-e-3b}{2b}}$.

3) His ipsis exemplis EVLERVS phaenomenon, ab ipso, vti supra iam dixi (§. XXX. 4.), obseruatum illustrat, quod integrale certis casibus a logarithmo liberum prodire queat; etiamsi fuerit k numerus integer: nec vero hos casus regula generali comprehendit, nec animaduertisse videtur, tum integrale *semper finite* exprimi posse. Pro priori enim exemplo (1) integrale hac serie exhibet (l. c. §. 980.):

$$y = \mathfrak{A}x + \mathfrak{C}x^5 + \mathfrak{D}x^7 + \mathfrak{E}x^9 + \mathfrak{F}x^{11} + \ldots$$

posito:

$$2 \cdot 6\mathfrak{D} + 4\mathfrak{C}(5b + e) = 0$$
$$4 \cdot 8\mathfrak{E} + 6\mathfrak{D}(7b + e) = 0$$
$$6 \cdot 10\mathfrak{F} + 8\mathfrak{E}(9b + e) = 0$$
etc. etc.

haecque adiicit verba: "integrale *adeo finite* exprimi, si $e = -(2i + 5)b$, pro i sumendo numeros 0, 1, 2, 3, 4 etc."; quae *limitatio* ex nostra formula superflua est. Similiter pro exemplo altero (2) integrale completum per seriem infinitam exprimit, cum hic contra expressio finita inuenta sit.

Problema X.

§. XXXVII. Existentibus $\frac{\pi - p}{n} = r$ et $\frac{\pi^I - p^I}{n} = r^I$ numeris integris affirmatiuis, reperire expressionem finitam integralis completi aequationis differentialis:

$$o = x^2 (a + bx^n) d^2y + x(c + ex^n) dy dx + (f + gx^n) y dx^2.$$

Solutio.

Ob p, p^I; π, π^I; inuicem permutabiles quantitatem $k = \frac{p - p^I}{n}$ hic affirmatiue sumere licet (§. XXXI. 1.). Iam *duo problematis casus* sunt discernendi:

1) Si fuerit k numerus *non integer*, aut *integer maior numero* r^I, tum ex formulis superioribus sponte prodit integrale completum hoc (§. XXX. 1.):

$$y = Ax^p \left(1 + \frac{r(k - r^I)}{k + 1} \beta x^n + \frac{r(r - 1)}{1 . 2} \frac{(k - r^I)(k - r^I + 1)}{(k + 1)(k + 2)} \beta^2 x^{2n} + \dots\right)$$
$$+ A^I x^{p^I} \left(1 + \frac{r^I(k + r)}{k - 1} \beta x^n + \frac{r^I(r^I - 1)}{1 . 2} \frac{(k + r)(k + r - 1)}{(k - 1)(k - 2)} \beta^2 x^{2n} + \dots\right)$$

Quae formula est algebraica et finita, quippe binarum serierum, quas ea inuoluit, prior ob r, altera ob r^I abrumpit; et haec quidem, si k fuerit numerus integer maior quam r^I, abrumpet citius, ac termini propter denominatores euanescentes in infinitum abeant. Loco seriei prioris poni etiam potest, tanquam integralis completi prima pars, existente k numero integro $> r^I$, series itidem finita haec (cf. XXXIII.):

$$Ax^{p^I} (1 + \beta x^n)^{1 + r + r^I} \left(1 - \frac{(r + 1)(k - r^I - 1)}{k - 1} \beta x^n + \frac{(r + 1)(r + 2)(k - r^I - 1)(k - r^I - 2)}{1 . 2 \dots (k - 1)(k - 2)} \beta^2 x^{2n} - \dots\right).$$

2) Alia ratione tractandus est *alter casus*, quo est k *numerus integer, non maior numero* r^I. Tum formulae §. XXXI. traditae in vsum adhibendae sunt, ex iisque obtinetur integrale completum hoc:

$$y = -A^I \frac{r^I(r^I - 1) \dots (r^I - k + 1) . (r + k) \dots (r + 1)}{1 . 2 \dots k . (k - 1) \dots 1} . \beta^k n \log x \left(x^p + \frac{r(k - r^I)}{k + 1} \beta x^{p + n} + \frac{r(r - 1)}{1 . 2} \frac{(k - r^I)(k - r^I + 1)}{(k + 1)(k + 2)} \beta^2 x^{p + 2n} + \dots\right)$$
$$+ A^I \left(x^{p - kn} + \frac{r^I(k + r)}{k - 1} \beta x^{p - kn + n} + \frac{r^I(r^I - 1)}{1 . 2} \frac{(k + r)(k + r - 1)}{(k - 1)(k - 2)} \beta^2 x^{p - kn + 2n} + \dots + \frac{r^I(r^I - 1) \dots (r^I - k + 2)}{1 . 2 \dots k - 1} \frac{(k + r) \dots (r + 2)}{(k - 1) \dots 1} \beta^{k - n} x^{p - n}\right)$$
$$+ \mathfrak{B} x^p + \mathfrak{B}^I \beta x^{p + n} + \mathfrak{B}^{II} \beta^2 x^{p + 2n} + \mathfrak{B}^{III} \beta^3 x^{p + 3n} + \dots$$

 vbi

vbi A^I et $\mathfrak{H}$ funt Conftantes arbitrariae, et reliqui coëfficientes $\mathfrak{H}^I$, $\mathfrak{H}^{II}$, $\mathfrak{H}^{III}$ etc. definiuntur ope formularum fupra (§. XXXI. 4.) exhibitarum, quas hoc loco repetere fuperfluum eft: legem, ex qua ifti coëfficientes progrediuntur, haec aequatio exprimit:

$$(k+\mu+1)(\mu+1)\mathfrak{H}^{\mu+1} - (k-r^I+\mu)(r-\mu)\mathfrak{H}^{\mu} =$$
$$\mathfrak{A}^I \frac{r(r-1)\ldots(r-\mu+1)}{1.2\ldots\mu}\cdot\frac{(k-r^I)(k-r^I+1)\ldots(k-r^I+\mu-1)}{(k+1)\ldots(k+\mu)}$$
$$\cdot\left((k+2\mu+2)\frac{(r-\mu)}{\mu+1}\,\frac{(k-r^I+\mu)}{k+\mu+1}+k-r-r^I+2\mu\right)$$
$$=\pm\beta^k A^I\cdot\frac{(r+k)(r+k-1)\ldots(r-\mu+1)}{1.2\ldots k+\mu}\cdot\frac{r^I(r^I-1)\ldots(r^I-k-\mu+1)}{1.2\ldots\mu:1.2\ldots(k-1)}$$
$$\cdot\left((k+2\mu+2)\frac{(r-\mu)}{\mu+1}\,\frac{(k-r^I+\mu)}{k+\mu+1}+k-r-r^I+2\mu\right)$$

Expreffio integralis finita eft, quippe trium ferierum, quae illam conftituunt, prima ob r abrumpit, altera finito terminorum numero conftat: tertiam quoque, ab $\mathfrak{H}x^p$ incipientem, abrumpere, accuratior confideratio oftendit.

Namque fit *primo* $k > r^I - r$, tum coëfficientes $\mathfrak{H}^I, \mathfrak{H}^{II}, \ldots \mathfrak{H}^{r^I-k}$ per Conftantes $\mathfrak{H}$ et A^I determinantur; reliqui autem a fola $\mathfrak{A}^I$ vel A pendentes ex his aequationibus prodeunt: $(r^I+1)(r^I-k+1)\mathfrak{H}^{r^I-k+1} =$
$$\frac{\mathfrak{A}^I r(r-1)\ldots(r-r^I+k+1)}{1.2\ldots r^I-k}\cdot\frac{(k-r^I)(k-r^I+1)\ldots(-1)}{(k+1)(k+2)\ldots r^I}\cdot(r^I-k-r)\quad\text{fiue}$$
$$\mathfrak{H}^{r^I-k+1}=\pm\mathfrak{A}^I\cdot\frac{r(r-1)\ldots(r-r^I+k)}{(r^I-k+1)\cdot(k+1)(k+2)\ldots(r^I+1)}$$
$$=\pm\beta^k A^I\cdot\frac{r^I(r^I-1)\ldots(r^I-k+2)}{1.2\ldots k-1}\cdot\frac{(k+r)(k+r-1)\ldots(k+r-r^I)}{1.2\ldots(r^I+1)}$$
(ex valore τῶ $\mathfrak{A}^I$ §. XXXI. 4.); porro:
$$(r^I+2)(r^I-k+2)\mathfrak{H}^{r^I-k+2}-1.(r-r^I+k-1)\mathfrak{H}^{r^I-k+1}=0$$
$$(r^I+3)(r^I-k+3)\mathfrak{H}^{r^I-k+3}-2.(r-r^I+k-2)\mathfrak{H}^{r^I-k+2}=0$$
$$(r^I+4)(r^I-k+4)\mathfrak{H}^{r^I-k+4}-3.(r-r^I+k-3)\mathfrak{H}^{r^I-k+3}=0$$
etc. etc.

vbi propter feriem numerorum $r-r^I+k-1$, $r-r^I+k-2$, ... ad 0 perducentem, tandem ad coëfficientes euanefcentes perueniri euidens eft.

Quodfi iam *fecundo* ponatur $k =$ vel $< r^I - r$, tum coëfficientes $\mathfrak{H}^I$, $\mathfrak{H}^{II}$, ... $\mathfrak{H}^r$ ex conftantibus $\mathfrak{H}$ et A^I definiuntur, reliqui autem ex fola A^I fequentem in modum:

 (k+r+1)

$$(k+r+1)(r+1)\mathfrak{H}^{r+1} = \mathfrak{A}^{I} \cdot \frac{(\kappa - r^{I})(\kappa - r^{I}+1)\ldots(\kappa - r^{I} + r - 1)}{(\kappa+1)(\kappa+2)\ \ldots\ (\kappa+r)} \cdot (k+r-r^{I})$$

fiue $\mathfrak{H}^{r+1} = \pm A^{I} \cdot \frac{(\kappa+r)(\kappa+r-1)\ldots(r+2)}{1.2\ \ldots\ \kappa-1} \cdot \frac{r^{I}(r^{I}-1)\ldots(r^{I}-r-\kappa)}{1.2\ \ldots\ r+\kappa+1}$;

porro: $(k+r+2)(r+2)\mathfrak{H}^{r+2} - (r^{I}-r-k-1).1\,\mathfrak{H}^{r+1} = 0$

$(k+r+3)(r+3)\mathfrak{H}^{r+3} - (r^{I}-r-k-2).2\,\mathfrak{H}^{r+2} = 0$

$(k+r+4)(r+4)\mathfrak{H}^{r+4} - (r^{I}-r-k-3).3\,\mathfrak{H}^{r+3} = 0$

etc. etc.

quas aequationes itidem ad coëfficientes euanefcentes perducere fponte apparet.

Hinc tandem concludere licet, formulam pro integrali completo inuentam femper effe finitam.

Exemplum 1.

§. XXXVIII. 1) Sit propofita aequatio:

$$0 = x^2(1+bx^3)d^2y + x(5+ex^3) + (-12+gx^3)ydx^2;$$

erit, ob $-12 = -p(p-1) - 5p$, $p = 2$ et $p^{I} = -6$, $k = \frac{8}{3}$; porro $\frac{e}{b} = \frac{c}{a} - n(r+r^{I})$ (§. XXVII. 2.) $= 5 - 3(r+r^{I})$; $\frac{\pi - 2}{3} = r$, feu $\pi = 2+3r$, hinc $\frac{g}{b} = -\pi(\pi-1) - \frac{e}{b}\pi = -(2+3r)(1+3r) - (3r+2)(5-3r-3r^{I})$ $= -(3r+2)(6-3r^{I})$.

Quibus valoribus fuppofitis, prodit aequationis differentialis

$$0 = x^2(1+bx^3)d^2y + x(5+(5-3r-3r^{I})bx^3)dydx - 3(4-(3r+2)(r^{I}-2)bx^3)ydx^2$$

integrale completum hoc:

$$y = A\Big(x^2 + \frac{r(8-3r^{I})}{11}bx^5 + \frac{r(r-1)}{1.2}\,\frac{(8-3r^{I})(8-3r^{I}+3)}{11.14}b^2x^8 + \frac{r(r-1)(r-2)}{1.2.3}\,\frac{(8-3r^{I})(8-3r^{I}+3)(8-3r^{I}+6)}{11.14.17}b^3x^{11} + \ldots\Big)$$

$$+ A^{I}\Big(x^{-6} + \frac{r^{I}(8+3r)}{5}bx^{-3} + \frac{r^{I}(r^{I}-1)}{1.2}\,\frac{(8+3r)(8+3r-3)}{5.2}b^2x^0 + \frac{r^{I}(r^{I}-1)(r^{I}-2)}{1.2.3}\,\frac{(8+3r)(8+3r-3)(8+3r-6)}{5.2.-1}b^3x^3 + \ldots\Big)$$

vbi, cum r et r^{I} fupponantur numeri integri affirmatiui, vtraque feries abrumpit.

2) In Calc. Integr. P. P. LE SEUR et JACQUIER Vol. II. p. 428. huius exempli cafus particularis refolutus extat, nimirum pro $r = 0$, $r^{I} = 1$, $b = 1$; tunc habetur aequationis differentialis: $0 = x^2$

$$o = x^2\,(1 + x^3)\,d^2y + x(5 + 2x^3)\,dy\,dx - 6(2 + x^3)\,y\,dx^2$$

integrale completum: $y = Ax^2 + A^{I}\left(x^{-6} + \tfrac{8}{5}x^{-3}\right)$.

3) Seruato valore $r = 0$, pro quouis valore r^{I} prodit aequationis:

$$o = x^2(1 + bx^3)\,d^2y + x(5 + (5 - 3r^{I})\,bx^3)\,dy\,dx - 6(2 - (r^{I} - 2)\,x^3)\,y\,dx^2$$

integrale hoc:

$$y = Ax^2 + 8A^{I}\left(\frac{x^{-6}}{8} + \tfrac{1}{5}r^{I}bx^{-3} + \tfrac{1}{2}\,\frac{r^{I}(r^{I}-1)}{1.2}\,b^2x^{0} - \tfrac{1}{1}\,\frac{r^{I}(r^{I}-1)(r^{I}-2)}{1.2.3}\,b^3x^3 - \tfrac{1}{4}\,\frac{r^{I}\ldots(r^{I}-3)}{1\ldots4}\,b^4x^6 - \ldots\right).$$

Exemplum 2.

§. XXXIX. 1) Pro aequatione

$$o = x^2\,(1 + bx^2)\,d^2y + x(-5 + ex^2)\,dy\,dx + (5 + gx^2)\,y\,dx^2$$

est $5 = -p(p-1) + 5p$, hinc $p = 5$, $p^{I} = 1$, $k = 2$; porro $\frac{e}{b} = -5 - 2(r^{I} + r)$; $\frac{g}{b} = (p + nr)(p^{I} + nr^{I}) = (5 + 2r)(1 + 2r^{I})$. Quare obtinetur aequationis differentialis

$$o = x^2(1 + bx^2)\,d^2y - x(5 + (5 + 2(r^{I} + r))\,bx^2)\,dy\,dx + (5 + (5 + 2r)(1 + 2r^{I})\,bx^2)\,y\,dx^2$$

integrale completum hoc:

$$y = -(r+2)(r+1)\,r^{I}(r^{I}-1)\,b^2A^{I}\log.x.\left(x^5 + \frac{r(2-r^{I})}{3}\,bx^7 + \frac{r(r-1)}{1.2}\,\frac{(2-r^{I})(3-r^{I})}{3.4}\,b^2x^9 + \ldots\right)$$

$$+ A^{I}(x + r^{I}(r+2)\,bx^3) + \mathfrak{H}x^5 + \mathfrak{H}^{I}bx^7 + \mathfrak{H}^{II}b^2x^9 + \ldots$$

vbi coëfficientes $\mathfrak{H}^{I}$, $\mathfrak{H}^{II}$, $\mathfrak{H}^{III}$ . . . ex his aequationibus definiuntur:

$$1.3.\mathfrak{H}^{I} + (r^{I}-2)\,r.\,\mathfrak{H} = +\frac{(r+2)(r+1)}{1.2}\,r^{I}(r^{I}-1)\,b^2A^{I}\left(\frac{4r(2-r^{I})}{3} + 2 - r - r^{I}\right);$$

$$2.4.\mathfrak{H}^{II} + (r^{I}-3)(r-1)\,\mathfrak{H}^{I} = -\frac{(r+2)(r+1)\,r}{1.2.3}\,\frac{r^{I}(r^{I}-1)(r^{I}-2)}{1}\,b^2A^{I}\left(\frac{6(r-1)}{2}\,\frac{(3-r^{I})}{4} + 4 - r - r^{I}\right);$$

$$3.5.\mathfrak{H}^{III} + (r^{I}-4)(r-2)\,\mathfrak{H}^{II} = +\frac{(r+2)(r+1)\,r(r-1)}{1.2.3.4}\,\frac{r^{I}(r^{I}-1)(r^{I}-2)(r^{I}-3)}{1.2}\,b^2A^{I}\left(\frac{8(r-2)}{3}\,\frac{(4-r^{I})}{5} + 6 - r - r^{I}\right)$$

etc. etc.

2) Posito

2) Posito $r^I = 0$ vel $= 1$, integrale a logarithmo $\lg x$ liberum prodibit. Pro $r^I = 0$ habetur:

$$y = A^I x + \mathfrak{H}\left(x^5 + \tfrac{2}{3} r b x^7 + \frac{2\cdot 3}{3\cdot 4}\frac{r(r-1)}{1\cdot 2} b^2 x^9 + \frac{2\cdot 3\cdot 4}{3\cdot 4\cdot 5}\frac{r(r-1)(r-2)}{1\cdot 2\cdot 3} b^3 x^{11} + \ldots\right)$$

$$= A^I x + 2\mathfrak{H} x^5 \left(\tfrac{1}{2} + \tfrac{1}{3} r b x^2 + \tfrac{1}{4}\frac{r(r-1)}{1\cdot 2} b^2 x^4 + \tfrac{1}{5}\frac{r(r-1)(r-2)}{1\cdot 2\cdot 3} b^3 x^6 + \ldots\right)$$

siue etiam $= A^I x + A x (1 + b x^2)^{1+r} \cdot (1 - (r+1) b x^2)$ (cf. §. XXXVII. 1.).

Pro $r^I = 1$ est:

$$y = A^I (x + (r+2) b x^3) + \mathfrak{H} x^5 \left(1 + \frac{r}{3} b x^2 + \frac{r(r-1)}{3\cdot 4} b^2 x^4 + \frac{r(r-1)(r-2)}{3\cdot 4\cdot 5} b^3 x^6 + \ldots\right)$$

siue $= A^I (x + (r+2) b x^3) + A x (1 + b x^2)^{r+2}$, integrale completum aequationis differentialis:

$$0 = x^2 (1 + b x^2) d^2 y - x(5 + (7 + 2r) b x^2) dy dx + (5 + 3(5 + 2r) b x^2) y dx^2.$$

Huius aequationis casum specialem, pro $r = 0$, resoluit EVLERVS l. c. §. 981.

Scholion.

§. XL. Formam integralis pro casu altero problematis praecedentis (§. XXXVII. 2.) etiam ex integralis completi expressione supra inuenta (§. XXVII. 2.) directe deducere licet. Habetur nimirum $y = \chi^{\frac{p^I}{n}} (1+\beta\chi)^{1+r+r^I} d^r \left(\chi^{k+r} (1+\beta\chi)^{-r^I-1} z\right)$, existente $z = N + M \int \chi^{-k-r-1} (1+\beta\chi)^{r^I} d\chi$. Quare integrale z euoluendum est.

1) Sumamus *primo* $k =$ vel $< r^I - r$, tum binomio $(1+\beta\chi)^{r^I}$ in seriem conuerso, integrale illud necessario logarithmo ipsius x ($= \int x^{-1} dx$) implicabitur, hancque induet formam:

$$z = a\chi^{-k-r} + b\chi^{-k-r+1} + c\chi^{-k-r+2} + \ldots + e \log.\chi + f + g\chi + h\chi^2 + \ldots$$

vnde sequitur $y = \chi^{\frac{p^I}{n}} (1+\beta\chi)^{1+r+r^I} d^r (1+\beta\chi)^{-r^I-1}$

$$\left(e\chi^{k+r} \log.\chi + a + b\chi + c\chi^2 \ldots + f\chi^{k+r} + \ldots\right).$$

Differentiali r^{to} actu euoluto haec prodibit forma:

$$y = e \log.\chi \cdot \chi^{\frac{p^I}{n}} (1+\beta\chi)^{1+r+r^I} d^r \left((1+\beta\chi)^{-r^I-1} \cdot \chi^{k+r}\right)$$

$$+ \mathfrak{B}\chi^{\frac{p^I}{n}} + \mathfrak{C}\chi^{\frac{p^I}{n}+1} + \mathfrak{D}\chi^{\frac{p^I}{n}+2} + \ldots$$

Cuius

Cuius expressionis parte prima ex supra (§. XXV.) demonstratis transformata, et posito x^n loco χ, $p - kn$ pro p^I, obtinetur

$$y = B \log. x \,.\left(x^p + \frac{r(k-r^I)}{k+1}\beta x^{p+n} + \frac{r(r-1)}{1.2}\,\frac{(k-r^I)(k-r^I+1)}{(k+1)(k+2)}\beta^2 x^{p+2n} + \ldots\right)$$
$$+ \mathfrak{B}x^{p-kn} + \mathfrak{C}x^{p-kn+n} + \mathfrak{D}x^{p-kn+2n} + \ldots$$

2) Quodsi *secundo* fuerit $k > r^I - r$, at $< r^I$, tum integrale $z = \int \chi^{-k-r-1}(1+\beta\chi)^{r^I} d\chi$ in seriem finitam euoluere licebit (EVLER. Inst. C. I. Vol. I. p. 103.); quo facto manifestum erit, in expressione $\chi^{k+r}(1+\beta\chi)^{-r^I-1} z$ maximum exponentem τῦ χ esse $= k + r - r^I - 1$, *id est* $< r$, hincque differentiale r^{tum} euanescere. Quare prodibit $y = 0$; qui valor quanquam aequationi differentiali per se satisfaciat, haud tamen nouum integrale particulare praebet, ex quo iuncto cum altero ex $z = N$ orto definire liceret completum. Cùm igitur pro hoc casu, existente nimirum $k > r^I - r$ et $< r^I$, formula nostra determinando integrali completo minus apta esse videatur, hic casus ad priorem (1) reducendus est: quod quidem fit, permutatis tantum π et π^I, manentibus p, p^I; sic r abit in $r^I - k$, r^I in $r + k$; hinc $r^I - r$ in $2k + r - r^I$, quod iam est $> k$.

Hinc apparet, formam integralis (1) etiam pro numeris integris k maioribus quam $r^I - r$ locum habere. Coëfficientes $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ ... definire liceret ex substitutione valoris τῦ y (1) in ipsa aequatione differentiali proposita. De quo vero cum ex superioribus abunde iam constet, amplior euolutio superflua est.

Sufficiat hoc loco, praeter methodum §. XXXI. expositam, aliam insuper viam directam monstrasse, qua ad formam integralis pro numeris k integris, communiter sine demonstratione assumtam (§. XXX. 3.), certa inuestigatione peruenire liceat. Quae quidem hactenus dicta ad numeros r integros spectantia, sine negotio ad quosuis τῦ r valores extendi posse, sponte intelligitur (§. XXIX. 1.).

PROBLEMA XI.

§. XLI. Existentibus $\frac{\pi - p}{n} = r = -\varrho$, et $\frac{\pi^I - p^I}{n} = r^I = -\varrho^I$ numeris integris negatiuis, inuenire expressionem finitam integralis completi aequationis differentialis:

$$0 = x^2(a + bx^n)d^2y + x(c + ex^n)dydx + (f + gx^n)ydx^2.$$

Solutio.

Sicuti in solutione praecedentis problematis (§. XXXVII.), ita hic quoque *duo casus* sunt discernendi.

1) Si

1) Si numerus k (quem affirmatiue fumere licet) fuerit vel non integer, vel integer *non minor* numero ϱ, tum ex fupra demonftratis (§. XXVII. 3. XXX. 1.) prodit integrale completum

$$y = A x^{p^I}(1+\beta x^n)^{1-\varrho-\varrho^I}\Big(1 + \frac{(\varrho-1)(\varrho^I+k-1)}{k-1}\beta x^n + \frac{(\varrho-1)(\varrho-2)}{1.2}\frac{(\varrho^I+k-1)(\varrho^I+k-2)}{(k-1)(k-2)}\beta^2 x^{2n} + \ldots\Big)$$

$$+ A^I x^p (1+\beta x^n)^{1-\varrho-\varrho^I}\Big(1 - \frac{(\varrho^I-1)(\varrho-k-1)}{k+1}\beta x^n + \frac{(\varrho^I-1)(\varrho^I-2)}{1.2}\frac{(\varrho-k-1)(\varrho-k-2)}{(k+1)(k+2)}\beta^2 x^{2n} + \ldots\Big)$$

cuius vtraque pars abrumpit. Exiftente k numero integro $=$ aut $> \varrho$, loco partis pofterioris poni etiam poteft haee feries:

$$A^I x^{p^I}\Big(1 - \frac{\varrho^I(k-\varrho)}{k-1}\beta x^n + \frac{\varrho^I(\varrho^I+1)}{1.2}\frac{(k-\varrho)(k-\varrho-1)}{(k-1)(k-2)}\beta^2 x^{2n} - \ldots\Big)$$

quae cum priori parte coniuncta itidem praebet integrale completum finite expreffum.

2). Quod *alterum cafum* attinet, k nimirum exiftente numero integro $< \varrho$, quanquam eius folutio ex principiis §. XXXI. vel XL. adhibitis deriuari queat: fatius tamen videtur, eundem ad cafum alterum problematis praecedentis (§. XXXVII. 2.) reducere.

Pofito $y = x^{1-\frac{c}{a}}(a+bx^n)^{1+\frac{c}{na}-\frac{e}{nb}}.v$ (§. X.), aequatio

$$0 = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)ydx^2$$

in hanc transformatur:

$$0 = x^2(a+bx^n)d^2v + x(C+Ex^n)dvdx + (f+Gx^n)ydx^2,$$

vbi funt coëfficientes: $C = 2a - c$; $E = 2(n+1)b - e$; $G = n(n+1)b - ne + g$.

Loco p, π, k, r, ponantur pro aequatione transformata P, Π, K, R; tum aequationibus: $f = -ap(p-1) - cp$, $g = -b\pi(\pi-1) - e\pi$, refpondebunt hae: $f = -aP(P-1) - C\Pi$, $G = -b\Pi(\Pi-1) - E\Pi$, quas cum illis comparando facile colligitur, fore $P = -p$, $\Pi = -n - \pi$. Hinc ob p, p^I et π, π^I inuicem permutabiles, ponere licet $P = -p^I$, $P^I = -p$; $\Pi = -n - \pi^I$, $\Pi^I = -n - \pi$; vnde porro fit $K = k$, $R = -1 - r^I = -1 + \varrho^I$, $R^I = -1 - r = -1 + \varrho$. Quare cum in aequatione propofita r et r^I fint numeri negatiui, in transformata contra R et R^I erunt affirmatiui. Haec igitur aequatio ex praecedenti problemate integrari poteft; ficque, ponendo pro literis minoribus p, p^I ... literas refpondentes maiores, earumue valores modo expreffos, prodit ex §. XXXVII. 2. aequationis transformatae integrale completum hoc:

$$v = -A^I . \frac{(\rho-1)(\rho-2)\ldots(\rho-k)}{1.2\ldots k} . \frac{(\rho^I+k-1)(\rho^I+k-2)\ldots\rho^I}{(k-1)\ldots1} . \beta^k n \log.x . x^{-p^I}$$

$$\left(1 + \frac{(\rho^I-1)(k-\rho+1)}{k+1}\beta x^n + \frac{(\rho^I-1)(\rho^I-2)}{1.2}\frac{(k-\rho+1)(k-\rho+2)}{(k+1)(k+2)}\beta^2 x^{2n} + \ldots\right)$$

$$+ A^I x^{-p^I-kn}\left(1 + \frac{(\rho-1)(k+\rho^I-1)}{k-1}\beta x^n + \frac{(\rho-1)(\rho-2)}{1.2}\frac{(k+\rho^I-1)(k+\rho^I-2)}{(k-1)(k-2)}\beta^2 x^{2n}\right.$$

$$\left. + \ldots + \frac{(\rho-1)\ldots(\rho-k+1)}{1.2\ldots k-1} . \frac{(k+\rho^I-1)\ldots(\rho^I+1)}{(k-1)\ldots1}\beta^{k-1} x^{kn-n}\right)$$

$+ x^{-p^I}(\mathfrak{H} + \mathfrak{H}^I\beta x^n + \mathfrak{H}^{II}\beta^2 x^{2n} + \mathfrak{H}^{III}\beta^3 x^{3n} + \ldots)$, vbi legem, ex qua coëfficientes $\mathfrak{H}^I$, $\mathfrak{H}^{II}$, $\mathfrak{H}^{III}$ . . . progrediuntur, haec aequatio ostendit:

$$(k+\mu+1)(\mu+1)\mathfrak{H}^{\mu+1} - (k-\rho+1+\mu)(\rho^I-1-\mu)\mathfrak{H}^{\mu} =$$

$$\pm\beta^k A^I . \frac{(\rho^I+k-1)(\rho^I+k-2)\ldots(\rho^I-\mu)}{1.2\ldots k+\mu} . \frac{(\rho-1)(\rho-2)\ldots(\rho-k-\mu)}{1.2\ldots\mu . 1.2\ldots k-1}$$

$$. \left((k+2\mu+2)\frac{(\rho^I-\mu-1)}{\mu+1} . \frac{(k-\rho+\mu+1)}{k+\mu+1} + k-\rho-\rho^I+2\mu+2\right).$$

Expressionem integralis finitam esse, eadem ratione ac §. XXXVII. 2. apparet. Ex v sponte prodit integrale completum aequationis propositae:

$$y = x^{1-\frac{c}{a}}(1+\beta x^n)^{1+\frac{c}{na}-\frac{e}{nb}} . v = x^{p+p^I}(1+\beta x^n)^{1-\rho-\rho^I} . v.$$

Problema XII.

§. XLII. Definire conditiones, sub quibus integrale *completum* aequationis differentialis: $o = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)ydx^2$, vel algebraice vel saltem per Arcus circulares et Logarithmos exhibere liceat, dum supponatur $r =$ numero integro affirmatiuo siue negatiuo.

Solutio.

1) Sit *primo* $r =$ numero integro affirmatiuo, tum integrale completum hac formula exprimitur: $y = \chi^{\frac{p^I}{n}}(1+\beta\chi)^{1+r+r^I}\dfrac{d^r\left(\chi^{k+r}(1+\beta\chi)^{-r^I-1}.z\right)}{d\chi^r}$,

existente $z = N + M\int\chi^{-k-r-1}(1+\beta\chi)^{r^I}d\chi$ (§. XXVII. 2.). Itaque y duabus partibus constat, quarum prima, pro $z = N$, algebraica est (§. XXIV. XXV.); altera inuoluit integrale $\int\chi^{-k-r-1}(1+\beta\chi)^{r^I}d\chi$. Quare id iam agitur, vt inuestigetur,

tur, quando hoc integrale vel algebraice vel per quantitates transcendentes notas exhibere liceat. Cum vero differentiale $\chi^{-k-r-1}(1+\beta\chi)^{r^I}d\chi$ cum formula satis nota $x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ (*) conueniat, facile colligitur, illud tribus casibus praedicta ratione integrabile esse, si nimirum trium numerorum: r^I; k; r^I-k; vnus fuerit integer siue affirmatiuus siue negatiuus.

2) Sit *secundo*, $r=$ numero integro negatiuo $=-\varrho$, tum habetur pro integrali completo (§. XXVII. 3.):

$$y=\chi^{\frac{p}{n}}d^{\varrho-1}\left(\chi^{-k+\varrho-1}(1+\beta\chi)^{-\varrho^I}.z\right):d\chi^{\varrho-1}, \text{ posito } z= N+M\int\chi^{k-\varrho}(1+\beta\chi)^{\varrho^I-1}d\chi.$$

Hinc simili ratione concluditur, integrale z, indeque et ipsum integrale completum y vel algebraice vel saltem per Arcus circulares et Logarithmos exprimi posse, si trium numerorum: ϱ^I; k; ϱ^I+k; vnus fuerit integer, siue affirmatiuus siue negatiuus. Quae conditiones cum praecedentibus (1) conspirant.

3) Exinde haec conclusio communis infertur: Integrale completum aequationis differentialis propositae tribus casibus, vel algebraice, vel saltem per Arcus circulares et Logarithmos assignari posse: si 1) r et r^I; vel 2) r et k; vel 3) r et $k-r^I$ fuerint numeri integri, nullo ad eorum signa respectu habito. Quod quidem integrale duabus partibus constat, quarum prima algebraica, eaque supra iam euoluta est (§. XXVII. 2. 3.), altera ad differentiale secundum regulas notas integrandum reducitur. Supponitur autem, numeros r^I et k, si non integros, saltem fractos rationales esse.

Corollarium 1.

§. XLIII. 1) Conditiones modo inuentae comprehendunt etiam casus §§. XXXIII. XXXIV. XXXV. XXXVII. XLI. euolutos, quibus integrale completum vel expressionibus mere algebraicis, vel assumto insuper log. x, exhiberi posse, iam ex formulis §pho praecedente adhibitis concludere licet. Ceteris casibus, quibus istae conditiones locum habent, integrale aliis quantitatibus logarithmicis vel circularibus exprimitur.

2) Quanquam hae formulae integrale completum exhibentes in genere pro sufficientibus habendae sint, in earum tamen applicatione difficultates occurrere possunt, quae accuratius excutiendae sunt.

Supponitur *primum*, sumendo $z=\int\chi^{-k-r-1}(1+\beta\chi)^{r^I}d\chi$, vel $=\int\chi^{k-\varrho}(1+\beta\chi)^{\varrho^I-1}d\chi$, obtineri integrale particulare, quod cum altero algebraico, ex

(*) *Euler* Instit. Calc. Integr. Vol. I. Sect. I. Cap. II. *Kaestner* Analys. infinit. §. 399. pag. 343 sq.

ex $z = N$ deriuato, iungere liceat. At vero ex eo, quod §. XL. 2. adhibendo ipsas hasce formulas, obseruatum est, colligere licet, loco integralis istius particularis prodire nonnumquam $y = 0$, qui valor ad integrale completum eruendum inutilis est. Quae igitur difficultas iam soluenda est.

Illud quidem incommodum locum habere potest, si vel 1) $r^{I} - k - r$ pro (§. XLII. 1.), vel 2) $k - \varrho + \varrho^{I}$ pro (§. XLII. 2.), fuerit numerus integer negatiuus, vbi integrale z algebraice exprimitur. Est nimirum pro (1) seu pro affirmatiuo r,

$$z = \int \chi^{-k-r-1}(1+\beta\chi)^{r^{I}} d\chi =$$

$$\frac{-1}{k+r}(1+\beta\chi)^{r^{I}+1}\chi^{-k-r}\Big(1 - \frac{(k+r-r^{I}-1)}{k+r-1}\beta\chi + \frac{(k+r-r^{I}-1)(k+r-r^{I}-2)}{(k+r-1)(k+r-2)}\beta^2\chi^2 + \ldots\Big);$$

$$\text{hinc } y = \chi^{\frac{p^{I}}{n}}(1+\beta\chi)^{1+r+r^{I}} d^{r}\Big(1 - \frac{(k+r-r^{I}-1)}{k+r-1}\beta\chi + \frac{(k+r-r^{I}-1)(k+r-r^{I}-2)}{(k+r-1)(k+r-2)}\beta^2\chi^2 + \ldots\Big).$$

Series, cuius differentiale r^{tum} hic occurrit, finita est, et in vltimo termino maximus exponens τȢ χ, $= k + r - r^{I} - 1$. Quodsi igitur numerus $k - r^{I}$ negatiuus est, vel $= 0$, tum is exponens erit $< r$, et differentiale illud $r^{tum} = 0$; hinc quoque y euanescet. Simili quoque ratione pro (2) seu pro negatiuo $r = -\varrho$, est

$$z = \int \chi^{k-\varrho}(1+\beta\chi)^{\varrho^{I}-1} d\chi =$$

$$\frac{1}{k-\varrho+1}(1+\beta\chi)^{\varrho^{I}}\chi^{k-\varrho+1}\Big(1 - \frac{(-k+\varrho-1-\varrho^{I})}{-k+\varrho-2}\beta\chi + \frac{(-k+\varrho-1-\varrho^{I})(-k+\varrho-2-\varrho^{I})}{(-k+\varrho-2)(-k+\varrho-3)}\beta^2\chi^2 + \ldots\Big)$$

$$\text{et } y = \chi^{\frac{p}{n}} d^{\varrho-1}\Big(1 - \frac{(\varrho-k-\varrho^{I}-1)}{\varrho-k-2}\beta\chi + \frac{(\varrho-k-\varrho^{I}-1)(\varrho-k-\varrho^{I}-2)}{(\varrho-k-2)(\varrho-k-3)}\beta^2\chi^2 - \ldots\Big),$$

quod integrale rursus erit $= 0$, dum sit $k + \varrho^{I} > 0$, i. e. $r^{I} - k$ numerus integer negatiuus. Exinde apparet, y duplici casu prodire $= 0$: 1) si r et $r^{I} - k$ sint numeri integri affirmatiui, et $r^{I} - k < r$; 2) si r et $r^{I} - k$ sint numeri integri negatiui, et $-(r^{I} - k) < -r$.

Cum ita constet, quando hoc incommodum locum habere possit, facile iam illud euitare licet. Permutatis nimirum π et π^{I}, r abit in $r^{I} - k$; r^{I} in $r + k$, siue, cum k valorem seruet, $r^{I} - k$ in r. Qui noui τῶν r et $r^{I} - k$ valores, ex pristinorum mutua permutatione orti, ita sunt comparati, vt priores conditiones, sub quibus $y = 0$ prodibat, non amplius locum habeant.

3) Prae-

3) Praeter incommodum iam sublatum, *alterum* insuper applicationem solutionis praecedentis (§. XLII.) impedire potest.

Supponitur nimirum, integrale particulare ex $z = N$ ortum algebraice exprimi posse per series supra inuentas (§. XXVII. 2. 3.). At si fuerit 1) r numerus integer affirmatiuus, k integer negatiuus, et $-k \leqq r$, vel 2) r integer negatiuus, k integer affirmatiuus, et $k < -r$; tum illae series ob terminos infinitos inutiles sunt (§. XXX. 2.).

Quae difficultas sic tolli potest. Permutatis inuicem p et p^I, k abit in $-k$; r in $k+r$; r^I in $r^I - k$. Hic nouus valor τȢ r signum pristinum seruat, i. e. pro (1) rursus erit numerus integer affirmatiuus; pro (2) integer negatiuus; contra valor τȢ k priori oppositus erit. Qua igitur permutatione facta series istae vtiliter adhiberi possunt, sistentque integralia particularia algebraice expressa.

4) Quanquam sic *alterutrum* incommodum (2, 3) seorsim tolli queat, dubium tamen suboriri posset, num id etiam semper efficere liceat, vt *neutrum* solutionem praecedentis problematis (§. XLII.) impediat. Quod dubium diluitur hac obseruatione. Si nimirum alterutrum incommodum sine altero adsit, et prius tollitur, permutando inuicem π et π^I (2), vel p et p^I (3), tum hoc sublato, nec pro nouis valoribus τῶν π, p, alterum incommodum locum habebit. Id quidem ex consideratione valorum, quos r et r^I recipiunt, manifestum est; hincque sponte colligitur, si vel vtrumque incommodum simul occurrat, vtrumque etiam, vnum post alterum, tolli semper posse.

Corollarium 2.

§. XLIV. Quo iam ex hactenus sigillatim demonstratis conclusio generalis formetur, casus, quibus integrale completum ex formulis inuentis vel algebraice vel saltem per quantitates logarithmicas et circulares exhibere licet, sic discerni possunt:

A) Si praeter numerum r, integrum, siue affirmatiuum siue negatiuum, trium insuper numerorum: 1) k, vel 2) $k - r^I$, vel 3) r^I, vnus fuerit itidem integer, affirmatiuus, seu negatiuus; tum integrale completum semper, si non algebraice, saltem per quantitates transcendentes notas assignabitur.

B) Ex hisce casibus praecipue notandi sunt tres specialiores, dum conditiones modo commemoratae (A), quae signorum nullum respectum habent, quoad haec signa numerorum integrorum arctius limitentur.

1) Si binorum numerorum integrorum r et k alteruter sit affirmatiuus, cum alter negatiuus sit, simulque $r+k$ eiusdem signi cum k;

2) Si binorum numerorum integrorum r et $k - r^I$ vterque sit affirmatiuus, vel vterque negatiuus;

3) Si eodem modo numeri integri r et r^I in signis conueniant; tum integrale completum quantitatibus mere algebraicis exprimitur, nisi quod casu tertio si fuerit

rit k numerus integer, nec simul conditio (1) locum habeat, logarithmus τῦ x in expressionem integralis ingrediatur. Horum casuum integrationem completam supra §§. XXXIII. XXXIV. 1. XXXV. XXXVII. XLI. satis euoluimus. Pro ceteris τῶν r, r^{I} et k valoribus, qui conditionibus (A), nec tamen conditionibus (B) accommodati sunt, integrale completum quantitates logarithmicas alias, praeter log. x, vel circulares inuoluet: eiusque determinatio pendet ab integratione formulae differentialis secundum regulas notas haud difficulter absoluenda (§. XLII.).

Scholion.

§. XLV. Alii auctores, qui integrationem aequationis nostrae differentialis, posito $\frac{\pi - p}{n} = r =$ numero integro, pertractarunt, plerumque tantum in integrali particulari per series abrumpentes expresso subsistentes, haud satis accurate euoluerunt, quomodo integrale completum inuestigandum sit.

1) Quae EVLERVS in Institt. Calc. Integr. (l. c.) de integrali completo tradit, tantum ad expressionem integralis per series infinitas pertinent. Idem in commentatione prima supra (§. II.) laudata (Comment. Petrop. T. X.), exhibito integrali particulari pro r = numero integro affirmatiuo, addit, exinde completum petendum esse ope regulae generalis, qua ex integrali particulari v aequationis $o = P d^2 y + Q dy dx + R y dx^2$ eliciatur completum $y = C v \int \frac{e^{-\int \frac{Q dx}{P}} . dx}{vv}$ (*). Quodsi haec regula ad casum nostrum specialem applicetur,

$$\text{erit } \int \frac{Q dx}{P} = \int \frac{(c + e x^n) dx}{x (a + b x^n)} = \frac{c}{a} \log. x + \left(\frac{e}{nb} - \frac{c}{na}\right) \log. (a + b x^n);$$

$$\text{vnde fit } y = C v \int \frac{x^{-\frac{c}{a}} (a + b x^n)^{\frac{c}{na} - \frac{e}{nb}} dx}{vv} = \mathfrak{C} v \int \frac{\chi^{\frac{2p}{n} - k - 1} (1 + \beta\chi)^{r + r^{I}} d\chi}{vv}.$$

Est autem

$$v = \chi^{\frac{p}{n}} \left(1 + \frac{r(k - r^{I})}{k + 1} \beta\chi + \frac{r(r-1)}{1 . 2} \frac{(k - r^{I})(k - r^{I} + 1)}{(k+1)(k+2)} \beta^2 \chi^2 + \dots\right),$$

pro affirmatiuo r; pro negatiuo $r = -\varrho$,

$$v = \chi^{\frac{p^{I}}{n}} (1 + \beta\chi)^{-\varrho - \varrho^{I} + 1} \left(1 + \frac{(\varrho - 1)(\varrho^{I} + k - 1)}{k - 1} \beta\chi + \frac{(\varrho - 1)(\varrho - 2)}{1 . 2} \frac{(\varrho^{I} + k - 1)(\varrho^{I} + k - 2)}{(k - 1)(k - 2)} \beta^2 \chi^2 + \dots\right).$$

Hinc

(*) Cf. infra §. LXXI. 1.

Hinc erit y vel $= D v \int \frac{\chi^{-k-1}(1+\beta\chi)^{r+r^{1}}\, d\chi}{\left(1+\frac{r(k-r^{1})}{k+1}\beta\chi+\frac{r(r-1)}{1.2}\,\frac{(k-r^{1})(k-r^{1}+1)}{(k+1)(k+2)}\beta^{2}\chi^{2}+\dots\right)^{2}}$, vel

$$= D v \int \frac{\chi^{k-1}(1+\beta\chi)^{\varrho+\varrho^{1}-2}}{\left(1+\frac{(\varrho-1)(\varrho^{1}+k-1)}{k-1}\beta\chi+\dots\right)^{2}},$$ prouti r fuerit numerus affirmatiuus vel negatiuus. Exinde intelligitur, hac ratione pendere determinationem integralis completi ab integratione formulae differentialis admodum complicatae; cum adhibitis contra nostris formulis, integralis completi expressio inuoluat tantum integrale differentialis omnino simplicioris, sc. $\int \chi^{-k-r-1}(1+\beta\chi)^{r^{1}} d\chi$ pro affirmatiuo r, et $\int \chi^{k-\varrho}(1+\beta\chi)^{\varrho^{1}-1} d\chi$ pro negatiuo $r = -\varrho$. Prior quidem formula differentialis, si vel ea ab irrationalitate liberata fuerit, difficulter tamen ex praeceptis vsitatis tractatur, haec enim praecepta supponunt resolutionem denominatoris in factores simplices vel quadraticos, qui quomodo inueniendi sint, denominatore inuoluente seriem indefinitam, haud liquet. Quare ista integrale completum eliciendi regula haud sufficiens esse videtur.

2) In commentatione recentiori (Nou. Comm. Petrop. XVII.) EVLERVS regulam omnino simplicem tradit de integrali completo algebraice determinando. Commemoratis binis casibus, quibus integrale particulare per series abrumpentes exprimere liceat, cum nimirum in nostris signis $\frac{1-p}{n} = r$ fuerit numerus integer vel affirmatiuus vel negatiuus, haec addit verba (§. XI. pag. 134. l. c.): "Si insuper $\frac{ae-bc}{ab}$ fuerit numerus integer, "vtroque modo integratio absolui poterit, vnde integrale completum algebraice obtinebi"tur." Cuius afferti rationem §pho IX. profert, quae sic concipi potest. Est pro quouis n, non tantum pro $n = 1$, $\frac{ae-bc}{nab} = -r-r^{1}$; posito igitur $\frac{ae-bc}{nab} = \alpha$, qua litera numerus integer, et quidem ex EVLERI mente affirmatiuus, denotetur, erit $r^{1} = -r-\alpha$. Hinc r^{1} obtinebit valorem integrum negatiuum, et cum r supponatur esse numerus integer affirmatiuus, vterque valor, r et r^{1}, suppeditabit integrale particulare, sicque "pro eadem aequatione gemina integralia exhiberi queunt" (l. c.), ex quorum combinatione sequitur integrale completum. Ex hoc fundamento patet, regulam commemoratam breuiter sic esse enuntiandam: Positis binorum numerorum integrorum r et r^{1} vno affirmatiuo, altero negatiuo, integrale completum semper algebraice obtineri. Quam vero ipsam regulam minus veram esse, ex iis elucet, quae de integrali completo supra demonstrata et §. XLIV. breuiter exposita sunt. Quanquam argumentatio ab EVLE-

RO

RO adhibita omnino sit speciosa, eam tamen haud solidam esse, sic probatur. Ex valore integro affirmatiuo r prodit integrale particulare hoc:

$$y = x^p\left(1 + \frac{r(k-r^1)}{k+1}\beta x^n + \frac{r(r-1)}{1\cdot 2}\frac{(k-r^1)(k-r^1+1)}{(k+1)(k+2)}\beta^2 x^{2n} + \ldots\right).$$

Ex valore negatiuo r^1 sequitur alterum integrale particulare:

$$y = x^p(1+\beta x^n)^{1+r+r^1}\left(1 + \frac{(r^1+1)(r+k+1)}{k+1}\beta x^n + \frac{(r^1+1)(r^1+2)}{1\cdot 2}\frac{(r+k+1)(r+k+2)}{(k+1)(k+2)}\beta^2 x^{2n} + \ldots\right).$$

Iam vero supra obseruatum est (§. XXVIII. 2.), haec bina integralia, etiamsi diuerse expressa, reuera tamen identica esse. Hinc EVLERI argumentum, quod illorum diuersitatem supponit, fundamento destituitur.

Cum itaque vel tantus Analysta in diiudicanda quaestione de integrali completo, a vero aberrauerit, nec ceteri auctores hanc quaestionem satis euoluerint, eandem omni cura denuo examinandam censui; indeque ea, quae hoc maxime consilio hactenus exposita sunt, quanquam paullo prolixiora, iis tamen haud prorsus superflua videbuntur, qui accuratiorem et, quantum fieri potest, absolutam cognitionem amant. Satis constat, aequationem differentialem secundi gradus tum demum pro resoluta habendam esse, cum integrale binas constantes arbitrarias inuoluat, i. e. completum sit. Integralia particularia non aeque late patent, ac aequationes differentiales ipsae; quin omnino fieri potest, cum de solutione certi problematis, v. c. geometrici, agitur, vt integrale quoddam particulare tali problemati nequaquam satisfaciat, verum solutio demum ex integrali completo petenda sit, dum constantes sic definiantur, vti conditiones peculiares problemati additae postulant.

Reliquum iam est, vt eiusmodi casus problematis (§. XLII.), qui problematibus tribus praecedentibus (§§. XXXIII. XXXVII. XLI.) haud subsunt, quibusque integrale completum transcendenter exprimitur, nonnullis exemplis illustremus.

Exemplum I.

§. XLVI. 1) Sit $r = 1$; $r^1 = -2$; $k = -\frac{3}{2}$; erit $\frac{c}{a} = 1 - 2p - \frac{3}{2}n$; $\frac{e}{b} = 1 - 2p - \frac{n}{2}$; $\frac{f}{a} = p(p+\frac{3}{2}n)$; $\frac{g}{b} = (p+n)(p-\frac{n}{2})$. Inde haec prodit aequatio differentialis:

$$0 = x^2(a+bx^n)d^2y + x\left(a\left(1-2p-\frac{3n}{2}\right) + b\left(1-2p-\frac{n}{2}\right)x^n\right)dy\,dx + \left(ap\left(p+\frac{3n}{2}\right) + b(p+n)\left(p-\frac{n}{2}\right)x^n\right)y\,dx^2.$$

2) Inte-

2) Integralis completi pars algebraica (§. XLII. 1.) est $= \mathfrak{A}\, x^{p}(1 - \beta x^{n})$. Ad determinandam alteram partem euoluendum est integrale $z = \int \chi^{-k-r-1}(1+\beta\chi)^{r} d\chi$ $= \int \frac{d\chi}{\chi^{\frac{1}{2}}(1+\beta\chi)^{2}}$, quod, posito $\beta\chi = u^{2}$, abit in $\frac{2}{\sqrt{\beta}} \int \frac{du}{(1+u^{2})^{2}}$. Est autem $\int \frac{du}{(1+u^{2})^{2}} = \frac{u}{2(1+u^{2})} + \frac{1}{2}\int \frac{du}{1+u^{2}} = \frac{u}{2(1+u^{2})} + \frac{1}{2}$ Arc. tang. u; hinc fit $z.\sqrt{\beta} = \frac{(\beta\chi)^{\frac{1}{2}}}{1+\beta\chi} +$ A. tang. $(\beta\chi)^{\frac{1}{2}}$. Quare habetur integralis completi pars altera

$$= M\chi^{\frac{p^{I}}{n}}(1+\beta\chi)^{1+r+r^{I}} \frac{d^{r}\left(\chi^{k+r}(1+\beta\chi)^{-r^{I}-1}.z\right)}{d\chi^{r}}$$

$$= \frac{M}{\sqrt{\beta}}\chi^{\frac{p}{n}+\frac{3}{2}} d\left(\sqrt{\beta} + \chi^{-\frac{1}{2}}(1+\beta\chi)\,\text{A.tg.}\,(\beta\chi)^{\frac{1}{2}}\right) : d\chi$$

$$= \frac{M}{\sqrt{\beta}}\chi^{\frac{p}{n}+\frac{3}{2}} \left\{\chi^{-\frac{1}{2}}(1+\beta\chi)\frac{\frac{1}{2}\beta}{(\beta\chi)^{\frac{1}{2}}(1+\beta\chi)} + \text{A. t.}\,(\beta\chi)^{\frac{1}{2}} \left(\chi^{-\frac{1}{2}}\beta - (1+\beta\chi)\chi^{-\frac{3}{2}}\right)\right\}$$

$$= \frac{M}{\sqrt{\beta}}\chi^{\frac{p}{n}}\left(\tfrac{1}{2}\sqrt{\beta}\chi - \text{A. tang.}\,\sqrt{\beta}\chi\right).$$

3) Exinde pro aequatione differentiali (1) haec obtinetur integralis completi expressio: $y = N x^{p}(1 - \beta x^{n}) + \mathfrak{M} x^{p}(\sqrt{\beta} x^{n} - 2\,\text{A.tang.}\,\sqrt{\beta} x^{n})$; vti N et $\mathfrak{M}$ sunt Constantes arbitrariae, et $\beta = \frac{b}{a}$.

Hoc etiam exemplo comprobatur, EVLERI assertum de integrali completo algebraice determinando erroneum esse (§. XLV. 2.). Est nimirum pro hac aequatione $r =$ numero integro, simulque $\frac{ae - bc}{nab} = 1$; nihilominus tamen integrale completum transcendenter exprimitur.

Exemplum 2.

§. XLVII. Sit $r = 0$, $r^I = \frac{1}{2}$, $k = 0$, erit $\frac{c}{a} = 1 - 2p$; $\frac{e}{b} = 1 - 2p - \frac{n}{2}$; $\frac{f}{a} = p^2$; $\frac{g}{b} = p(p + \frac{n}{2})$; inde aequatio integranda haec:

$$0 = x^2(a + bx^n)d^2y + x(a(1-2p) + (1 - 2p - \frac{n}{2})x^n)dydx + (ap^2 + bp(p + \frac{n}{2})x^n)y.$$

Cuius integrale completum y prodit $= Nx^p + M\chi^{\frac{p}{n}} \int \chi^{-1}(1 + \beta\chi)^{\frac{1}{2}} d\chi$.

Est autem, posito $1 + \beta\chi = u^2$, $\int \frac{d\chi(1+\beta\chi)^{\frac{1}{2}}}{\chi} = 2\int \frac{udu \,.\, u}{u^2 - 1}$

$= 2\int du\left(1 + \frac{1}{u^2 - 1}\right) = 2u + \log. \frac{u - 1}{u + 1}$; hinc fit

$$y = Nx^p + Mx^p\left\{2\sqrt{(1 + \beta x^n)} + \log.\left(\frac{\sqrt{(1+\beta x^n)} - 1}{\sqrt{(1+\beta x^n)} + 1}\right)\right\}$$

$$= \mathfrak{M}x^p \,.\, \left(\mathfrak{N} + \sqrt{(1 + \beta x^n)} + \log.(\sqrt{(1 + \beta x^n)} - 1) - \frac{n}{2}\log. x\right).$$

Exemplum 3.

PROBLEMA.

§. XLVIII. Reperire integrale *completum* aequationis differentialis:

$$0 = \alpha^2 d^2y(1 + x^2) - \gamma^2 y dx^2.$$

Solutio.

1) Comparando hanc aequationem differentialem cum forma generali hactenus considerata, est $a = b = \alpha^2$; $c = e = 0$; $f = 0$; $g = -\gamma^2$; $n = 2$. Hinc aequationes pro p et π in has duas abeunt: $0 = p(p - 1)$; $\frac{\gamma^2}{\alpha^2} = \pi(\pi - 1)$. Aequationi primae satisfaciunt valores $p = 0$, et $p = 1$. Assumto valore priori erit $\frac{\pi - p}{n} = \frac{\pi}{2}$ $= r$, hinc $\frac{\gamma^2}{\alpha^2} = 2r(2r - 1)$; pro altero valore est $\frac{\pi - p}{n} = \frac{\pi - 1}{2}$ $= r$, hinc $\frac{\gamma^2}{\alpha^2} = (2r + 1)2r$. Iam constat, y per seriem abrumpentem exprimi, si r aequetur numero integro; inde sequitur, hac ratione aequationem integrabilem fore, si fuerit $\frac{\gamma^2}{\alpha^2} = s(s - 1)$, denotante s numerum integrum, sive parem $(2r)$, sive imparem $(2r + 1)$.

2) Po-

2) Posito *primum* $\frac{\gamma^2}{\alpha^2} = 2r(2r-1)$, erit $p^I = 1$; $k = -\frac{1}{2}$; $\pi^I = 1 - \pi$, $= 1 - 2r$; $r^I = -r$; hinc obtinetur

$$y = A\left(1 + \frac{r(2r-1)}{1}x^2 + \frac{r(r-1)}{1.2}\,\frac{(2r-1)(2r+1)}{1.3}x^4 + \frac{r(r-1)(r-2)}{1.2.3}\,\frac{(2r-1)(2r+1)(2r+3)}{1.3.5}x^6 + \ldots\right)$$

$$= Av.$$

3) Posito *secundo* $\frac{\gamma^2}{\alpha^2} = (2r+1)2r$, est $p = 1$, $p^I = 0$; $k = \frac{1}{2}$; $\pi^I = 1 - \pi$ $= -2r$; $r^I = -r$; hinc prodit

$$y = Ax\left(1 + \frac{r(2r+1)}{3}x^2 + \frac{r(r-1)}{1.2}\,\frac{(2r+1)(2r+3)}{3.5}x^4 + \frac{r(r-1)(r-2)}{1.2.3}\,\frac{(2r+1)(2r+3)(2r+5)}{3.5.7}x^6 + \ldots\right)$$

$$= Aw.$$

Quam seriem, aeque ac priorem (2), ob r numerum integrum abrumpere, manifestum est.

4) His tamen seriebus integralia tantum *particularia* aequationis differentialis propositae exprimuntur. Quaeritur igitur, quomodo integratio *completa* peragenda sit: quod quidem negotium difficilius esse videtur.

Sumatur *primo* $\frac{\gamma^2}{\alpha^2} = 2r(2r-1)$, tum, praeter integrale (2), habetur alterum integrale particulare hac serie expressum (§§. XXXI. 1. XXXVII. 1.):

$$y = Bx\left(1 + \frac{r(2r-1)}{3}x^2 + \frac{r(r+1)}{1.2}\,\frac{(2r-1)(2r-3)}{3.5}x^4 + \frac{r(r+1)(r+2)}{1.2.3}\,\frac{(2r-1)(2r-3)(2r-5)}{3.5.7}x^6 + \ldots\right)$$

$$= Bv^I.$$

Quae series cum infinita sit, videndum est, quomodo eius summa inueniri queat: quo facto erit integrale completum $= Av + Bv^I$. Simili deinde ratione casus *alter*, quo $\frac{\gamma^2}{\alpha^2} = 2r(2r+1)$ (3), tractandus est.

5) Quem in finem illa series v^I in duas partes distribuatur, quarum vna comprehendat terminos r priores, altera ceteros. Cuius alterius partis termini seorsim considerati hanc seriem constituunt:

$\frac{(2r-1)(2r-3)\ldots 1}{1.2.3\ldots r-1}$

$$\frac{(2r-1)(2r-3)\ldots 1}{1.2.3\ldots r-1}\left(\frac{(r+1)\ldots(2r-1)}{1.3\ \ldots\ 2r+1}x^{2r+1}-\frac{(r+2)\ldots 2r}{3.5\ldots 2r+3}x^{2r+3}+\frac{(r+3)\ldots(2r+1)}{5\ldots 2r+5}x^{2r+5}-\ldots\pm\frac{(r+m)(r+m+1)\ldots(2r+m-2)}{(2m-1)(2m+1)\ldots(2r+2m-1)}x^{2r+2m-1}\mp\ldots\right)$$

$=\frac{(2r-1)\ldots 1}{1.2\ldots r-1}.S.$ Termini generalis seu m^{ti} seriei S coëfficiens $\frac{(r+m)\ldots(2r+m-2)}{(2m-1)\ldots(2r+2m-1)}$, instar functionis fractae ipsius m, methodo vsitata (*) in fractiones simplices resolutus, fit $=\frac{C}{2m-1}+\frac{C^{I}}{2m+1}+\frac{C^{II}}{2m+3}+\ldots+\frac{C^{r}}{2m+2r-1}$, vbi numeratores tales prodeunt:

$$C=\frac{1}{2^{2r-1}}.\frac{(2r+1)(2r+3)\ldots(4r-3)}{1.2.3\ldots r}$$

$$C^{I}=-\frac{1}{2^{2r-1}}.\frac{1}{1}.\frac{(2r-1)(2r+1)\ldots(4r-5)}{1.2\ldots r-1}$$

$$C^{II}=+\frac{1}{2^{2r-1}}.\frac{1}{1.2}.\frac{(2r-3)\ldots(4r-7)}{1.2\ldots r-2}$$

. . . . .

$$C^{r-1}=\mp\frac{1}{2^{2r-1}}.\frac{1}{1.2\ldots r-1}.\frac{3.5\ldots 2r-1}{1}$$

$$C^{r}=\pm\frac{1}{2^{2r-1}}.\frac{1}{1.2\ldots r}.1.3.5\ldots 2r-3$$

quorum mutuam relationem etiam sic exprimere licet:

$$C^{r-1}=-\frac{r}{1}\frac{(2r-1)}{1}C^{r};$$

$$C^{r-2}=+\frac{r(r-1)}{1.2}\frac{(2r-1)(2r+1)}{1.3}C^{r};$$

$$C^{r-3}=-\frac{r(r-1)(r-2)}{1.2.3}\frac{(2r-1)(2r+1)(2r+3)}{1.3.5}C^{r};$$

etc. etc.

Hac adhibita resolutione termini generalis, series S in $r+1$ series partiales dispescitur: est nimirum S =

C

(*) *Kaestner* Analys. inf. §. 361. p. 291. edit. 2.

$$C\left(\frac{x^{2r+1}}{1}-\frac{x^{2r+3}}{3}+\frac{x^{2r+5}}{5}-\ldots\right)$$
$$+C^{I}\left(\frac{x^{2r+1}}{3}-\frac{x^{2r+3}}{5}+\frac{x^{2r+5}}{7}-\ldots\right)$$
$$+C^{II}\left(\frac{x^{2r+1}}{5}-\frac{x^{2r+3}}{7}+\frac{x^{2r+5}}{9}-\ldots\right)$$
$$+\quad.\quad.\quad.\quad.$$
$$+C^{r-1}\left(\frac{x^{2r+1}}{2r-1}-\frac{x^{2r+3}}{2r+1}+\frac{x^{2r+5}}{2r+3}-\ldots\right)$$
$$+C^{r}\left(\frac{x^{2r+1}}{2r+1}-\frac{x^{2r+3}}{2r+3}+\frac{x^{2r+5}}{2r+5}-\ldots\right)$$

Quarum ferierum prima fponte fummabilis eft, ceterae ad primam reducuntur; ficque habetur $S = Cx^{2r}$ A. tang. x

$$+C^{I}x^{2r-2}(-\text{A. tang. } x + x)$$
$$+C^{II}x^{2r-4}\left(\text{A. tang. } x - x + \frac{x^3}{3}\right)$$
$$+C^{III}x^{2r-6}\left(-\text{A. tang. } x + x - \frac{x^3}{3} + \frac{x^5}{5}\right)$$
$$+\quad.\quad.\quad.$$
$$+C^{r-1}x^2\left(\mp\text{A. tang. } x \pm x \mp \frac{x^3}{3} \pm \frac{x^5}{5} \ldots + \frac{x^{2r-3}}{2r-3}\right)$$
$$+C^{r}\left(\pm\text{A. tang. } x \mp x \pm \frac{x^3}{3} \ldots + \frac{x^{2r-1}}{2r-1}\right);$$

vel, terminos in A. tang. x ductos coniungendo, et reliquos fecundum poteftates τῆς x ordinando, $\pm S =$ A. tang. x . $(C^r - C^{r-1}x^2 + C^{r-2}x^4 \ldots \pm Cx^{2r})$

$$-x.C^r + x^3\left(C^{r-1}+\frac{C^r}{3}\right) - x^5\left(C^{r-2}+\frac{C^{r-1}}{3}+\frac{C^r}{5}\right)$$
$$+x^7\left(C^{r-3}+\frac{C^{r-2}}{3}+\frac{C^{r-1}}{5}+\frac{C^r}{7}\right)-\ldots\pm x^{2r-1}\left(C^I+\frac{C^{II}}{3}+\frac{C^{III}}{5}\ldots+\frac{C^r}{2r-1}\right).$$

6) Ex hac fummatione obtinetur integrale particulare alterum v^I (4), quod cum priori v (2) iunctum praebet integralis completi expreffionem hanc:

y =

$$y = \left(A \pm B \cdot \frac{1 \cdot 3 \ldots 2r-1}{1 \cdot 2 \ldots r-1} \cdot C^r A.\text{tang.}x\right)\left(1 + r\frac{(2r-1)}{1}x^2 + \frac{r(r-1)}{1 \cdot 2}\frac{(2r-1)(2r+1)}{1 \cdot 3}x^4 + \ldots\right)$$

$$+ Bx\left(1 + \frac{r(2r-1)}{3}x^2 + \frac{r(r+1)}{1 \cdot 2}\frac{(2r-1)(2r-3)}{3 \cdot 5}x^4 + \ldots + \frac{r(r+1)\ldots(2r-2)}{1 \cdot 2 \ldots r-1} \cdot \frac{(2r-1)\ldots 3}{3 \cdot 5 \ldots 2r-1}x^{2r-2}\right)$$

$$\mp B \cdot \frac{1 \cdot 3 \ldots 2r-1}{1 \cdot 2 \ldots r-1} \cdot x\left(C^r - \left(C^{r-1} + \frac{C^r}{3}\right)x^2 + \left(C^{r-2} + \frac{C^{r-1}}{3} + \frac{C^r}{5}\right)x^4 - \ldots \mp \left(C^I + \frac{C^{II}}{3} \ldots + \frac{C^r}{2r-1}\right)x^{2r-2}\right).$$

7) Qua igitur ratione inuenta est

Integratio completa

aequationis differentialis: $0 = \alpha^2 d^2 y(1 + x^2) - \gamma^2 y\, dx^2$, posito $\frac{\gamma^2}{\alpha^2} = 2r(2r-1)$.

Erit nimirum

$$y = (A + B\mathfrak{C} \cdot A.\text{tang.}x)\left(1 + \frac{r(2r-1)}{1}x^2 + \frac{r(r-1)}{1 \cdot 2}\frac{(2r-1)(2r+1)}{1 \cdot 3}x^4 + \frac{r \ldots (r-2)}{1 \cdot 2 \cdot 3}\frac{(2r-1)\ldots(2r+3)}{1 \cdot 3 \cdot 5}x^6 + \ldots\right)$$

$$+ Bx\left(D^I + D^{II}x^2 + D^{III}x^4 + D^{IV}x^6 + \ldots + D^R x^{2r-2}\right),$$

vbi A, B denotant Constantes arbitrarias, et coëfficientes $D^I, D^{II}, \ldots D^R$ his aequationibus definiuntur:

$$D^I = 1 - \mathfrak{C}$$

$$D^{II} = \frac{r}{1}\frac{(2r-1)}{3} + \mathfrak{C}\left(\frac{1}{3} - \frac{r}{1}\frac{(2r-1)}{1}\right)$$

$$D^{III} = \frac{r(r+1)}{1 \cdot 2}\frac{(2r-1)(2r-3)}{3 \cdot 5} - \mathfrak{C}\left(\frac{1}{5} - \frac{1}{3}\frac{r}{1}\frac{(2r-1)}{1} + \frac{1}{1}\frac{r(r-1)}{1 \cdot 2}\frac{(2r-1)(2r+1)}{1 \cdot 3}\right)$$

$$D^{IV} = \frac{r(r+1)(r+2)}{1 \cdot 2 \cdot 3}\frac{(2r-1)(2r-3)(2r-5)}{3 \cdot 5 \cdot 7} + \mathfrak{C}\left(\frac{1}{7} - \frac{1}{5}\frac{r}{1}\frac{(2r-1)}{1} + \frac{1}{3}\frac{r(r-1)}{1 \cdot 2}\frac{(2r-1)(2r+1)}{1 \cdot 3} - \frac{1}{1}\frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3}\frac{(2r-1)(2r+1)(2r+3)}{1 \cdot 3 \cdot 5}\right)$$

qua-

quarum lex progressus manifesta est. Litera $\mathfrak{C}$ ponitur $=$

$$\frac{1}{2^{2r-1}} \cdot \frac{1.3.5\ldots 2r-3}{1.2\ldots r} \cdot \frac{1.3\ldots 2r-1}{1.2\ldots r-1}.$$

8) Formulis hisce *independentibus*, quibus coëfficientes determinantur, haud inutile est adiungere formulas *recurrentes* (*), quibus quilibet ex praecedente definitur. Est nimirum:

$$1.3D^{II} - r(2r-1)\ D^{I} = \mathfrak{C}(1 - \frac{(2r-1)2r}{1})$$

$$2.5D^{III} - (r+1)(2r-3)D^{II} = \mathfrak{C}.\frac{(r-1)}{1}\frac{(2r+1)}{1}(1 - \frac{(2r-1)2r}{3})$$

$$3.7D^{IV} - (r+2)(2r-5)D^{III} = \mathfrak{C}.\frac{(r-1)(r-2)}{1.2}\frac{(2r+1)(2r+3)}{1.3}(1 - \frac{(2r-1)2r}{5})$$

$$4.9D^{V} - (r+3)(2r-7)D^{IV} = \mathfrak{C}.\frac{(r-1)(r-2)(r-3)}{1.2.3}\frac{(2r+1)(2r+3)(2r+5)}{1.3.5}(1 - \frac{(2r-1)2r}{7})$$

etc. etc.

Ad has aequationes perueni, substituendo formulam pro y (7) in aequatione differentiali proposita, vnde ipsae hae relationes inter coëfficientes D^{I}, D^{II}, ... prodeunt. Talis substitutio statim ab initio adhiberi, indeque alia ratio, integrale completum inueniendi, deduci posset. Posito nimirum $y = v.A.\mathrm{tang}.x + p$, aequatio differentialis nostra has binas aequationes suppeditat: $d^2v(1+x^2) - (2r-1)2r.vdx^2 = 0$;

$$d^2p(1+x^2) - (2r-1)2rpdx^2 + 2dx(dv - \frac{vxdx}{1+x^2}) = 0.$$

Quarum priori satisfacit integrale particulare (2), seu $v =$
$\mathfrak{B}(1 + r(2r-1)x^2 + \frac{r(r-1)}{1.2}\frac{(2r-1)(2r+1)}{1.2}x^4 + \ldots)$. Assumta deinde pro p hac serie: $p = A + A^{I}x + A^{II}x^2 + A^{III}x^3 + A^{IV}x^4 + \ldots$, ex altera aequatione differentiali, ob $\frac{v}{1+x^2} = \mathfrak{B}(1 + (r-1)(2r+1)x^2 + \frac{(r-1)(r-2)}{1.2}\frac{(2r+1)(2r+3)}{1.3}x^4 + \ldots)$, obtinentur pro coëfficientibus A^{II}, A^{III}, A^{IV} ... sequentes aequationes:

$$A^{II} = \frac{2r(2r-1)}{1.2}A$$

$$A^{IV} = \frac{(2r-2)(2r+1)}{3.4}A^{II}$$

$$A^{VI} = \frac{(2r-4)(2r+3)}{5.6}A^{IV}$$

etc. etc.

por-

(*) Vtrumque hoc formularum analyticarum genus *independentium* et *recurrentium*, re et verbis primum clarius distinxit *Hindenburgius* (cf. infra *Disquis.* III.).

porro:

$$2.3\,A^{III} - (2r-1)2r\,A^{I} = 2\mathfrak{B}(1-(2r-1)2r)$$
$$4.5\,A^{V} - (2r-3)(2r+2)A^{III} = 2\mathfrak{B}\frac{(r-1)}{1}\frac{(2r+1)}{1}\left(1-\frac{(2r-1)2r}{3}\right)$$
$$6.7\,A^{VII} - (2r-5)(2r+4)A^{V} = 2\mathfrak{B}\frac{(r-1)(r-2)}{1.2}\frac{(2r+1)(2r+3)}{1.3}\left(1-\frac{(2r-1)2r}{5}\right)$$

etc. etc.

Prior aequationum feries fponte ad coëfficientes euanefcentes deducit; nec minus altera, dum ponatur $A^{2r+1}=0$, vnde etiam coëfficientes fequentes cum indicibus imparibus euanefcunt, praecedentes vero per $\mathfrak{B}$ definiuntur, vti coëfficientes cum indicibus paribus per A. Sic habetur integrale completum, quippe quod duas Conftantes arbitrarias A et $\mathfrak{B}$ inuoluit. Hoc integrale cum prius inuento confpirat.

9) Ponatur iam *fecundo* $\frac{\gamma^2}{\alpha^2}=(2r+1)2r$, tum integrali particulari (3) iungendum eft integrale particulare alterum, hac ferie expreffum:

$$B\Big(1+r(2r+1)x^2+\frac{r(r+1)}{1.2}\frac{(2r+1)(2r-1)}{1.3}x^4+\frac{r(r+1)(r+2)}{1.2.3}\frac{(2r+1)(2r-1)(2r-3)}{1.3.5}x^6+\ldots\Big)$$

$=Bw^I$. Quam feriem infinitam eadem methodo fummare liceret, ac feriem v^I (5). Repetita tamen huius methodi applicatio fuperflua redditur, fequenti obferuatione, qua oftenditur, fummationem feriei w^I ad fummam iam inuentam feriei v^I reduci poffe. Eft nimirum

$$\frac{w^I}{1+x^2}=1+\frac{(r+1)}{1}\frac{(2r-1)}{1}x^2+\frac{(r+1)(r+2)}{1.2}\frac{(2r-1)(2r-3)}{1.3}x^4+\ldots$$
$$\frac{v^Ix}{1+x^2}=\quad x^2\quad+\frac{(r+1)(2r-3)}{3}x^4+\frac{(r+1)(r+2)}{1.2}\frac{(2r-3)(2r-5)}{3.5}x^6+\ldots$$

hinc fit $\frac{w^I-(2r-1)v^Ix}{1+x^2}=1+r(2r-1)x^2+\frac{r(r+1)}{1.2}\frac{(2r-1)(2r-3)}{1.3}x^4+\ldots$

$=\frac{dv^I}{dx}$; feu $w^I=(1+x^2)\frac{dv^I}{dx}+(2r-1)xv^I$. Iam habetur ex (7)

$$\frac{B\,dv^I}{dx}=\frac{B\mathfrak{C}}{1+x^2}\Big(1+r(2r-1)x^2+\frac{r(r-1)}{1.2}\frac{(2r-1)(2r+1)}{1.3}x^4+\ldots\Big)$$
$$+B\mathfrak{C}.x\,A.\,tang.\,x\Big(2.r(2r-1)+2.\frac{r(r-1)}{1}\frac{(2r-1)(2r+1)}{1.3}x^2+\ldots\Big)$$
$$+B(D^I+3D^{II}x^2+5D^{III}x^4+\ldots+(2r-1)D^Rx^{2r-2});$$

hinc

hinc fit $B w^I =$

$$B\mathfrak{C}.x.A.\text{tang}.x\left\{\begin{matrix}2r(2r-1)+2.\frac{r(r-1)}{1}\frac{(2r-1)(2r+1)}{1.3}x^2+\dots \\ +2r-1+(2r-1).\frac{r(2r-1)}{1}x^2+\dots\end{matrix}\right\}$$

$$+B\mathfrak{C}\left(1+\frac{r(2r-1)}{1}x^2+\frac{r(2r-1)}{1.2}\frac{(2r-1)(2r+1)}{1.3}x^4+\dots\right)$$

$$+B(1+x^2)(D^I+3D^{II}x^2+5D^{III}x^4+\dots+(2r-1)D^R x^{2r-2})$$

$$+(2r-1)Bx^2(D^I+D^{II}x^2+D^{III}x^4+\dots+D^R x^{2r-2});$$

cuius expressionis pars prima sic exhiberi potest:

$$B\mathfrak{C}x.A.\text{tang}.x.(2r-1)(2r+1)\left(1+\frac{r(2r+1)}{3}x^2+\frac{r(r-1)}{1.2}\frac{(2r+1)(2r+3)}{3.5}x^4+\dots\right)$$

10) Ex hac summatione sponte consequitur integrale completum $y = Aw + Bw^I$. Inde obtinetur

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aequationis differentialis: $0 = \alpha^2 d^2 y(1+x^2) - \gamma^2 y dx^2$,

posito $\frac{\gamma^2}{\alpha^2} = (2r+1)2r$. Est nimirum

$$y = (A+(4r^2-1)B\mathfrak{C}.A.t.x)x\left(1+\frac{r(2r+1)}{3}x^2+\frac{r(r-1)}{1.2}\frac{(2r+1)(2r+3)}{3.5}x^4+\dots\right).$$

$+B(1+\Delta^I x^2+\Delta^{II}x^4+\Delta^{III}x^6+\dots+\Delta^R x^{2r})$, vbi A et B sunt constantes arbitrariae, et coëfficientes $\Delta^I, \Delta^{II}, \dots \Delta^R$ ex his aequationibus definiuntur.

$$\Delta^I = \mathfrak{C}\frac{r(2r-1)}{1}+3D^{II}+2rD^I$$

$$\Delta^{II} = \mathfrak{C}\frac{r(r-1)}{1.2}\frac{(2r-1)(2r+1)}{1.3}+5D^{III}+2(r+1)D^{II}$$

$$\Delta^{III} = \mathfrak{C}\frac{r(r-1)(r-2)}{1.2.3}\frac{(2r-1)(2r+1)(2r+3)}{1.3.5}+7D^{IV}+2(r+2)D^{III}$$

etc. etc.

Quantitatum $\mathfrak{C}$; $D^I, D^{II}, \dots D^R$; valores iam (7) assignati sunt.

11) Coëfficientium $\Delta^I, \Delta^{II}, \dots \Delta^R$ valores etiam sequentem in modum exhiberi possunt:

$$\Delta^{I} = r(2r+1) - (4r^2-1)\mathfrak{C}$$

$$\Delta^{II} = \frac{r(r+1)}{1.2}\frac{(2r+1)(2r-1)}{1.3} + (4r^2-1)\mathfrak{C}\left(\tfrac{1}{3} - \frac{r(2r+1)}{3}\right)$$

$$\Delta^{III} = \frac{r(r+1)(r+2)}{1.2.3}\frac{(2r+1)(2r-1)(2r-3)}{1.3.5} - (4r^2-1)\mathfrak{C}\left(\tfrac{1}{5} - \tfrac{1}{3}\frac{r(2r+1)}{3} + \tfrac{1}{1}\frac{r(r-1)}{1.2}\frac{(2r+1)(2r+3)}{3.5}\right)$$

etc. etc.

Porro mutuam eorum relationem fic exhibere licet:

$$2.3\Delta^{II} - (r+1)(2r-1)\Delta^{I} = (4r^2-1)\mathfrak{C}(1-(2r+1)r)$$

$$3.5\Delta^{III} - (r+2)(2r-3)\Delta^{II} = (4r^2-1)\mathfrak{C}(r-1)\frac{(2r+3)}{3}\left(1 - \frac{(2r+1)r}{2}\right)$$

$$4.7D^{IV} - (r+3)(2r-5)D^{III} = (4r^2-1)\mathfrak{C}\frac{(r-1)(r-2)}{1.2}\frac{(2r+3)(2r+5)}{3.5}\left(1 - \frac{(2r+1)2r}{3}\right)$$

etc. etc.

Hae aequationes obtinentur, fi integrale completum methodo pro cafu priori adhibita quaeritur. Eaedem cum aequationibus (10) confentiunt.

Scholion.

§. XLIX. De aequatione differentiali §pho praecedente integrata egit L. EVLERVS in Nou. Act. Erudit. Lipf. Iun. 1744. (pp. 315-336.) (*), occafione fumta a problemate in Nou. Act. Nouembr. 1743 propofito, quo quaerebatur linea curua, "in cuius axe duo dentur eiusmodi puncta, vt, ductis inde ad quoduis peripheriae punctum binis rectis, areae ex vno puncto refectae perpetuo fint proportionales angulis ad alterum punctum formatis". Ex integralibus per feries abrumpentes expreffis (§. XLVIII. 2. 3.) reperiuntur innumerae curuae algebraicae, problemati fatisfacientes: vna nimirum linea ex quouis ordine. Praeter hasce curuas algebraicas, alteram claffem conftituunt innumerae curuae tranfcendentes, quae per quadraturam circuli conftrui queunt.

Hac data occafione EVLERVS obferuat (l. c. p. 324.), valores feriebus finitis v, w, expreffos fiftere integralia tantum particularia aequationis differentialis propofitae, tumque demum exinde prodire integralia completa, cum illa iungantur integralibus itidem particularibus, per feries infinitas v^I, w^I (§. XLVIII. 4. 9.) exhibendis. Pro $\frac{\gamma^2}{\alpha^2} = 2.1$, fummam feriei v^I per Arcum circularem determinat; additque, in genere fummationem ferierum v^I, w^I, pendere a quadratura circuli. Hanc vero fummationem generalem haud aggreditur, quippe quam mox ad "*calculos inextricabiles*" perducere exiftimat

(*) cf. *Kaeftneri* Analyf. infinit. §. 462. p. 404. Literis illic adhibitis z; t; a; c; refpondent hic, literae y; x; α; γ.

ſtimat (l. c. p. 325.). Quam ob cauſſam aliam methodum, integrale completum inueſtigandi, exponit, eandem iam ſupra §. XLV. 1. breuiter indicatam, qua ad aequationem propoſitam adplicata, obtinetur $y = Cv \int \frac{dx}{v^2}$ vel $= Cw \int \frac{dx}{w^2}$, vbi v et w denotant integralia particularia. Ex his formulis caſus tres ſimpliciores: $\frac{\gamma^2}{\alpha^2} = 2 \cdot 1$; $= 3 \cdot 2$; $= 4 \cdot 3$; euoluit; hincque porro concludit, in reliquis quoque caſibus integrationes $\int \frac{dx}{v^2}$, $\int \frac{dx}{w^2}$, conceſſa ſola Circuli quadratura abſolui poſſe. At vero cum

$$\int \frac{dx}{v^2} \text{ ſit} = \int \frac{dx}{\left(1 + \frac{r(2r-1)}{1} x^2 + \frac{r(r-1)}{1.2} \frac{(2r-1)(2r+1)}{1.3} x^4 + \ldots\right)^2},$$

$$\int \frac{dx}{w^2} = \int \frac{dx}{x^2 \left(1 + \frac{r(2r+1)}{3} x^2 + \frac{r(r-1)}{1.2} \frac{(2r+1(2r+3)}{3.5} x^4 + \ldots\right)^2},$$

cumque harum formularum denominatores in factores ſimplices vel duplices reſolui nequeant; fateor, me vix intelligere, quomodo haec integralia generaliter pro quouis r, methodo vſitata, nec adhibitis aliis artificiis, determinari poſſint: ne id quidem ſatis inde manifeſtum videtur, integrationes tantum pendere a quadratura circuli. Quare huic methodo alteram, §pho praecedenti expoſitam, praeferendam duxi, qua integrale completum ab EVLERO tantum pro $s = 1$; 2; 3; expreſſum, formula generali, ad quemuis numerum integrum $= s$ patente, exhibetur: quaque methodo ſimul ſummatio ſerierum v^1, w^1, quae calculos operoſiſſimos poſcere videbatur, via ſatis ſimplici eruta eſt.

Ceterum abſque hac ſummatione, integrale completum etiam ex formulis noſtris generalibus (§. XLII. 1.) deducere liceret. Eſt nimirum pro $\frac{\gamma^2}{\alpha^2} = 2r(2r-1)$,

$$y = \chi^{\frac{1}{2}}(1+\chi)\frac{d^r(\chi^{r-\frac{1}{2}}(1+\chi)^{r-1}z)}{d\chi^r}, \text{ poſito } \chi = x^2;$$

$z = N + M\int \chi^{-r-\frac{1}{2}}(1+\chi)^{-r} d\chi$; denotantibus N et M conſtantes arbitrarias, et ſumto in repetita differentiatione $d\chi$ pro differentiali conſtanti. Iam integrale $\int \chi^{-r-\frac{1}{2}}(1+\chi)^{-r} d\chi = 2\int \frac{dx}{x^{2r}(1+x^2)^r}$ ex regulis notis * ad $\int \frac{dx}{(1+x^2)^r}$, hincque porro ad $\int \frac{dx}{1+x^2} = $ A. tang. x reducitur. Exinde ſtatim ſequitur, integrale com-

* *Kaeſtner* Analyſ. infinit. §§. 400. 401. p. 347 ſq.

pletum praeter quantitates algebraicas inuoluere tantum A. tang. x. Ipsam tamen expressionem pro y hac ratione euoluere prolixius videtur; hinc satius foret, formam tantum integralis exinde sumere, et quantitates, quas ea continet, indeterminatas ex substitutione in ipsa aequatione differentiali proposita quaerere. Quam viam supra iam attigimus (§. XLVIII. 8.), vnde nunc haec commemorasse sufficiat. Quae hactenus dicta etiam obtinent pro $\frac{\nu^2}{\alpha^2} = 2r(2r+1)$, quo casu est $y = \frac{(1+\chi)\, d^r\left(\chi^{r+\frac{1}{2}}(1+\chi)^{r-1} z\right)}{d\chi^r}$,

posito $z = N + M\int \chi^{-r-\frac{3}{2}}(1+\chi)^{-r}\, d\chi$.

ARTICVLVS SECVNDVS.

Euolutio casus integrabilis secundi.

PROBLEMA.

§. L. Integrare aequationem differentialem:

$$0 = x(a+bx)\, d^2 z + \left(\frac{a}{2} + bx\right) dz\, dx + Gz\, dx^2.$$

Solutio.

1) Multiplicando per dz habetur

$$x(a+bx)\, dz\, d^2 z + \left(\frac{a+2bx}{2}\right) dz^2\, dx = -Gz\, dz\, dx^2,$$

i. e. $\frac{1}{2} d\left(x(a+bx)\, dz^2\right) = -\frac{G}{2}\, d(z^2\, dx^2)$.

Hinc, particulariter integrando, fit $x(a+bx)\, dz^2 = -Gz^2\, dx^2$, seu $\frac{dz}{z} = \pm\sqrt{-G}\cdot\frac{dx}{\sqrt{x(a+bx)}}$.

2) Ex integratione nota: $\int\frac{du}{\sqrt{(1+u^2)}} = \log.(u+\sqrt{(1+u^2)})$ sequitur $\int\frac{dx}{\sqrt{x(a+bx)}}$, (posito $\frac{bx}{a} = t^2$), $= \frac{2a}{b}\int\frac{t\,dt}{\sqrt{\frac{a}{b}t^2(a+at^2)}} = \frac{2}{\sqrt{b}}\int\frac{dt}{\sqrt{(1+t^2)}}$

$= \frac{2}{\sqrt{b}}\log.\left(\sqrt{\frac{bx}{a}} + \sqrt{\frac{(a+bx)}{a}}\right)$. Hinc ex (1) obtinetur $\log. z = \log.$ Const. $\pm 2\sqrt{-\frac{G}{b}}\cdot\log.(\sqrt{(a+bx)}+\sqrt{bx})$, siue $z = C.(\sqrt{(a+bx)}+\sqrt{bx})^{\pm 2\sqrt{-\frac{G}{b}}}$.

Ob duplicitatem signi, inde statim prodit integrale completum, duas constantes arbitrarias A, B, inuoluens, hoc:

$z =$

$$z = A\left(\sqrt{(a+bx)}+\sqrt{bx}\right)^{2\sqrt{-\frac{G}{b}}} + B\left(\sqrt{(a+bx)}+\sqrt{bx}\right)^{-2\sqrt{-\frac{G}{b}}}$$

siue etiam

$$z = A\left(\sqrt{(a+bx)}+\sqrt{bx}\right)^{2\sqrt{-\frac{G}{b}}} + \mathfrak{B}\left(\sqrt{(a+bx)}-\sqrt{bx}\right)^{2\sqrt{-\frac{G}{b}}},$$

ob $\sqrt{(a+bx)}-\sqrt{bx} = \frac{a}{\sqrt{(a+bx)}+\sqrt{bx}}$.

Aequationem (1) complete integrando, eadem expressio pro z reperitur, calculis tantum prolixioribus.

Corollarium.

§. LI. Casus $b = o$ peculiarem solutionem postulat, quippe tum integrale quantitates exponentiales inuoluet. Est nimirum (§. L. 1.) $\frac{dz}{z} = \pm\sqrt{-\frac{G}{a}}\cdot\frac{dx}{\sqrt{x}}$, hinc log. $z = \pm 2\sqrt{-\frac{G}{a}}\cdot x^{\frac{1}{2}}$; $z = e^{\pm 2\sqrt{-\frac{Gx}{a}}}$, siue, pro integrali completo, $z = A e^{2\sqrt{-\frac{Gx}{a}}} + B e^{-2\sqrt{-\frac{Gx}{a}}}$. Ceterum haec ipsa expressio etiam ex formula generali (§. L. 2.) deriuari potest: dummodo notetur, esse, pro quantitate ω euanescente seu infinite parua, $(1+\omega u)^{\frac{1}{\omega}} = e^u$ (cf. §. XXII.). Hinc fit

$$A\left(\sqrt{(a+bx)}+\sqrt{bx}\right)^{2\sqrt{-\frac{G}{b}}} = \mathfrak{A}\left(1+\sqrt{\frac{bx}{a}}\right)^{2\sqrt{-\frac{G}{b}}}$$

$$= \mathfrak{A}\cdot\left\{1+2\sqrt{-\frac{Gx}{a}}\cdot\frac{1}{2\sqrt{-\frac{G}{b}}}\right\}^{2\sqrt{-\frac{G}{b}}} = \mathfrak{A}\cdot e^{2\sqrt{-\frac{Gx}{a}}}.$$

Scholion.

§. LII. 1) Aequationem praecedenti problemate integratam sub casu integrabili *primo* non contineri, facile apparet. Quantitates nimirum p et π, supra ad exprimendam integrabilitatis conditionem adhibitae (§. XXVII.), ob $c = \frac{a}{2}$, $e = b$, $f = o$, sequentes valores recipiunt: $p = o$, $p^I = \frac{1}{2}$, vel vice versa; $\pi = \sqrt{-\frac{G}{b}}$, $\pi^I = -\sqrt{-\frac{G}{b}}$, vel vice versa. Iam vero cum quantitas G nulla determinatione limitetur, superior conditio, quod $\pi - p$ aequari debeat numero integro, pro aequatione §. L. integrata

integrata

tegrata locum non habet; nisi fuerit $2r - \frac{G}{b}$ numerus integer, par, vel impar, affirmatiuus vel negatiuus: sub qua hypothesi eandem aequationem ex formulis etiam pro casu primo repertis integrare liceret.

2) Cum igitur nouam aequationem integrabilem adepti simus, ex hac, tanquam generis sui simplicissima, aequationes generaliores itidem integrabiles deriuari possunt, dum eae ita sint comparatae, vt ope reductionum et transformationum Cap. I. ad illam reuocari queant. Quae iam reductiones in sequentibus applicandae, indeque noui casus integrabiles euoluendi sunt.

Scholion.

§. LIII. Aequatio differentialis (§. L.) sequenti etiam ratione integrari potest. Constat nimirum, aequationem secundi gradus: $d^2z + Pdzdx + Qzdx^2 = 0$ semper ad primum gradum deprimi, ponendo $z = e^{\int tdx}$, vnde fit $dt + t^2dx + Ptdx + Qdx = 0$ (*Euler* I. C. I. Vol. II. §. 852. p. 104.). Sic aequatio nostra in hanc abit:

$$x(a+bx)(dt+t^2dx) + (\tfrac{a}{2} + bx)tdx + Gdx = 0,$$

quae porro, multiplicando per t, et substituendo $x(a+bx)t = \omega$, in hanc mutatur: $\frac{1}{2}d\omega + t(x(a+bx)t^2 + G)dx = 0 = \frac{1}{2}d\omega + \frac{\omega^3 dx}{x^2(a+bx)^2} + \frac{G\omega dx}{x(a+bx)}$, cuius aequationis integratio aliunde satis nota est (*Euler* Inst. Calc. Integr. Vol. I. Sect. II. Cap. I. Probl. 53. §. 429. cf. *Kaestner* Analys. infinit. §. 412. p. 361. edit. 2.).

Problema.

§. LIV. Integrare aequationem differentialem: $0 = x(a+bx)d^2y + (c+ex)dydx + gydx^2$; posito $c = a(\frac{1}{2} + r - \varrho)$, $e = b(1 + 2r)$; denotantibus r et ϱ numeros quosuis integros, siue affirmatiuos, siue negatiuos.

Solutio.

Pro diuersitate signorum numeris r et ϱ competentium quatuor aequationis propositae species discernendae sunt: prouti nimirum 1) vterque fuerit affirmatiuus, vel 2) primus affirmatiuus, alter negatiuus, vel 3) vice versa, primus negatiuus, alter affirmatiuus, vel 4) vterque negatiuus. Denotent in sequentibus literae r et ϱ absolute sumtae numeros integros affirmatiuos: tum varietas casuum ita exhiberi potest, vt ponatur $c = a(\frac{1}{2} \pm r \mp \varrho)$, $e = b(1 \pm 2r)$; vbi capienda sunt pro casu primo signa τῶν r et ϱ superiora; pro quarto, inferiora; pro secundo, signum τȣ̃ r superius, τȣ̃ ϱ inferius; pro tertio, signum τȣ̃ r inferius, τȣ̃ ϱ superius. Qui iam casus singuli ex quatuor reductionibus, §. XIV. exhibitis, seorsim tractandi sunt.

1) Sit

1) Sit igitur *primo* $c = a(\frac{1}{2} + r - \varrho)$, $e = b(1 + 2r)$; tum, quo aequatio differentialis propofita ad aequationem iam integratam (§. L.) reducatur, ex reductione prima (§. XIV. 1.) ponendum eft $c + (\varrho - r)a = \frac{a}{2}$; $e - 2rb = b$; $g + r(r+1)b - re = G$. Hinc c et e recipiunt valores ipfos affumtos, ac fit $G = g + br(r + 1 - 1 - 2r) = g - r^2 b$; quare $\sqrt{-\frac{G}{b}} = \sqrt{(r^2 - \frac{g}{b})}$, quae radix defignetur litera μ. Exinde habetur pro aequatione reducta: $z = \mathfrak{A}(\sqrt{(a+bx)} + \sqrt{bx})^{2\mu} + \mathfrak{B}(\sqrt{(a+bx)} - \sqrt{bx})^{2\mu}$ (§. L. 2.); et pro integrali aequationis propofitae, $y = x^{1-\frac{c}{a}} \frac{d^\varrho\left(x^{\varrho - 1 + \frac{c}{a}} d^r z\right)}{dx^{r+\varrho}}$

$$= x^{\frac{1}{2} - r + \varrho} \frac{d^\varrho\left(x^{r - \frac{1}{2}} d^r z\right)}{dx^{r+\varrho}}.$$

2) Sumendo pro cafu *fecundo*, fignum τῶ r fuperius, τῶ ϱ inferius, ponendum eft ex reductione (§. XIV. 2.), $c - (\varrho + r)a = \frac{a}{2}$, $e - 2rb = b$; inde fponte erit $c = a(\frac{1}{2} + r + \varrho)$, $e = b(1 + 2r)$. Hinc pro integratione aequationis differentialis noftrae obtinetur: $y = (a + bx)^{1 + \frac{c}{a} - \frac{e}{b}} \frac{d^\varrho\left((a+bx)^{\frac{e}{b} - \frac{c}{a} + \varrho - 1} d^r z\right)}{dx^{r+\varrho}}$

$= (a + bx)^{\frac{1}{2} - r + \varrho} \frac{d^\varrho\left((a+bx)^{r - \frac{1}{2}} d^r z\right)}{dx^{r+\varrho}}$, vbi z priorem valorem (1) feruat, nec non litera μ.

3) Pro cafu *tertio*, feu pro figno τῶ r inferiori, τῶ ϱ fuperiori, adhibita reductione (§. XIV. 3.) fequitur $c + (r + \varrho)a = \frac{a}{2}$, $e + 2rb = b$, quod confentit cum valoribus affumtis, $c = a(\frac{1}{2} - r - \varrho)$, $e = b(1 - 2r)$. Hinc fit $y =$

$x^{\frac{1}{2} + r + \varrho} \frac{d^\varrho\left((a+bx)^{\frac{1}{2} + r} d^r\left(x^{-\frac{1}{2}}(a+bx)^{-\frac{1}{2}} z\right)\right)}{dx^{\varrho + r}}$. Iam vero eft $d(\sqrt{(a+bx)} \pm \sqrt{bx})^{2\mu}$

$= 2\mu(\sqrt{(a+bx)} \pm \sqrt{bx})^{2\mu - 1} \cdot \frac{1}{2} b \left(\frac{1}{\sqrt{(a+bx)}} \pm \frac{1}{\sqrt{bx}}\right) dx =$

$\pm \mu b^{\frac{1}{2}} x^{-\frac{1}{2}} (a+bx)^{-\frac{1}{2}} dx \cdot (\sqrt{(a+bx)} \pm \sqrt{bx})^{2\mu}$. Quare habetur

d^r

$$d^r\left(x^{-\frac{1}{2}}(a+bx)^{-\frac{1}{2}}(\gamma(a+bx)\pm\gamma bx)^{2\mu}\right)$$
$$=\pm\frac{d^r d(\gamma(a+bx)\pm\gamma bx)^{2k}}{\mu b^{\frac{1}{2}}dx}=\pm\frac{d^{r+1}(\gamma(a+bx)\pm\gamma bx)^{2k}}{\mu b^{\frac{1}{2}}dx}.$$

Hinc formula integralis concinnius ſic exprimi poteſt:

$y=x^{\frac{1}{2}+r+\varrho}\frac{d^\varrho\left((a+bx)^{\frac{1}{2}+r}d^{r+1}z\right)}{dx^{\varrho+r+1}}$, vbi pro z rurſus idem valor, ac in caſu primo et ſecundo, ſupponitur.

4) Pro caſu tandem *quarto*, ſeu ſignis τῶν r et ϱ inferioribus, reductio (§. XIV. 4.) praebet: $c+(r-\varrho)a=\frac{a}{2}$, $e+2rb=b$; ſeu $c=a(\frac{1}{2}-r+\varrho)$, $e=b(1-2r)$. Tum prodit $y=(a+bx)^{\frac{1}{2}+r+\varrho}\frac{d^\varrho\left(x^{\frac{1}{2}+r}d^r\left(x^{-\frac{1}{2}}(a+bx)^{-\frac{1}{2}}z\right)\right)}{dx^{\varrho+r}}$, quae expreſſio, ad modum praecedentis (3) in hanc transformatur: $y=(a+bx)^{\frac{1}{2}+r+\varrho}\frac{d^\varrho\left(x^{\frac{1}{2}+r}d^{r+1}z\right)}{dx^{\varrho+r+1}}$.

Quoad valorem τȣ z nec non literae μ, hic etiam caſus cum primo conſentit.

Corollarium.

§. LIV. Aequatio $0=x^2(a+bx)d^2y+x(c+ex)dydx+(f+gx)ydx^2$ ſemper in aliam transformari poteſt, pro qua ſit $f=0$ (§. VII.). Inde ad illam quoque aequationem integrationes praecedentis problematis extendi poſſunt. Iſta quidem transformatio triplici modo peragi poteſt (§. VI. VII.); hoc autem loco prima transformatio ſufficit, cum reliquae binae haud nouas integrationes praebeant. Qua igitur ratione ſoluere licebit ſequens

PROBLEMA.

§. LV. Integrare aequationem differentialem:

$$0=x^2(a+bx)d^2y+x(c+ex)dydx+(f+gx)ydx^2,$$

poſito $e=b(\frac{c}{a}+r+\varrho+\frac{1}{2})$, $f=\frac{a}{4}(\frac{c}{a}-r+\varrho-\frac{1}{2})(\frac{c}{a}+r-\varrho-\frac{1}{2})$, denotantibus r et ϱ numeros quosuis integros, ſiue affirmatiuos ſiue negatiuos.

Solutio.

Aequatio propoſita ope ſubſtitutionis $y=x^p.v$ in hanc transformatur (§. VI. 4.):
$$0=x^2(a+bx)d^2v+x(c+2pa+(e+2pb)x)dvdx+(f+pc+p(p-1)a+(g+pe+p(p-1)b)x)vdx^2.$$

Quam

Quam transformatam ad aequationem §. XIII. integratam reuocare licet, dum quantitas p rite definiatur. Iam, vti in problemate praecedenti, quatuor casus discernendi sunt, qui ita simul exhiberi possunt, vt ponatur $e = b(\frac{c}{a} \pm r \pm \varrho + \frac{1}{2})$,

$f = \frac{a}{4}(\frac{c}{a} \mp r \pm \varrho - \frac{1}{2})(\frac{c}{a} \pm r \mp \varrho - \frac{1}{2})$, denotantibus r et ϱ numeros affirmatiuos.

1) *Primo*, pro signis τῶν r et ϱ superioribus, ponendum est:

$$c + 2pa = a(\tfrac{1}{2} + r - \varrho)$$
$$e + 2pb = b(1 + 2r)$$
$$f + pc + p(p-1)a = 0.$$

Ex prima aequatione fit $p = \frac{1}{4} + \frac{r-\varrho}{2} - \frac{c}{2a}$; hinc ex altera $e = b(1 + 2r - \frac{1}{2} - r + \varrho + \frac{c}{a})$ $= b(\frac{c}{a} + r + \varrho + \frac{1}{2})$; ex tertia $f = -ap(p - 1 + \frac{c}{a}) =$

$a(\frac{c}{2a} - \frac{r+\varrho}{2} - \frac{1}{4})(\frac{c}{2a} + \frac{r-\varrho}{2} - \frac{1}{4}) = \frac{a}{4}(\frac{c}{a} - r + \varrho - \frac{1}{2})(\frac{c}{a} + r - \varrho - \frac{1}{2})$; qui valores pro e et f cum assumtis conueniunt. Quare ex (§. LIII. 1.) aequationis transformatae integrale prodit hoc: $v = x^{\frac{1}{2} - r + \varrho} d^{\varrho}(x^{r - \frac{1}{2}} d^{r} z) : dx^{\varrho + r}$, existente $z =$ $\mathfrak{A}(\surd(a+bx) + \surd bx)^{2\mu} + \mathfrak{B}(\surd(a+bx) - \surd bx)^{2\mu}$. Est autem $\mu =$ $\surd(r^2 - \frac{g}{b} - \frac{pe}{b} - p(p-1))$, siue, ob $p = \frac{1+2r}{2} - \frac{e}{2b}$, $\mu =$ $\surd\left(r^2 - \frac{g}{b} - (\frac{1+2r}{2} - \frac{e}{2b})(-\frac{1+2r}{2} + \frac{e}{2b})\right) =$ $\surd(r^2 - \frac{g}{b} - r^2 + (\frac{e}{2b} - \frac{1}{2})^2) = \surd\left(\frac{1}{4}(\frac{e}{b} - 1)^2 - \frac{g}{b}\right)$. Ex v sponte sequitur integrale aequationis propositae, $y = x^{p} v =$

$$x^{\frac{3}{4} - \frac{r+\varrho}{2} - \frac{c}{2a}} d^{\varrho}\left(x^{r - \frac{1}{2}} d^{r} z\right) : dx^{\varrho + r}.$$

2) *Secundo* pro signis, τῷ r superiori, τῷ ϱ inferiori, aequationes tres (1) ita tantum mutandae sunt, vt pro ϱ ponatur $-\varrho$; tum e et f valores debitos recipiunt, $e =$ $b(\frac{c}{a} + r - \varrho + \frac{1}{2})$, $f = \frac{a}{4}(\frac{c}{a} - r - \varrho - \frac{1}{2})(\frac{c}{a} + r + \varrho - \frac{1}{2})$. Pro aequatione transformata est, ex §. LIII. 2, $v = (a+bx)^{\frac{1}{2} - r + \varrho} d^{\varrho}((a+bx)^{r - \frac{1}{2}} d^{r} z) : dx^{\varrho + r}$; hinc pro aequatione proposita $y =$

$$x^{\frac{1}{4} + \frac{r+\varrho}{2} - \frac{c}{2a}}(a+bx)^{\frac{1}{2} - r + \varrho} d^{\varrho}\left((a+bx)^{r - \frac{1}{2}} d^{r} z\right) : dx^{\varrho + r}.$$ Quantitates z et μ valores (1) seruant.

 3) *Ter-*

3) *Tertio* pro signis, τῦ r inferiori, τῦ ϱ superiori, ex aequationibus (1), scripto pro r, — r, prodeunt:

$$e = b\left(\frac{c}{a} - r + \varrho + \tfrac{1}{2}\right),\quad f = \frac{a}{4}\left(\frac{c}{a} + r + \varrho - \tfrac{1}{2}\right)\left(\frac{c}{a} - r - \varrho - \tfrac{1}{2}\right);$$

hinc fit ex (§. LIII. 3.) pro aequatione transformata:

$$v = x^{\frac{1}{2}+r+\varrho}\, d^{\varrho}\left((a+bx)^{\frac{1}{2}+r}\, d^{r+1} z\right) : dx^{\varrho+r+1},$$

et pro aequatione proposita:

$$y = x^{\frac{3}{4}+\frac{r+\varrho}{2}-\frac{c}{2a}}\, d^{\varrho}\left((a+bx)^{\frac{1}{2}+r}\, d^{r+1} z\right) : dx^{\varrho+r+1}.$$

4) Tandem pro signis τῶν r et ϱ inferioribus, simili ratione obtinemus:

$$e = b\left(\frac{c}{a} - r - \varrho + \tfrac{1}{2}\right);\quad f = \frac{a}{4}\left(\frac{c}{a} + r - \varrho - \tfrac{1}{2}\right)\left(\frac{c}{a} - r + \varrho - \tfrac{1}{2}\right);$$

v ex (§. LIII. 4.) $= (a+bx)^{\frac{1}{2}+r+\varrho}\, d^{\varrho}\left(x^{\frac{1}{2}+r}\, d^{r+1} z\right) : dx^{\varrho+r+1}$;

hinc $y = x^{\frac{1}{4}+\frac{\varrho-r}{2}-\frac{c}{2a}}(a+bx)^{\frac{1}{2}+r+\varrho}\, d^{\varrho}\left(x^{\frac{1}{2}+r}\, d^{r+1} z\right) : dx^{\varrho+r+1}$.

Quantitas z eodem modo, ac prioribus tribus casibus, definitur; est nimirum

$z = \mathfrak{A}\left(\sqrt{(a+bx)} + \sqrt{bx}\right)^{2\mu} + \mathfrak{B}\left(\sqrt{(a+bx)} - \sqrt{bx}\right)^{2\mu}$, existente $\mu = \sqrt{\left(\tfrac{1}{4}\left(\frac{e}{b} - 1\right)^2 - \frac{g}{b}\right)}$.

Problema.

§. LVI. Integrare aequationem differentialem:

$0 = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)ydx^2$; posito $e = b\left(\frac{c}{a} + (\tfrac{1}{2}+r+\varrho)n\right)$; $f = \frac{a}{4}\left(\left(\frac{c}{a} - 1\right)^2 - n^2\left(\tfrac{1}{2} - r + \varrho\right)^2\right)$; denotantibus r et ϱ numeros quosuis integros, siue affirmatiuos, siue negatiuos.

Solutio.

Aequatio proposita in aliam transformari potest, pro qua n sit $= 1$ (§. IV.). Posito nimirum $x^n = \chi$, illa abit in hanc: $0 = \chi^2(a+b\chi)d^2y + \chi\left(\frac{a(n-1)+c}{n} + \left(\frac{b(n-1)+e}{n}\right)\chi\right)dyd\chi + \left(\frac{f}{n^2} + \frac{g}{n^2}\chi\right)yd\chi^2$, vbi iam $d\chi$ pro differentiali constanti habetur. Dum igitur haec aequatio transformata ex praecedenti problemate integrari queat, proposita quoque integrabilis erit. Pro diuersitate signo-

ſignorum in aequationibus: $e = b\left(\frac{c}{a} + (\frac{1}{2} \pm r \pm \varrho)n\right)$,

$f = \frac{a}{4}\left((\frac{c}{a} - 1)^2 - n^2(\frac{1}{2} \mp r \pm \varrho)^2\right)$, rurſus quatuor caſus diſcernendi ſunt.

1) *Primo* ex (§. LV. 1.) ponendum eſt $\frac{b(n-1)+e}{n} = b(\frac{n-1}{n} + \frac{c}{na} + r + \varrho + \frac{1}{2})$,

et $\frac{f}{n^2} = \frac{a}{4}(\frac{n-1}{n} + \frac{c}{na} - r + \varrho - \frac{1}{2})(\frac{n-1}{n} + \frac{c}{na} + r - \varrho - \frac{1}{2})$. Hinc fit $e =$

$b\left(\frac{c}{a} + (\frac{1}{2} + r + \varrho)n\right)$; $f = \frac{a}{4}\left(\frac{c}{a} - 1 + n(\frac{1}{2} - r + \varrho)\right)\left(\frac{c}{a} - 1 - n(\frac{1}{2} - r + \varrho)\right)$

$= \frac{a}{4}\left((\frac{c}{a} - 1)^2 - n^2(\frac{1}{2} - r + \varrho)^2\right)$.

Pro his igitur valoribus τῶν e et f, ſeu pro ſignis τῶν r et ϱ ſuperioribus, fit integrale aequationis transformatae, indeque etiam propoſitae, ſiue $y =$

$$\chi^{\frac{1}{2} + \frac{\varrho - r}{2} - \frac{(n-1)}{2n} - \frac{c}{2na}} d^\varrho\left(\chi^{r - \frac{1}{2}} d^r z\right) : d\chi^{\varrho + r} =$$

$$\chi^{\frac{2+n}{4n} + \frac{\varrho - r}{2} - \frac{c}{2na}} d^\varrho\left(\chi^{r - \frac{1}{2}} d^r z\right) : d\chi^{\varrho + r}.$$ Eſt autem $z =$

$\mathfrak{A}(\surd(a + b\chi) + \surd b\chi)^{2\mu} + \mathfrak{B}(\surd(a + b\chi) - \surd b\chi)^{2\mu}$, poſito $\mu =$

$\surd\left(\frac{1}{4}(\frac{n-1}{n} + \frac{e}{nb} - 1)^2 - \frac{g}{n^2 b}\right) = \frac{1}{n}\surd\left(\frac{1}{4}(\frac{e}{b} - 1)^2 - \frac{g}{b}\right)$.

2) *Secundo* ex (§. LVI. 2.) fit $e = b\left(\frac{c}{a} + (\frac{1}{2} + r - \varrho)n\right)$;

$f = \frac{a}{4}\left((\frac{c}{a} - 1)^2 - n^2(\frac{1}{2} - r - \varrho)^2\right)$; et pro his valoribus, ſiue ſignis, τῷ r ſuperiori, τῷ ϱ inferiori, obtinetur

$$y = \chi^{\frac{1}{2} + \frac{r + \varrho}{2} - \frac{(n-1)}{2n} - \frac{c}{2na}} \cdot (a + b\chi)^{\frac{1}{2} + \varrho - r} d^\varrho\left((a + b\chi)^{r - \frac{1}{2}} d^r z\right) : d\chi^{\varrho + r}$$

$$= \chi^{\frac{2-n}{4n} + \frac{\varrho + r}{2} - \frac{c}{2na}} \cdot (a + b\chi)^{\frac{1}{2} + \varrho - r} d^\varrho\left((a + b\chi)^{r - \frac{1}{2}} d^r z\right) : d\chi^{\varrho + r};$$

vbi z priorem valorem (1) ferunt.

3) *Tertio* ex (§. LVI. 3.) pro $e = b\left(\frac{c}{a} + (\frac{1}{2} - r + \varrho)n\right)$; $f =$

$\frac{a}{4}\left((\frac{c}{a} - 1)^2 - n^2(\frac{1}{2} + r + \varrho)^2\right)$, ſiue pro ſignis, τῷ r inferiori, τῷ ϱ ſuperiori, reperitur integrale $y =$

$$\chi^{\frac{3}{4}+\frac{r+\varrho}{2}-\frac{(n-1)}{2n}-\frac{c}{2na}} \cdot d^{\varrho}\left((a+b\chi)^{\frac{1}{2}+r} d^{r+1} z\right) : d\chi^{\varrho+r+1}$$

$$=\chi^{\frac{2+n}{4n}+\frac{r+\varrho}{2}-\frac{c}{2na}} \cdot d^{\varrho}\left((a+b\chi)^{\frac{1}{2}+r} d^{r+1} z\right) : d\chi^{\varrho+r+1}.$$

4) *Quarto* tandem ex (§. LVI. 4.) prodeunt ſigna τῶν r et ϱ inferiora, ſiue $e = b\left(\frac{c}{a}+(\frac{1}{2}-r-\varrho)n\right)$; $f=\frac{a}{4}\left((\frac{c}{a}-1)^2-n^2(\frac{1}{2}+r-\varrho)^2\right)$; et $y =$
$\chi^{\frac{2-n}{4n}+\frac{\varrho-r}{2}-\frac{c}{2na}}(a+b\chi)^{\frac{1}{2}+r+\varrho} d^{\varrho}\left(\chi^{\frac{1}{2}+r} d^{r+1} z\right) : d\chi^{\varrho+r+1}$; vbi iterum z, χ, μ, valores habent, tribus praecedentibus caſibus communes. In differentialibus r^{to} et ϱ^{to}, quae formulas integrales ingrediuntur, $d\chi$ pro differentiali conſtanti haberi debet, non dx, vti in ipſa aequatione propoſita.

Corollarium 1.

§. LVII. 1) Aequatio differentialis modo integrata, quae tria problemata praecedentia (§§. L. LIII. LV.), ceu caſus particulares, complectitur, ſiſtit iam *caſum integrabilem ſecundum generaliorem.* Conditiones integrabilitatis ſic etiam exprimi poſſunt, vt ponatur $e = b\left(\frac{c}{a}+(\frac{1}{2}+s)n\right)$, $f=\frac{a}{4}\left((\frac{c}{a}-1)^2-n^2(\frac{1}{2}-\sigma)^2\right)$; vbi s et σ denotant numeros integros tam affirmatiuos quam negatiuos, quorum vero vel vterque par, vel impar vterque eſſe debet.

2) Iam quatuor ſpecies aequationis differentialis, quarum quaeuis peculiarem integrationem poſtulat (§. LVI. 1. 2. 3. 4.), ita diſcerni poſſunt, vt pro prima et quarta ſit numerus s, abſolute ſpectatus, id eſt nullo ad ſignum reſpectu habito, maior numero σ itidem abſolute ſpectato; pro ſecunda et tertia contrarium obtineat. Porro prima et quarta ſpecies inter ſe ita differunt, vt pro illa ſit s numerus affirmatiuus, pro hac negatiuus; ſimiliter pro ſecunda numerus σ affirmatiuus, pro tertia negatiuus eſt. Poſito nimirum $\pm r \pm \varrho = s$, $\pm r \mp \varrho = \sigma$, exit $\pm r = \frac{s+\sigma}{2}$, $\pm \varrho = \frac{s-\sigma}{2}$, ſiue $r = \pm\frac{(s+\sigma)}{2}$, $\varrho = \pm\frac{(s-\sigma)}{2}$. Quae expreſſiones pro r et ϱ ſemper numeros integros affirmatiuos, quales in praecedenti ſolutione (§. LVI.) ſupponuntur, praebent: dum ſigna debito modo accipiantur, i. e. ſuperiora pro caſu primo; inferiora pro quarto; pro ſecundo ſignum τȣ̃ r ſuperius, τȣ̃ ϱ inferius; vice verſa pro caſu tertio.

Corollarium 2.

§. LVIII. Casus, quo in aequatione differentiali coëfficiens b euanescit, peculiarem solutionem exigit: cuius rationem iam ex §. LI. petere licet. Tum etiam e erit $= 0$, at quotiens $\frac{e}{b} = \frac{c}{a} + (\frac{1}{2} \pm r \pm \varrho)$ n finitam magnitudinem habet. Hinc quantitas $\mu = \frac{1}{n}\sqrt{\left(\frac{1}{4}(\frac{e}{b} - 1)^2 - \frac{g}{b}\right)}$ (§. LVI. 1.) in infinitum crescit, ponique potest $= \frac{1}{n}\sqrt{-\frac{g}{b}}$. Exinde, adhibita formula exponentiali (§. LI.), expressio pro z (§. LVI. 1.) in hanc abit: $z = \mathfrak{A}e^{\frac{2}{n}\sqrt{-\frac{gx}{a}}} + \mathfrak{B}e^{-\frac{2}{n}\sqrt{-\frac{gx}{a}}}$. Qui igitur valor in formulis pro y supra (§. LVI.) inuentis, atque ad hunc etiam casum patentibus, substituendus est.

Scholion.

§. LIX. 1) Casum integrabilem *secundum*, praecedenti problemate (§. LVI.) euolutum, a casu integrabili *primo*, supra demonstrato (§§. XXI. XXVII.), reuera diuersum esse, sequens vtriusque comparatio ostendet: ex qua simul conditionum integrabilitatis pro casu secundo nouam expressionem petere licebit.

Denotent, vti supra (l. c.), p, p^I; π, π^I; radices aequationum: $f = -ap(p-1) - cp$; $g = -b\pi(\pi-1) - e\pi$; fit porro $\frac{p-p^I}{n} = k$; at $\frac{\pi - p}{n}$ et $\frac{\pi^I - p^I}{n}$ exprimantur literis maioribus R, R^I, loco minorum supra vsurpatarum.

Iam cum pro casu secundo (§. LVI.) esse debeat $f = \frac{a}{4}\left(\frac{c}{a} - 1 + n(\frac{1}{2} - r + \varrho)\right) \cdot \left(\frac{c}{a} - 1 - n(\frac{1}{2} - r + \varrho)\right)$; hoc valore posito $= -ap(p - 1 + \frac{c}{a}) = app^I$, prodit p vel $p^I = -\frac{c}{2a} + \frac{1}{2} - \frac{n}{2}(\frac{1}{2} - r + \varrho)$, et p^I vel $p = -\frac{c}{2a} + \frac{1}{2} + \frac{n}{2}(\frac{1}{2} - r + \varrho)$; hinc $\frac{p - p^I}{n} = k = \mp(\frac{1}{2} - r + \varrho)$. Cum porro sit (§. LVI.), $\frac{e}{b} - \frac{c}{a} = (\frac{1}{2} + r + \varrho)n = -(R + R^I)n$ (§. XXVII. 2.), erit $R + R^I = -(\frac{1}{2} + r + \varrho)$.

Quare binae conditiones, sub quibus casus *secundus* integrabilis locum habet, ita exhiberi possunt, vt ponatur 1) $R + R^I = -(\frac{1}{2} + r + \varrho)$, 2) $k = \mp(\frac{1}{2} - r + \varrho)$; denotantibus r et ϱ numeros quosuis integros siue affirmatiuos, siue negatiuos.

At vero pro casu *primo*, conditio integrabilitatis huc redit, vt sit R (vel R^I) numerus integer, vel affirmatiuus, vel negatiuus. Hinc discrimen inter vtrumque casum manifestum est: istae enim binae conditiones (pro secundo) locum habere possunt, quin postrema (pro casu primo) simul obtineat.

2) Quan-

2) Quanquam autem nec casus primus secundum, nec hic illum includat, fieri tamen potest, vt ambo pro *vna* aequatione differentiali simul locum habeant. Tum integrationem vel ex (§. LVI.), vel ex formulis pro casu primo, suscipere licebit: quae binae solutiones inter se consentiant necesse est. Habetur pro casu secundo, $k - R^I = R + 2r$ vel $= R + 1 + 2\varrho$; hinc, existente R numero integro, etiam $k - R^I$ numerus integer erit. Iam pro hac ipsa τῶν R et $k - R^I$ conditione, supra casu primo (§ XLII. A. 2. B. 2.) integrale completum finite exhiberi posse, vidimus. Cum vero hoc integrale non, nisi R et $k - R^I$ in signis conueniant, algebraicum sit, sed alioquin quantitates transcendentes, logarithmicas vel circulares, inuoluat: cum ex altera parte formulae pro casu secundo inuentae (§. LVI.), semper expressionem integralis algebraicam suppeditent; iste binarum solutionum consensus turbari videtur.

Ad quam difficultatem tollendam, attendendum est, formulas casus secundi, etiamsi in genere, ob binas constantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, integrale completum exprimant, nonnunquam tamen integrale tantum particulare exhibere. Est nimirum (§. LVI.),

$$z = \mathfrak{A}(\sqrt{b\chi} + \sqrt{(a+b\chi)})^{2\mu} + \mathfrak{C}(\sqrt{b\chi} - \sqrt{(a+b\chi)})^{2\mu}$$
$$= (\mathfrak{A}+\mathfrak{C})\Big(b^\mu\chi^\mu + \frac{2\mu(2\mu-1)}{1.2} b^{\mu-1}\chi^{\mu-1}(a+b\chi)$$
$$+ \frac{2\mu\ldots(2\mu-3)}{1\ldots4} b^{\mu-2}\chi^{\mu-2}(a+b\chi)^2 + \ldots\Big)$$
$$+ (\mathfrak{A}-\mathfrak{C})(a+b\chi)^{\frac{1}{2}} b^{-\frac{1}{2}}\chi^{-\frac{1}{2}}\Big(2\mu b^\mu\chi^\mu + \frac{2\mu\ldots2\mu-2}{1.2.3} b^{\mu-1}\chi^{\mu-1}(a+b\chi) + \ldots\Big);$$

vnde $d^r z$ hanc formam induet: $d^r z = (\mathfrak{A}+\mathfrak{C})(.\chi^{\mu-r} + .\chi^{\mu-r-1} + .\chi^{\mu-r-2} + \ldots)$ $+ (\mathfrak{A}-\mathfrak{C})(a+b\chi)^{\frac{1}{2}-r}\chi^{-\frac{1}{2}}(.\chi^\mu + .\chi^{\mu-1} + .\chi^{\mu-2} + \ldots)$; vbi coëfficientes breuitatis caussa punctis notati sunt. Quodsi nunc 2μ est numerus integer, tum binae series in $\mathfrak{A}+\mathfrak{C}$ et $\mathfrak{A}-\mathfrak{C}$ ductae, quibus tam z quam $d^r z$ exprimitur, abrumpunt; tumque fieri potest, vt, si $d^r z$ multiplicetur in $\chi^{r-\frac{1}{2}}$ (§. LVI. 1.), vel in $(a+b\chi)^{r-\frac{1}{2}}$ (§. LVI. 2.), vel $d^{r+1}z$ in $(a+b\chi)^{r+\frac{1}{2}}$ (§. LVI. 3.), vel in $\chi^{r+\frac{1}{2}}$ (§. LVI. 4.), vt, inquam, pro differentiali ϱ^{to} istiusmodi producti, alterutra series, vel ea quae $\mathfrak{A}+\mathfrak{C}$, vel altera quae $\mathfrak{A}-\mathfrak{C}$ factorem habet, e calculo exeat, sicque vna tantum Constans arbitraria supersit, id est, reuera integrale tantum particulare obtineatur. Iam iste exponens 2μ est $=$ $\frac{2}{n}\sqrt{\left(\frac{1}{4}\left(\frac{e}{b} - 1\right)^2 - \frac{g}{b}\right)} = \frac{2}{n}\sqrt{\left(\frac{1}{4}(\pi+\pi^I)^2 - \pi\pi^I\right)} = \pm\frac{(\pi-\pi^I)}{n} = \pm$ $(R + k - R^I)$ (§. XXVII.), $= \pm(2R + 2r)$ vel $= \pm(2R + 2\varrho + 1)$. Hinc, R existente numero integro, etiam 2μ erit numerus integer. Exinde intelligitur, sub hac ipsa hypothesi numeri integri R, formulas (§. LVI.) integrale quidem algebraicum, at non necessario completum, praebere. Ex quo haud amplius obscurum esse potest, quomodo dissensus ille apparens

rens tolli queat; nec necesse esse videtur, prolixius ostendere, quod tum demum, cum R et $k - R^1$ sint numeri integri, signis oppositi, ex formulis casus secundi, loco integralis completi, particulare tantum consequatur.

Articvlvs tertivs.
Euolutio casus integrabilis tertii.

Problema.

§. LX. Integrare aequationem differentialem:

$$o = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)ydx^2;$$

posito $c = a\left(\frac{e}{b} - (\tfrac{1}{2} + r + \varrho)n\right)$; $g = \frac{b}{4}\left(\left(\frac{e}{b} - 1\right)^2 - n^2(\tfrac{1}{2} - r + \varrho)^2\right)$; denotantibus r et ϱ numeros quosuis integros, tam affirmatiuos, quam negatiuos.

Solutio.

Diuidendo per x^n, aequatio proposita in hanc abit:

$o = x^2(b + ax^{-n})d^2y' + x(e + cx^{-n})dydx + (g + fx^{-n})ydx^2$; id est, in in aliam, eiusdem quidem formae, in qua tamen coëfficientes a, b; c, e; f, g; inuicem permutati sunt, et exponens (n) oppositum valorem (—n) habet. Quae iam aequatio variata si ad casum integrabilem secundum, seu ad Problema (§. LVI.) reuocetur: pro aequatione proposita nouae conditiones integrabilitatis innotescunt, eae nimirum ipsae, quas problematis nunc soluendi inscriptio enuntiat. Sic exoritur nouus casus integrabilis, *tertius*. Cuius integratio ex (§. LVI.) sponte consequitur: dummodo illic ponatur,

loco a	b
b	a
c	e
e	c
f	g
g	f
n	—n

Ad discernendam quadruplicem varietatem respectu signorum, iterum statuatur $c = a\left(\frac{e}{b} - (\tfrac{1}{2} \pm r \pm \varrho)n\right)$;

$$g = \frac{b}{4}\left(\left(\frac{e}{b} - 1\right)^2 - n^2(\tfrac{1}{2} \mp r \pm \varrho)^2\right);$$

vbi nunc r et ϱ denotant numeros integros affirmatiuos.

1) Iam *primo*, pro signis τῶν r et ϱ superioribus, seu $c = a\left(\frac{e}{b} - (\tfrac{1}{2} + r + \varrho)n\right)$; $g = \frac{b}{4}\left(\left(\frac{e}{b} - 1\right)^2 - (\tfrac{1}{2} - r + \varrho)^2 n^2\right)$; prodit integrale hoc (§. LVI. 1.):

$y = \chi^{\frac{n-2}{4n} + \frac{\varrho - r}{2} + \frac{e}{2nb}} . d^{\varrho}\left(\chi^{r - \frac{1}{2}} d^r z\right) : d\chi^{\varrho + r}$; vbi est $\chi = x^{-n}$; $z = \mathfrak{A}(\sqrt{(b + a\chi)} + \sqrt{a\chi})^{2\mu} + \mathfrak{B}(\sqrt{(b + a\chi)} - \sqrt{a\chi})^{2\mu}$; $\mu = -\frac{1}{n}\sqrt{\left(\tfrac{1}{4}\left(\frac{c}{a} - 1\right)^2 - \frac{f}{a}\right)}$; μ pro lubitu etiam cum signo opposito accipi potest. In differentialibus r^{to} et ϱ^{to} $d\chi$ pro differentiali constanti sumi debet. 2) Se-

2) *Secundo* pro signis, τῦ r superiori, τῦ ϱ inferiori, seu $c = a\left(\frac{e}{b} - (\tfrac{1}{2} + r - \varrho)n\right)$, $g = \frac{b}{4}\left(\left(\frac{e}{b} - 1\right)^2 - (\tfrac{1}{2} - r - \varrho)^2 n^2\right)$; obtinetur (§. LVI. 2.) $y =$

$$\chi^{-\frac{n-2}{4n} + \frac{\varrho + r}{2} + \frac{e}{2nb}} \cdot (b + a\chi)^{\frac{1}{2} + \varrho - r} d\varrho\left((b + a\chi)^{r - \frac{1}{2}} d^r z\right) : d\chi^{\varrho + r}.$$

3) *Tertio*, pro signis, τῦ r inferiori, τῦ ϱ superiori, siue $c = a\left(\frac{e}{b} - (\tfrac{1}{2} - r + \varrho)n\right)$, $g = \frac{b}{4}\left(\left(\frac{e}{b} - 1\right)^2 - (\tfrac{1}{2} + r + \varrho)^2 n^2\right)$; habetur $y =$

$$\chi^{\frac{n-2}{4n} + \frac{r + \varrho}{2} + \frac{e}{2nb}} \cdot d\varrho\left((b + a\chi)^{\frac{1}{2} + r} d^{r+1} z\right) : d\chi^{\varrho + r}.$$

4) *Quarto* tandem, pro signis τῶν r vel ϱ inferioribus, siue $c = a\left(\frac{e}{b} - (\tfrac{1}{2} - r - \varrho)n\right)$, $g = \frac{b}{4}\left(\left(\frac{e}{b} - 1\right)^2 - (\tfrac{1}{2} + r - \varrho)^2 n^2\right)$; prodit integrale:

$$y = \chi^{-\frac{n-2}{4n} + \frac{\varrho - r}{2} + \frac{e}{2nb}} (b + a\chi)^{\frac{1}{2} + r + \varrho} d\varrho\left(\chi^{\frac{1}{2} + r} d^{r+1} z\right) : d\chi^{\varrho + r}.$$ Valores τῶν χ, z, μ, pro singulis quatuor speciebus aequationis propositae, iidem manent (1).

Corollarium 1.

§. LXI. 1) Cum casus integrabilis *tertius*, praecedenti problemate (§. LX.) euolutus, immediate ex *secundo* (§. LVI.) consequatur; ambo pro *vno* haberi possunt, dummodo obseruetur, aequationem propositam *sub duplici semper forma* exhiberi posse, nimirum

$$\text{vel } 1)\ o = x^2 (a + bx^n) d^2y + x(c + ex^n) dy dx + (f + gx^n) y dx^2$$

$$\text{vel } 2)\ o = x^2 (b + ax^{-n}) d^2y + x(e + cx^{-n}) dy dx + (g + fx^{-n}) y dx^2.$$

Quam obseruationem supra iam (§. XXIX. 2.) commemorauimus.

2) Cum sit (§. LX.) pro casu tertio, $\frac{e}{b} - \frac{c}{a} = (\tfrac{1}{2} + r + \varrho)n$, siue $e = b\left(\frac{c}{a} + (\tfrac{1}{2} + r + \varrho)n\right)$; prima aequatio conditionalis pro binis casibus, secundo et tertio *eadem est:* ex altera autem aequatione $\frac{g}{b}$ ex $\frac{e}{b}$ pro tertio casu eadem ratione determinatur, ac pro secundo $\frac{f}{a}$ ex $\frac{c}{a}$. Haec altera aequatio conditionalis pro tertio casu etiam sic exprimi potest: $g = \frac{b}{4}\left(\frac{c}{a} - 1 + 2rn\right)\left(\frac{c}{a} - 1 + (2\varrho + 1)n\right)$.

Corol-

Corollarium 2.

§. LXII. Quodsi ponatur in aequatione differentiali (§. LX.) $b = o$, esse etiam debet $e = o$; eritque $\frac{e}{b} = \frac{c}{a} + (\frac{1}{2} + r + \varrho)n$, $g = o$. Hinc aequatio differentialis hanc formam induit: $o = ax^2d^2y + cxdydx + fydx^2$. Pro cuius integrali habetur $y = \chi^{\frac{n-2}{4n} + \frac{o-r}{2} + \frac{c}{2na} + \frac{1}{2}(\frac{1}{2}+r+\varrho)} . d^\varrho(\chi^{r-\frac{1}{2}}d^r z) : d\chi^{\varrho+r}$; iam z abit in $\mathfrak{A}\chi^\mu$, hinc fit $d^r z = \mathfrak{A}^I\chi^{\mu-r}d\chi^r$, $d^\varrho(\chi^{r-\frac{1}{2}}d^r z) : d\chi^{\varrho+r} = \mathfrak{A}^I d^\varrho(\chi^{\mu-\frac{1}{2}}) : d\chi^\varrho = \mathfrak{A}^{II}\chi^{\mu-\varrho-\frac{1}{2}}$; vnde erit $y = \mathfrak{A}^{II}\chi^{-\frac{1}{2n}+\frac{c}{2na}+\mu} = \mathfrak{A}^{II}x^{\frac{1}{2}-\frac{c}{2a}-\mu n}$, ob $\chi = x^{-n}$; siue, cum radix $-\mu n = \surd(\frac{1}{4}(\frac{c}{a}-1)^2 - \frac{f}{a})$ cum vtroque signo accipi queat, integrale completum prodit: $y = \mathfrak{A}^{II}x^\alpha + \mathfrak{B}^{II}x^\beta$, existentibus $\alpha, \beta, = -\frac{1}{2}(\frac{c}{a} - 1) \pm \surd(\frac{1}{4}(\frac{c}{a} - 1)^2 - \frac{f}{a})$, id est, radicibus aequationis quadraticae $o = \alpha^2 + (\frac{c}{a} - 1)\alpha + \frac{f}{a}$. Quo igitur modo integrationem aliunde notam (*) aequationis differentialis $o = ax^2d^2y + cxdydx + fydx^2$, ex formulis generalibus pro casu integrabili tertio, corollarii instar, deducere licet.

Scholion.

§. LXIII. 1) Comparatione inter casus integrabiles, *tertium* et *primum*, instituta, adhibitis iisdem ratiociniis, ac supra (§. LIX.), conditiones integrabilitatis pro casu tertio sic exprimi possunt, vt ponatur

1) $R + R^I = -(\frac{1}{2} + r + \varrho)$
2) $R - R^I + k = \pm(\frac{1}{2} - r + \varrho)$.

Hinc, ambo casus reuera diuersos esse, euidens est. At si fuerit $R =$ numero integro, tum vterque casus simul locum habet; estque etiam $k =$ numero integro. Integrationes ex formulis pro casu tertio et primo, tunc inter se consentient; nisi quod formulae (§. LX.) exhibeant nonnunquam integrale tantum particulare, cum nimirum completum quantitates transcendentes inuoluat (cf. §. XLIV. A. 1; B. 1.).

2) Conditiones integrabilitatis pro casu tertio (§. LX.) sic etiam exprimere licet, vt ponatur

1) $c =$

(*) *Euler*. Inst. Calc. Int. Vol. II. §. 847. pag. 98.

$$1)\ c = a\left(\frac{e}{b} - \left(\tfrac{1}{2} + s\right)n\right)$$

$$2)\ g = \frac{b}{4}\left(\left(\frac{e}{b} - 1\right)^2 - \left(\tfrac{1}{2} - \sigma\right)^2 n^2\right),$$

denotantibus s et σ numeros quosuis integros, tam affirmatiuos, quam negatiuos, quorum tamen summa aequari debet, numero pari. Iam quatuor formularum integralium (§. LX. 1; 2; 3; 4;) adplicanda est,

prima: si s est numerus positiuus, maior quam σ (vel $-\sigma$, existente σ numero negatiuo);

secunda: si σ est numerus positiuus, et $\sigma > \pm s$;

tertia: si σ est numerus negatiuus, et $-\sigma > \pm s$;

quarta: si s est numerus negatiuus, et $-s > \pm \sigma$.

Valores τῶν r et ϱ petuntur ex his aequationibus: $r = \pm\frac{(s+\sigma)}{2}$, $\varrho = \pm\frac{(s-\sigma)}{2}$, signis debite acceptis, quo nimirum pro r et ϱ prodeant numeri affirmatiui (cf. §. LVII. 2.).

Articvlvs qvartvs.

Euolutio casus integrabilis quarti.

Problema.

§. LXIV. Integrare aequationem differentialem:

$$0 = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)ydx^2,$$

posito $f = \frac{a}{4}\left(\left(\frac{c}{a} - 1\right)^2 - \left(\tfrac{1}{2} - r - \varrho\right)^2 n^2\right)$

$$g = \frac{b}{4}\left(\left(\frac{e}{b} - 1\right)^2 - \left(\tfrac{1}{2} - r + \varrho\right)^2 n^2\right),$$

denotantibus r, ϱ, numeros quosuis integros, tam affirmatiuos, quam negatiuos.

Solutio.

Aequatio proposita, ex transformatione tertia, supra (Cap. I. §. X.) demonstrata, abit in hanc:

$$0 = X^2(a+bX^n)d^2v + X\left(-c + a\left(\frac{e}{b} + 2nq - n + 1\right) + \left(e + 2b(p+q)n\right)X^n\right)dvdX$$
$$+\left(anq\left(\frac{e}{b} - \frac{c}{a} + n(q-1)\right) + \left(g + n(p+q)\left(e + b(n(p+q)-1)\right)\right)X^n\right)vdX^2,$$

posito $y = x^{np}(a+bx^n)^q v$; $a+bx^n = -bX^n$; et $f + pnc + apn(pn-1) = 0$. Quodsi iam haec aequatio transformata ad casum integrabilem tertium (§. LX.), vel, quod perinde

inde eſt, eadem per X^n diuiſa, ad caſum ſecundum (§. LVI.) reuocetur: tum nouae conditiones integrabilitatis pro aequatione propoſita oriuntur, ſicque nouus exoritur caſus integrabilis, isque *quartus*. Ad deſignandam ſignorum varietatem, more hactenus ſeruato, ponatur $f = \frac{a}{4}\left((\frac{c}{a} - 1)^2 - (\frac{1}{2} \mp r \mp \varrho)^2 n^2\right)$,

$$g = \frac{b}{4}\left((\frac{e}{b} - 1)^2 - (\frac{1}{2} \mp r \pm \varrho)^2 n^2\right);$$

vbi iam r et ϱ denotant numeros integros affirmatiuos.

1) Iam *primo*, pro ſignis τῶν r et ϱ ſuperioribus, vt aequatio transformata ex (§. LX. 1.) integrabilis fiat, poni debet: $-c + a(\frac{e}{b} + 2nq - n + 1) = a(\frac{e}{b} + 2(p+q)n - (\frac{1}{2} + r + \varrho)n)$; et $g + n(p+q)(e + b(n(p+q) - 1)) = \frac{b}{4}\left((\frac{e}{b} + 2(p+q)n - 1)^2 - (\frac{1}{2} - r + \varrho)^2 n^2\right)$. Ex prima aequatione fit $-c + a(-n+1) = a(2pn - (\frac{1}{2} + r + \varrho)n)$, ſiue $2pn = -\frac{c}{a} + 1 + (-\frac{1}{2} + r + \varrho)n$; hinc ex altera, $g + n(p+q)(e-b) + bn^2(p+q)^2 = \frac{b}{4}\left((\frac{e}{b} - 1)^2 + 4(p+q)n(\frac{e}{b} - 1) + 4(p+q)^2 n^2 - n^2(\frac{1}{2} - r + \varrho)^2\right)$, vnde $g = \frac{b}{4}\left((\frac{e}{b} - 1)^2 - (\frac{1}{2} - r + \varrho)^2 n^2\right)$. Porro eſt $f = -pna(\frac{c}{a} + pn - 1) = \frac{a}{2}\left(\frac{c}{a} - 1 + (\frac{1}{2} - r - \varrho)n\right)\left(\frac{c}{2a} - \frac{1}{2} - (\frac{1}{2} - r - \varrho)n\right) = \frac{a}{4}\left((\frac{c}{a} - 1)^2 - (\frac{1}{2} - r - \varrho)^2 n^2\right)$. Coëfficientibus igitur ſic determinatis, vti ii in problemate ſupponuntur, habetur integrale aequationis transformatae $v = \chi^{\frac{n-2}{4n} + \frac{\varrho - r}{2} + \frac{e}{2nb} + p + q} \cdot d^{\varrho}(\chi^{r - \frac{1}{2}} d^r z) : d\chi^{\varrho + r}$, vbi eſt $\chi = X^{-n} = -\frac{b}{a + bx^n}$; hinc prodit pro ipſa aequatione propoſita, $y = x^{np}(a + bx^n)^q v$, ſiue, ob $x^n = -\frac{b - a\chi}{b\chi}$, et omiſſo factore conſtanti, $y = (b + a\chi)^p \chi^{-p} \chi^{-q} v =$
$(b + a\chi)^{\frac{2-n}{4n} + \frac{r+\varrho}{2} - \frac{c}{2na}} \cdot \chi^{\frac{n-2}{4n} + \frac{\varrho - r}{2} + \frac{e}{2nb}} d^{\varrho}(\chi^{r - \frac{1}{2}} d^r z) : d\chi^{\varrho + r}$. Eſt autem $z = \mathfrak{A}(\sqrt{(b + a\chi)} + \sqrt{a\chi})^{2\mu} + \mathfrak{B}(\sqrt{(b + a\chi)} - \sqrt{a\chi})^{2\mu}$; et $\mu =$
$\frac{1}{n}\sqrt{\left(\frac{1}{4}(-\frac{c}{a} + \frac{e}{b} + 2nq - n)^2 - nq(\frac{e}{b} - \frac{c}{a} + n(q-1))\right)}$

=

$$=\frac{1}{n}\gamma\Big(\tfrac{1}{4}(-\frac{c}{a}+\frac{e}{b}-n)^2+nq(-\frac{c}{a}+\frac{e}{b}-n)+n^2q^2$$
$$-nq\,(\frac{e}{b}-\frac{c}{a}+n\,(q-1))\Big)=\frac{1}{2n}(\frac{e}{b}-\frac{c}{a}-n).$$

2) *Secundo*, pro signis, τῦ r superiori, τῦ ϱ inferiori, ex (§. LX. 2.) simili ratione prodit: $g=\frac{b}{4}\Big((\frac{e}{b}-1)^2-(\tfrac{1}{2}-r-\varrho)^2n^2\Big)$,

$f=\frac{a}{4}\Big((\frac{c}{a}-1)^2-(\tfrac{1}{2}-r+\varrho)^2n^2\Big)$; tum pro aequatione transformata fit $v=$

$$\chi^{-\frac{n-2}{4n}+\frac{\varrho+r}{2}+\frac{e}{2nb}+p+q}(b+a\chi)^{\frac{1}{2}+\varrho-r}.\,d^{\varrho}\big((b+a\chi)^{r-\frac{1}{2}}d^{r}z\big):d\chi^{\varrho+r};$$

hinc integrale aequationis propositae $y=(b+a\chi)^{p}\chi^{-p-q}.\,v=$

$$(b+a\chi)^{\frac{2+n}{4n}+\frac{\varrho-r}{2}-\frac{c}{2na}}.\chi^{-\frac{2-n}{4n}+\frac{r+\varrho}{2}+\frac{e}{2nb}}d^{\varrho}\big((b+a\chi)^{r-\frac{1}{2}}d^{r}z\big):d\chi^{\varrho+r}.$$

3) *Tertio*, pro signis, τῦ r inferiori, τῦ ϱ superiori, est: $g=$
$=\frac{b}{4}\Big((\frac{e}{b}-1)^2-(\tfrac{1}{2}+r+\varrho)^2n^2\Big)$; $f=\frac{a}{4}\Big((\frac{c}{a}-1)^2-(\tfrac{1}{2}+r-\varrho)^2n^2\Big)$;

porro $v=\chi^{\frac{n-2}{4n}+\frac{r+\varrho}{2}+\frac{e}{2nb}+p+q}.\,d^{\varrho}\big((b+a\chi)^{\frac{1}{2}+r}d^{r+1}z\big):d\chi^{\varrho+r+1}$;

hinc $y=(b+a\chi)^{\frac{2-n}{4n}+\frac{\varrho-r}{2}-\frac{c}{2na}}.\chi^{\frac{n-2}{4n}+\frac{r+\varrho}{2}+\frac{e}{2nb}}d^{\varrho}\big((b+a\chi)^{\frac{1}{2}+r}d^{r+1}z\big):d\chi^{\varrho+r+1}$.

4) *Quarto* tandem, pro signis τῶν r et ϱ inferioribus, ex (§. LX. 4.) prodit:

$$g=\frac{b}{4}\Big((\frac{e}{b}-1)^2-(\tfrac{1}{2}+r-\varrho)^2n^2)\Big),$$

$$f=\frac{a}{4}\Big((\frac{c}{a}-1)^2-(\tfrac{1}{2}+r+\varrho)^2n^2)\Big);$$ pro aequatione transformata

$$v=\chi^{-\frac{n-2}{4n}+\frac{\varrho-r}{2}+\frac{e}{2nb}+p+q}(b+a\chi)^{\frac{1}{2}+r+\varrho}d^{\varrho}\big(\chi^{r+\frac{1}{2}}d^{r+1}z\big):d\chi^{\varrho+r+1};$$

hinc pro aequatione proposita, $y=$

$$(b+a\chi)^{\frac{n+2}{4n}+\frac{r+\varrho}{2}-\frac{c}{2na}}.\chi^{-\frac{n-2}{4n}+\frac{\varrho-r}{2}+\frac{e}{2nb}}d^{\varrho}\big(\chi^{r+\frac{1}{2}}d^{r+1}z\big):d\chi^{\varrho+r+1}.$$

Valores τῶν χ, z, et μ (1), pro quadruplici signorum diversitate iidem manent.

Scho-

Scholion.

§. LXV. 1) Ad comparationem inter casum integrabilem quartum, et casus praecedentes, instituendam, conditiones integrabilitatis pro illo sic exhiberi possunt:

$$1)\ k = \pm(\tfrac{1}{2} - r - \varrho), \quad 2)\ R - R^{I} + k = \pm(\tfrac{1}{2} - r + \varrho).$$

Hinc apparet differentia casus quarti a primo; qui tamen simul locum habere possunt, si R fuerit numerus integer: quo posito etiam R^{I} integer erit. Sub hac hypothesi supra (§. XLIV. A. 3; B. 3.) integrale completum, ex formulis pro casu primo, semper finite exprimi, vidimus: cum contra formulae (§. LXIV.) nonnunquam exhibeant integrale tantum particulare algebraicum, quoties nimirum completum transcendens fuerit.

2) Conditiones integrabilitatis pro casu quarto, sic etiam exprimere licet, vt ponatur $f = \frac{a}{4}\left((\frac{e}{a} - 1)^2 - (\frac{1}{2} - s)^2 n^2\right)$,

$g = \frac{b}{4}\left((\frac{e}{b} - 1)^2 - (\frac{1}{2} - \sigma)^2 n^2\right)$; vbi denotant s et σ numeros quosuis integros tam affirmatiuos, quam negatiuos, quorum tamen vterque vel par, vel impar esse debet. Quaenam quatuor formularum integralium (§. LXIV. 1; 2; 3; 4;), pro diuersa indole numerorum s, σ, applicanda sit, ex §. LXIII. 2. notum est.

Scholion.

§. LXVI. Sic igitur inuenti sunt quatuor casus generales, a se inuicem diuersi, qui innumeras comprehendunt aequationes speciales, ex formulis hactenus demonstratis (§§. XXI et seq.; LVI; LX; LXIV;) complete integrabiles.

Iam cum supra (§. II.) commemoratum sit, praeter casum primum, satis notum, ab EVLERO exhibitos fuisse nouem casus singulares, haud superfluum videtur, ostendere, quomodo hi ipsi casus ex nostris formulis tractandi sint. Quum tamen isti casus quodammodo ceu indiuiduales spectari queant, iuuat praemittere sequens problema, in quo formae aequationum integrabilium latius patentes, ceu corollaria particularia deducuntur: ex quo problemate deinde casus EVLERI, tanquam exempla, resoluere licebit.

PROBLEMA.

§. LXVII. Ex quatuor casibus generalibus (§§. XXI; LVI; LX; LXIV;) deducere aequationes differentiales magis particulares, earumque integralia completa.

Solutio.

Ex casu primo (§. XXI.).

1). Pro casu primo est $f = -ap(p-1) - cp$, $g = -b(p+nr)(p+nr-1) - e(p+nr)$, denotante r numerum integrum, siue affirmatiuum, siue negatiuum. Hinc prodit $\frac{f}{a} - \frac{g}{b} = -p(p-1) - \frac{c}{a}p + p(p-1) + pnr + nr(p+nr-1)$

$= p$

$= p(2nr - \frac{c}{a} + \frac{e}{b}) + nr(nr - 1 + \frac{e}{b})$. Quodsi iam haec aequatio generalis eo limitetur, vt ponatur $2nr - \frac{c}{a} + \frac{e}{b} = 0$, tum erit $\frac{f}{a} - \frac{g}{b} = nr(nr - 1 + \frac{e}{b})$ $= nr(\frac{c}{a} - nr - 1)$. Exinde obtinetur aequatio particularis haec:

$$0 = x^2(a + bx^n)d^2y + x(c + ex^n)dydx + (f + gx^n)ydx^2,$$

inter cuius coëfficientes hae relationes obtinent:

$$e = b\frac{(c - 2ran)}{a}, \qquad g = b\frac{(f - nr(c - (nr+1)a))}{a}.$$

Ad complete integrandam hanc aequationem differentialem, obseruandum est, esse $\frac{c}{a} - \frac{e}{b} = n(r + r^I) = 2nr$, hinc $r^I = r$. Quare, existentibus r et r^I numeris integris, integrale completum finite assignari poterit (§§. XXXVII. XLI.). Est nimirum pro affirmatiuo r,

$$y = \mathfrak{A}x^p\left(1 + r\frac{(k-r)}{k+1}\beta x^n + \frac{r(r-1)}{1.2}\frac{(k-r)(k-r+1)}{(k+1)(k+2)}\beta^2x^{2n} + \ldots\right)$$
$$+ \mathfrak{B}x^{p^I}\left(1 + r\frac{(k+r)}{k-1}\beta x^n + \frac{r(r-1)}{1.2}\frac{(k+r)(k+r-1)}{(k-1)(k-2)}\beta^2x^{2n} + \ldots\right);$$

pro negatiuo $r = -\varrho$,

$$y = \mathfrak{A}x^{p^I}(1+\beta x^n)^{1-2\varrho}\left(1 + (\varrho-1)\frac{(\varrho+k-1)}{k-1}\beta x^n + \frac{(\varrho-1)(\varrho-2)}{1.2}\frac{(\varrho+k-1)(\varrho+k-2)}{(k-1)(k-2)}\beta^2x^{2n} + \ldots\right)$$
$$+ \mathfrak{B}x^{p}(1+\beta x^n)^{1-2\varrho}\left(1 + (\varrho-1)\frac{(k-\varrho+1)}{k+1}\beta x^n + \frac{(\varrho-1)(\varrho-2)}{1.2}\frac{(k-\varrho+1)(k-\varrho+2)}{(k+1)(k+2)}\beta^2x^{2n} + \ldots\right);$$

vbi p, p^I sunt binae radices aequationis quadraticae: $f = -ap(p-1) - cp$, et $k = \frac{p - p^I}{n}$, $\beta = \frac{b}{a}$. Quando integrale logarithmum ipsius x inuoluat, et quomodo tum illud exprimendum sit, ex formulis (§§. XXXVII; XLI;) satis constat. Ex (§. XXVIII. 3.) aliam integralis completi expressionem deducere liceret, valoribus affirmatiuis et negatiuis τοῦ r aeque aptam.

2) Alia ex casu primo sequitur aequatio integrabilis, si ponatur $p = -\frac{c}{2a}$: tum erit $f = \frac{c(c-2a)}{4a}$; $g = \frac{(c - 2ran)(2ae - b(c + 2a(1 - rn)))}{4aa}$. Pro his igitur

valoribus

valoribus τῶν f et g, exiſtente r numero quouis integro, aequationis

$$0 = x^2 (a + bx^n) d^2y + x (c + ex^n) dydx + (f + gx^n) y dx^2$$

integrale particulare ſemper algebraice exhibere licet; completum ſaltem ad integrationem formulae differentialis ſatis notae reducitur (§. XLII. 1.).

Permutatis inuicem coëfficientibus a, b; c, e; f, g; et poſito $-n$ loco n, (§. LXI. 1.) eodem modo integrabilis eſt aequatio noſtra differentialis, dum fuerit

$$g = \frac{e(e - 2b)}{4b}, \quad f = \frac{(e + 2rbn)(2bc - a(e + 2b(1 + rn)))}{4bb}.$$

Cum ſit $p + p^I = 1 - \frac{c}{a}$, ex poſitione $p = -\frac{c}{2a}$ ſequitur $p^I = 1 - \frac{c}{2a} = 1 + p$, ſiue $p - p^I = -1$. Eſt autem $p - p^I = nk$; hinc, ſumto $n = \pm 1$, erit $k = \mp 1$. Tum, ob r et k numeros integros, integrale completum finite exprimi poteſt (§§. XLII. XLIV. A. 1.); et quidem tranſcendenter, ſeu per logarithmos et Arcus, niſi pro $n = +1$, $k = -1$, ſit $r = 0$; vel $r = -1$, aut $= 0$, pro $n = -1$, $k = +1$: quibus binis caſibus integrale completum algebraice datur (§§. XXXIII. XXXIV.). Sic obtinentur hae binae aequationes, complete integrabiles, vel algebraice, vel per quantitates tranſcendentes notas: $0 = x^2(a + bx) d^2y + x(c + ex) dydx + (f + gx) y dx^2$, poſito:

$$f = \frac{c(c - 2a)}{4a}, \quad g = \frac{(c - 2ra)(2ae - b(c + 2(1 - r)a))}{4aa};$$

et $0 = x^2 (A + Bx) d^2y + x(C + Ex) dydx + (F + Gx) y dx^2$, poſito:

$$G = \frac{E(E - 2B)}{4B}, \quad F = \frac{(E + 2rB)(2BC - A(E + 2(1 + r)A))}{4BB}.$$

Poſterior aequatio ex poſitione $n = -1$ oritur, dum loco a; b; c; e; f; g; ponantur B; A; E; C; G; F.

Prior aequatio ex (§. V. 1.) hanc etiam ſuppeditat aequationem, eodem modo integrabilem: $0 = x^2 (a + bx^n) d^2y + x(c + ex^n) dydx + (f + gx^n) y dx^2$, poſito:

$$f = \frac{(c + a(n - 1))(c - a(n + 1))}{4a}, \quad g = \frac{(c + a(n - 1) - 2anr)(2ae - b(c + a(n + 1) - 2anr))}{4aa};$$

quae aequatio ex caſu primo immediate deduci potuiſſet, ſumendo $-p = \frac{c + a(n - 1)}{2a}$; vnde ſit $p - p^I = -n$, id eſt, $k = -1$, hincque aequatio, ob r et k numeros integros, complete integrabilis. Simili ratione poſterior aequatio generalior reddi, vel etiam ex modo inuenta, alia itidem integrabilis, ex (§. LXI. 1.) deriuari poteſt.

Ex caſu II. et III.

3) Ponatur pro caſu *ſecundo* (§. LVI.) $g = 0$, erit $e = b(\frac{c}{a} + (\frac{1}{2} + r)n)$, $f = \frac{a}{4}\left((\frac{c}{a} - 1)^2 - (\frac{1}{2} - r)^2 n^2\right) = \frac{a}{4}\left(\frac{c}{a} - 1 + (\frac{1}{2} - r)n\right)\left(\frac{c}{a} - 1 - (\frac{1}{2} - r)n\right)$.

Tum-

Tum integrale aequationis differentialis:

$$o = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)ydx^2,$$

pro quouis affirmatiuo r, prodit: $y = \chi^{\frac{2-n}{4n}+\frac{r}{2}-\frac{c}{2na}} d^r z : d\chi^{\varrho}$, existente $z = \mathfrak{A}(\sqrt{}(a+b\chi)+\sqrt{}b\chi)^{2\mu} + \mathfrak{B}(\sqrt{}(a+b\chi)-\sqrt{}b\chi)^{2\mu}$, $\mu = \frac{1}{n}\sqrt{}(\frac{1}{4}(\frac{e}{b}-1)^2-\frac{g}{b})$, $\chi = x^n$.

4) Ex praecedenti integratione, diuidendo aequationem differentialem per x^n, et permutando cofficientes (§. LXI. 1.); vel etiam ex casu *tertio* (§. LX.), ponendo $\varrho = o$, sequitur integratio aequationis nostrae differentialis, si fuerit

$$e = b(\frac{c}{a}+(\frac{1}{2}+r)n), \qquad g = \frac{b}{4}(\frac{c}{a}-1+2rn)(\frac{c}{a}-1+n);$$

estque integrale completum pro affirmatiuo r (§. LXI. 1.): $y = \chi^{-\frac{n-2}{4n}+\frac{r}{2}+\frac{e}{2nb}} . d^r z : d\chi^r$; vbi $z = \mathfrak{A}(\sqrt{}(b+a\chi)+\sqrt{}a\chi)^{2\mu} + \mathfrak{B}(\sqrt{}(b+a\chi)-\sqrt{}a\chi)^{2\mu}$, $\mu = \frac{1}{n}\sqrt{}(\frac{1}{4}(\frac{c}{a}-1)^2-\frac{f}{a})$, $\chi = x^{-n}$.

Ex casu IV.

5) Ponatur pro casu *quarto* (§. LXIV.) $\varrho = o$, erit $f = \frac{a}{4}((\frac{c}{a}-1)^2-(\frac{1}{2}-r)^2n^2)$, $g = \frac{b}{4}((\frac{e}{b}-1)^2-(\frac{1}{2}-r)^2n^2)$; tumque pro affirmatiuo r, integrale completum: $y = (b+a\chi)^{\frac{2-n}{4n}+\frac{r}{2}-\frac{c}{2na}} . \chi^{-\frac{n-2}{4n}+\frac{r}{2}+\frac{e}{2nb}} d^r z : d\chi^r$; existente $z = \mathfrak{A}(\sqrt{}(b+a\chi)+\sqrt{}a\chi)^{2\mu} + \mathfrak{B}(\sqrt{}(b+a\chi)-\sqrt{}a\chi)^{2\mu}$, $\mu = \frac{1}{2n}(\frac{e}{b}-\frac{c}{a}-n)$, $\chi = -\frac{b}{a+bx^n}$.

Exempla.

§. LXVIII. 1) Posito (§. LXVII. 1.) $r = o$, erit $e = \frac{bc}{a}$, $g = \frac{bf}{a}$; tumque habetur aequationis differentialis $o = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)ydx^2$, integrale completum hoc: $y = \mathfrak{A}x^p + \mathfrak{B}x^{p^I}$, vbi p et p^I sunt radices aequationis quadraticae: $f = -ap(p-1)-cp$.

2) Po-

2) Posito ibidem $r = -1$, seu $\varrho = 1$, erit $e = \frac{b(c+2an)}{a}$; $g = \frac{b(f+nc-n(n-1)a)}{a}$; et integrale completum, $y = \frac{\mathfrak{A}x^{p} + \mathfrak{B}x^{p^{\mathrm{I}}}}{a+bx^{n}}$.

3) Posito (§. LXVII. 2.) $r = 0$, erit $f = \frac{(c+(n-1)a)(c-(n+1)a)}{4a}$, $g = \frac{(c+(n-1)a)(2ae-b(c+(n+1)a))}{4aa}$; et integrale completum ex (§. XXXIII.), $y = x^{\frac{-c-(n-1)a}{2a}}\left(\mathfrak{A} + \mathfrak{B}(a+bx^{n})^{1+\frac{c}{na}-\frac{e}{nb}}\right)$.

4) Hinc, adhibita obseruatione (§. LXI. 1.) sponte sequitur altera integratio: existentibus nimirum, $g = \frac{(e+(n-1)b)(e-(n+1)b)}{4b}$, $f = \frac{(e-(n+1)b)(2bc-a(e-(n-1)b))}{4bb}$, erit

$$y = \mathfrak{A}x^{\frac{-e+(n+1)b}{2b}} + \mathfrak{B}x^{\frac{e-(n-1)b}{2b}-\frac{c}{a}}(a+bx^{n})^{1+\frac{c}{na}-\frac{e}{nb}}.$$

5) Sumendo (§. LXVII. 3.) $r = 0$, habetur $e = \frac{b(an+2c)}{2a}$; $f = \frac{a}{4}\left(\left(\frac{c}{a}-1\right)^{2}-\frac{1}{4}n^{2}\right) = \frac{(2c+(n-2)a)(2c-(n+2)a)}{16a}$; tumque prodit aequationis differentialis,

$$0 = x^{2}(a+bx^{n})d^{2}y + x(c+ex^{n})dydx + (f+gx^{n})ydx^{2},$$

integrale completum: $y = x^{\frac{2-n}{4}-\frac{c}{2a}}.z$; existente $z = \mathfrak{A}(\sqrt{(a+bx^{n})}+\sqrt{bx^{n}})^{2\mu} + \mathfrak{B}(\sqrt{(a+bx^{n})}-\sqrt{bx^{n}})^{2\mu}$, $\mu = \frac{1}{n}\sqrt{\left(\frac{1}{4}\left(\frac{e}{b}-1\right)^{2}-\frac{g}{b}\right)}$.

6) Sumendo ibidem, $r = 1$, est $e = \frac{b(3an+2c)}{2a}$; $f = \frac{a}{4}\left(\left(\frac{c}{a}-1\right)^{2}-\frac{1}{4}n^{2}\right)$; tum prodit integrale completum $y = \chi^{\frac{2-n}{4n}+\frac{1}{2}-\frac{c}{2na}}\frac{dz}{d\chi}$, siue, actu differentiando z, vbi constantes $\mathfrak{A}$ et $\mathfrak{B}$ pro lubitu accipere licet (§. LIII. 3.) $y = \frac{x^{\frac{2-n}{4}-\frac{c}{2a}}.z}{\sqrt{(a+bx^{n})}}$; z et μ valores (5) seruant.

 7) Po-

7) Posito (§. LXVII. 4.) $r = 0$, erit $e = \frac{b(an+2c)}{2a}$; $g = \frac{b(c-a)(c+(n-1)a)}{4aa}$; et

$y = x^{\frac{n+2}{4} - \frac{e}{2b}} . z$, existente $z = \mathfrak{A}(\sqrt{(b+a\chi)}+\sqrt{a\chi})^{2\mu} + \mathfrak{B}(\sqrt{(b+a\chi)}-\sqrt{a\chi})^{2\mu}$

$= \frac{\mathfrak{A}(\sqrt{(a+bx^n)}+\sqrt{a})^{2\mu} + \mathfrak{B}(\sqrt{(a+bx^n)}-\sqrt{a})^{2\mu}}{x^{n\mu}}$,

$\mu = \frac{1}{n}\sqrt{(\frac{1}{4}(\frac{c}{a}-1)^2 - \frac{f}{a})}$.

8) Posito ibidem, $r = 1$, prodit $e = b\frac{(3an+2c)}{2a}$; $g = b\frac{(c+(2n-1)a)(c+(n-1)a)}{4aa}$, et integrale completum $y = \chi^{\frac{-n-2}{4n}+\frac{1}{2}+\frac{e}{2nb}}\frac{dz}{d\chi}$, siue rite euoluendo dz, $y = \frac{x^{\frac{3n+2}{4}-\frac{e}{2b}}.z}{\sqrt{(a+bx^n)}}$; vbi pro z et μ valores (7) substituuntur.

9) Posito (§. LXVII. 5.) $r = 0$, habetur

$f = \frac{a}{4}((\frac{c}{a}-1)^2 - \frac{1}{4}n^2) = \frac{(2c-(n-2)a)(2c-(n+2)a)}{16a}$;

$g = \frac{b}{4}((\frac{e}{b}-1)^2 - \frac{1}{4}n^2) = \frac{(2e-(n-2)b)(2e-(n+2)b)}{16b}$; tum integrale completum erit $y = (b+a\chi)^{\frac{2-n}{4n}-\frac{c}{2na}}.\chi^{\frac{-n-2}{4n}+\frac{e}{2nb}}.z$; at ob $\chi = \frac{-b}{a+bx^n}$, et $b+a\chi = \frac{b^2x^n}{a+bx^n}$, fit $z = \frac{\mathfrak{A}(b\sqrt{x^n}+\sqrt{-ab})^{2\mu}+\mathfrak{B}(b\sqrt{x^n}-\sqrt{-ab})^{2\mu}}{(a+bx^n)^{\mu}}$;

hinc, cum μ etiam negatiue accipi queat $= -\frac{e}{2nb}+\frac{c}{2na}+\frac{1}{2}$, constantibus insuper per $\sqrt{b}$ diuisis, prodit $y = x^{\frac{2-n}{4}-\frac{c}{2a}}(\mathfrak{A}(\sqrt{bx^n}+\sqrt{-a})^{2\mu}+\mathfrak{B}(\sqrt{bx^n}-\sqrt{-a})^{2\mu})$,

vbi $2\mu = 1+\frac{c}{na}-\frac{e}{nb}$.

Haec-

Haecce nouem exempla, posito n = 1, referunt ipsos casus ab EVLERO exhibitos, supra §. II. commemoratos. Exemplis nimirum nostris 1; 2; 3; 4; 5; 6; 7; 8; 9 respondent casus illi, hoc quisque ordine: 1; 2; 9; 7; 6; 5; 4; 3; 8. Quanquam EVLERVS singulos hosce casus peculiari methodo tractet, eosque a casu generali primo secernat, quem ex seriebus abrumpentibus resoluit: ex praecedenti tamen §pho apparet, quatuor horum casuum, 1; 2; 9; 7; esse tantum corollaria particularia casus generalis I. Deinde etiam casus 4 et 3 ex 6 et 5, vel vice versa, hi ex illis immediate consequuntur, dum ad duplicem aequationis differentialis formam (§. LXI. 1.) respiciatur.

Ex his iam intelligitur, quomodo varietas, quam nouem isti casus offerre videntur, simplicior reddi, ac negotium integrationis valde contrahi queat. Collatis porro casibus nostris generalibus II; III; IV; manifestum est, casus illos speciales habendos esse pro exemplis singularibus problematum longe latius patentium, ex quibus innumeras praeterea aequationes differentiales itidem integrabiles deducere liceat.

Ceterum EVLERVS, singulos nouem casus recensendo (l. c. pp. 146-153.), pro integralibus exhibuit aequationes differentiales primi gradus, quarum tamen vlterior resolutio ex regulis notis haud liquet; quare ipsa integralia completa finite euoluere operae pretium mihi videbatur. Pro quatuor casibus, 3; 4; 5; 6; praeter istas aequationes complicatas, alias insuper aequationes differentiales primi gradus, easque separatas, inuenit. Pro iisdem casibus, nec non pro 8vo, etiam in Commentatione supra §. III. laudata, integralia completa exprimuntur, quae cum nostris consentiunt.

Scholion.

§. LXIX. 1) Ill. LORGNA in Commentatione inscripta: Indagini nel Calcolo integrale (Memorie di Matem. e Fisica della Societa Italiana. Tom. II. P. I. Verona 1784. pag. 197 seq.), inter alias meditationes etiam aequationem differentialem hactenus consideratam, ob egregium eius vsum, ad examen reuocat, Proposit. IX. (§. XIX.), quae ita se habet: "Svolgere infiniti casi d'integrabilita dell' equazione $M\,dx^2 = x^2(a + bx^n)\,d^2y + x(e + fx^n)\,dy\,dx + (g + hx^n)\,y\,dx^2$, indipendenti dall' esponente n." Casus ab ipso euoluti huc redeunt, vt ponatur

1) $m^2a + m(a - e) + g = 0$; $m^2b + m(b - f) + h = 0$;
vel 2) $(2m + 4)a - e = 0$; $(2m + 4)b - f = 0$; pro $g = e$, $h = f$.
3) $(m + 1)a - e = 0$; $(m + 1)b - f = 0$; pro $g = -2a$, $h = -2b$.
4) $2am + g = 0$; $2bm + h = 0$; pro $e = 2a$, $f = 2b$.

Litera m denotat numerum quemuis integrum.

Quod iam hosce quatuor casus attinet, primum obseruandum est, in aequatione differentiali semper poni posse $M = 0$, quippe ex cognita integratione pro $M = 0$, sequitur integratio, etiamsi fuerit $M =$ cuiuis functioni variabilis x. Deinde ex supra demon-

 stratis

ſtratis (§. LXVII. 1.) facile colligitur, iſtos caſus ex caſu noſtro generali I ſponte fluere, poſito $r = 0$. Pro caſu (1) id quidem manifeſtum eſt; verum etiam pro caſibus 2; 3; 4; idem exinde probatur, quod ſit $\frac{f}{e} = \frac{b}{a} = \frac{h}{g}$. Exinde ſimul apparet, caſus hosce latius extendi, quippe valores τȣ m non ad numeros integros affirmatiuos reſtringendi, ſed pro lubitu accipiendi ſunt. Ceterum absque auxilio formularum ſuperiorum, integrabilitas quatuor caſuum ſequenti ratiocinio, quod mihi quidem ſimplicius eſſe videtur, demonſtrari poteſt. Pro caſu (1) ponatur $y = z x^{-m}$, tum aequatio differentialis in hanc abit: $M x^m d x^2 = x^2 (a + b x) d^2 z + x (e - 2 m a + (f - 2 m b) x^n) d z d x + (m (m + 1) a - m c + g + (m (m + 1) b - m f + h) x^n) z d x^2$. Quodſi nunc vltimi membri coëfficientes ad 0 redigantur, aequationem transformatam integrabilem eſſe, ponendo $d z = u d x$, ſatis notum eſt. Porro pro caſibus 2; 3; 4; tota aequatio per $a + b x^n$ diuiſibilis eſt, tumque eam integrabilem eſſe, aliunde conſtat. Quod ratiocinia attinet, quibus LORGNA caſus 2; 3; 4; demonſtrat, fateor, me eorum veritatem haud intelligere: quippe ex repetitis, quibus vtitur, ſubſtitutionibus, aliae potius transformationes, quam illic traditae, ſequi videntur. Hi autem ipſi caſus conſiderari etiam poſſunt tanquam corollaria caſus 1; quippe poſito $p^2 a + p(a - e) + g = 0$, erit ſimul $p^2 b + p(b - f) + h = 0$, ob $\frac{f}{b} = \frac{e}{a}$, $\frac{h}{b} = \frac{g}{a}$.

2) Propoſitione X (§. XX. p. 201.) idem auctor contemplatur aequationem generaliorem, $x^{\varphi+3}(a + b X) d^2 y + x^{\varphi+2}(e + f X) d y d x + x^{\varphi+1}(g + h X) y d x^2 - M d x^2 = 0$, denotantibus X et M quasuis functiones variabilis x. Ad cuius aequationis caſus integrabiles detegendos ponitur $y = \frac{v}{x^{\varphi+1}}$; tunc aequatio transformata, hincque etiam ipſa propoſita integrabilis eſt, cum fuerit

$$a(\varphi + 1)(\varphi + 2) - e(\varphi + 1) + g = 0$$
$$b(\varphi + 1)(\varphi + 2) - f(\varphi + 1) + h = 0.$$

Iam porro aequatio transformata tanquam data conſideratur, hincque ex ſimili ſubſtitutione et noua transformatione, nouae conditiones integrabilitatis oriuntur. Quae ſubſtitutiones ac transformationes cum in infinitum continuari queant, innumeri ſic obtinentur caſus integrabiles. Cui ratiocinio addere mihi liceat haec. Accuratior conſideratio, nec non actualis euolutio quantitatum, ab Auctore literis $e^{I}, e^{II}, \ldots e^{M}$; $f^{I}, f^{II}, \ldots f^{M}$; $g^{I}, g^{II}, \ldots g^{M}$; $h^{I}, h^{II} \ldots h^{M}$; deſignatarum, docet, omnes caſus integrabiles, qui hac methodo eruuntur, his binis aequationibus contineri:

$$a(m\varphi + m)(m\varphi + m + 1) - e(m\varphi + m) + g = 0$$
$$b(m\varphi + m)(m\varphi + m + 1) - f(m\varphi + m) + h = 0,$$

deno-

denotante m numerum quemuis integrum affirmatiuum. Iam vero, quin repetitis fubftitutionibus opus fit, vna ftatim fubftitutio $y = \frac{v}{x^p}$ fufficit; qua quidem adhibita aequatio differentialis propofita in hanc transformatur:

$$x^{\varphi+3}(a+bX)d^2v + x^{\varphi+2}(e-2pa+(f-2pb)X)dvdx$$
$$+x^{\varphi+1}(p(p+1)a-pe+g+(p(p+1)b-pf+h)X)vdx^2 - Mx^p dx^2 = o.$$

Quae aequatio fponte integrabilis fit, pofito $p(p+1)a - pe + g = o$, $p(p+1)b - pf + h = o$, ope fubftitutionis $dv = w\,dx$. Tum igitur etiam aequatio propofita integrabilis erit. Hicce cafus latius patere videtur, ac cafus infiniti ab Ill. LORGNA exhibiti, pro quibus m numero integro pofitiuo aequatur, cum pro illo p quosuis valores recipere queat.

3) Affertum modo commemoratum (1), quod ex cognita integratione aequationis

$$o = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)ydx^2,$$

fequatur etiam integratio aequationis generalioris,

$$Mdx^2 = x^2(a+bx^n)d^2y + x(c+ex^n)dydx + (f+gx^n)ydx^2,$$

denotante M quamuis functionem τῆς x, id, inquam, affertum pleniorem illuftrationem poftulat, quippe cum ea, quae fupra de cafibus integrabilibus demonftrata funt, ftatim ad aequationis differentialis formam generaliorem extendere licebit. Huc fpectat fequens

PROBLEMA.

§. LXX. Conceffa integratione aequationis differentialis $o = d^2v + Pdvdx + Qvdx^2$, inuenire integrale completum aequationis $Xdx^2 = d^2y + Pdydx + Qydx^2$.

Solutio.

Pofito $y = zv$, aequatio altera in hanc abit:

$$Xdx^2 = zd^2v + 2dzdv + vd^2z + P(zdv + vdz)dx + Qzvdx^2$$
$$= z(d^2v + Pdvdx + Qvdx^2) + vd^2z + (2dv + Pvdx)dz;$$

vnde, ob $d^2v + Pdvdx + Qvdx^2 = o$, fit $Xdx^2 = vd^2z + (2dv + Pvdx)dz$. Haec aequatio, fumendo $dz = z^I dx$, ad aequationem primi gradus reducitur: $Xdx = vdz^I + (2dv + Pvdx)z^I$. Quam ex regulis notis integrando, obtinetur $z^I = \frac{e^{-\int Pdx}}{vv}\int e^{\int Pdx}Xvdx$. Inde prodit $y = vz = v\int z^I dx =$

$$v\int\left(\frac{e^{-\int Pdx}dx}{vv}\int e^{\int Pdx}Xvdx\right).$$

Corol-

Corollarium.

§. LXXI. 1) Pro v sufficit accipere integrale particulare aequationis $o = d^2v + Pdvdx + Qvdx^2$, quo posito $= t$, erit $y = t\int\left(\frac{e^{-\int Pdx}dx}{tt}\int e^{\int Pdx}Xvdx\right)$, integrale completum alterius aequationis. Sumto $X = o$, abit y in v, hinc erit $v = Ct\int\frac{e^{-\int Pdx}dx}{tt}$, id est, ex integrali particulari t determinatur completum v (cf. §. XLV. 1.). Quod, ob constantem arbitrariam, integrali $\int\frac{e^{-\int Pdx}dx}{tt}$ addendam, sic exprimi potest: $v = \mathfrak{C}t + Ct\int\frac{e^{-\int Pdx}dx}{tt}$. Iam si denotet t^I alterum valorem particularem τȣ v, praeter t, erit etiam integrale completum $v = \mathfrak{C}t + Ct^I$. Quas expressiones aequando obtinetur $t^I = t\int\frac{e^{-\int Pdx}dx}{tt}$; $d\left(\frac{t^I}{t}\right) = \frac{e^{-\int Pdx}dx}{tt}$; $e^{\int Pdx} = \frac{dx}{ttd\left(\frac{t^I}{t}\right)}$. Hinc formula pro y in hanc transformatur, a quantitatibus exponentialibus liberam:

$$y = t\int\left(d\left(\frac{t^I}{t}\right)\int\frac{Xdx}{td\left(\frac{t^I}{t}\right)}\right).$$

2) Hanc expressionem ad aliam insuper formam, vsu commodiorem, reuocare licet. Est nimirum $\int(dr\int sdx) = r\int sdx - \int srdx$; hinc, sumto $r = \frac{t^I}{t}$, $s = \frac{X}{td\left(\frac{t^I}{t}\right)}$, sponte prodit $y = t^I\int\frac{Xdx}{td\left(\frac{t^I}{t}\right)} - t\int\frac{Xdx \,.\, t^I}{ttd\left(\frac{t^I}{t}\right)}$; siue, ob $d\left(\frac{t^I}{t}\right) = -\frac{t^I t^I}{tt}.d\left(\frac{t}{t^I}\right)$,

$$y = t^I\int\frac{Xdx}{td\left(\frac{t^I}{t}\right)} + t\int\frac{Xdx}{t^Id\left(\frac{t}{t^I}\right)}.$$

Quantitates t, t^I, denotant integralia particularia diuersa aequationis $o = d^2v + Pdvdx + Qvdx^2$, quae coniuncta praebent completum $v = \mathfrak{C}t + Ct^I$ (*).

(*) Formulam praecedentis §phi exhibuit *L. Eulerus* (I. C. I. Vol. II. §. 831.) Alteram vid. supe-rem (2) alia ratione demonstrauit *D'Alembertus* (Histoire de l'Academie Royale de Sciences. Annee MDCCLXIX. Paris MDCCLXXII.) Similes formulas pro aequationibus altiorum graduum tradidit Ill. *La Place* (Miscellanea Taurinensia Tom. IV. p. 186. etc.)

TRA-

TRACTATVS
DE REVERSIONE SERIERVM,
SIVE
DE RESOLVTIONE AEQVATIONVM PER SERIES.

TRACTATVS

DE REVERSIONE SERIERVM,

SIVE

DE RESOLVTIONE AEQVATIONVM PER SERIES.

CAP. I.

DE THEOREMATE LA GRANGIANO,

EXHIBENTE RESOLVTIONEM AEQVATIONIS: $y = x - z \,.\, \varphi x$ PER SERIEM INFINITAM.

PROBLEMA.

§. I. Propofita inter tres quantitates y, x, et z, aequatione hac: $y = x - z \,.\, \varphi x$, denotante φx certam functionem quantitatis x; exprimere quamuis functionem eiusdem quantitatis, veluti ψx, per feriem fecundum poteftates variabilis z progredientem.

Solutio.

1) Cum pro $z = 0$ fit $x = y$, hincque $\psi x = \psi y$, erit feriei quaefitae pro ψx membrum primum $= \psi y$. Ponatur itaque

$$(a)\ \psi x = \psi y + \overset{I}{Y} z^1 + \overset{II}{Y} z^2 + \overset{III}{Y} z^3 + \ldots + \overset{N}{Y} z^n + \ldots,$$

tum erunt coëfficientes affumti $\overset{I}{Y}, \overset{II}{Y}, \ldots \overset{N}{Y}, \ldots$ functiones quantitatis y, quippe x pendet a z et y, indeque quaeuis functio τȣ x praeter z inuoluet y. Ad quarum igitur functionum incognitarum determinationem problematis propofiti folutio redit.

2) Valor τȣ ψx pro iisdem z et y pendet 1) ab indole functionis figno ψ expreffae, 2) a forma aequationis propofitae, feu ab indole functionis φ. Hinc valores τῶν $Y^{I}, Y^{II}, Y^{III}, \ldots$ inuoluent figna functionalia φ et ψ, quorum illud ob aequationem datam pro certo ac determinato habendum, hoc vero arbitrarium eft: ita quidem, vt fi quaeuis alia functio quantitatis x, veluti Fx, quaeratur, eius valor immediate ex valore inuento pro ψx peti queat, permutando tantum fignum ψ cum F.

3) Cum ex aequatione propofita fit $x = y + z\varphi x$, erit ex theoremate *Tayloriano*, (a)

(b) ψx

(a) Hoc theorema grauiffimum, quanquam non *Taylori* nomine infignitum, demonftratum extat in *Kaeftneri Anfangsgr. der Analyf. des Unendl.* (2te Aufl. Goettingen im Verlag der Wittwe Vandenhoeck.

$$(b)\ \psi x = \psi y + z\varphi x \cdot \frac{d\psi y}{dy} + z^2(\varphi x)^2 \frac{d^2\psi y}{1.2.dy^2} + \ldots$$

$$+ z^n(\varphi x)^n \cdot \frac{d^n\psi y}{1.2\ldots n.dy^n} + \ldots$$

Ex cuius seriei comparatione cum serie assumta (a) (1), huius coëfficientes indeterminatos Y^{I}, Y^{II}, Y^{III}, ... successiue definire, eorumque legem generalem non minus simpliciter ac rigorose demonstrare licet.

Manifestum equidem est, si series (b), expressis valoribus τῶν φx, $(\varphi x)^2$, $(\varphi x)^3$, ... per y et z, in hanc abeat:

$$\psi x = \psi y + z\Upsilon^{I} + z^2\Upsilon^{II} + z^3\Upsilon^{III} \ldots + z^n\Upsilon^{N} + \ldots,$$

fore $\Upsilon^{I} = Y^{I}$, $\Upsilon^{II} = Y^{II}$, ... $\Upsilon^{N} = Y^{N}$, ...

4) Iam pro determinando coëfficiente Y^{I}, ponatur in serie (a) φ loco ψ, eritque $\varphi x = \varphi y + zU$, vbi reliqua membra praeter primum, quippe singula factorem z inuoluentia, per zU repraesentantur. Quo valore in serie (b) supposito, fit

$$\psi x = \psi y + z(\varphi y + zU)\frac{d\psi y}{dy} + z^2(\varphi x)^2 \frac{d^2\psi y}{1.2\,dy^2} + \ldots$$

$$= \psi y + z\varphi y \frac{d\psi y}{dy} + z^2(\ldots),$$

dum rursus membra post secundum omnia, signo (...), praemisso factore communi z^2, exprimuntur. Coëfficientem membri secundi aequando coëfficienti secundo seriei (a), obtinetur $Y^{I} = \varphi y \cdot \dfrac{d\psi y}{dy}$.

5) Cum itaque coëfficientes duo priores seriei generalis pro ψx noti iam sint, statim inde consequuntur coëfficientes correspondentes serierum pro φx et $(\varphi x)^2$ (2), eritque

$\varphi x =$

borsh. 1770. 8.) §. 151. p. 89. Idem nuper amplius pertractatum fuit ab *Hindenburgio* (*Archiv der reinen und angewandten Mathematik, II. Heft. 1794. pp. 201-219.*). Scriptis ibi laudatis alia insuper addi possunt: quod cum ad hunc locum minus pertineat, sufficiat nominasse duo: 1) *Principiorum Calculi diff. et integr. exposit. elementaris*, ad normam dissertationis ab Academia Scient. Reg. Pruss. a. 1786 praemii honore decoratae, elaborata Auctore *Sim. l'Huilier* etc. Tubing. ap. Ioh. Georg. Cottam 1795. 4; cuius operis praeclari cap. III. (pp. 48-58.) agit de theor. Taylor. 2) M. *Iac. Siegm. Beck* (nunc Profess. Halens.) *Demonstratio theor. Taylor.*, dissert. quae Halae 1791 prodiit 4; de cuius autem titulo mihi non certo constat, cum ea, quam ab amico extero acceperam, iam non sit ad manus, eademque in recentissima editione operis *Meuseliani.* (*Gelehrtes Deutschland Tom. 2. 1796.*) frustra quaeratur.

$$\varphi x = \varphi y + z\varphi y \frac{d\varphi y}{dy} + z^2 (\ldots)$$

$$(\varphi x)^2 = (\varphi y)^2 + z\varphi y \frac{d(\varphi y)^2}{dy} + z^2 (\ldots).$$

Quibus valoribus suppositis in serie (b), prodit

$$\psi x = \psi y + z(\varphi y + z\varphi y . \frac{d\varphi y}{dy} + z^2 \ldots) \frac{d\psi y}{dy}$$

$$+ z^2 ((\varphi y)^2 + z\varphi y \frac{d(\varphi y)^2}{dy} + z^2 \ldots) \frac{d^2\psi y}{1.2.dy^2}$$

$$+ z^3 (\ldots)$$

$$= \psi y + z\varphi y \frac{d\psi y}{dy} + z^2 (\varphi y . \frac{d\varphi y}{dy} . \frac{d\psi y}{dy} + (\varphi y)^2 \frac{d^2\psi y}{1.2.dy^2}) + z^3 (\ldots)$$

Cuius seriei membrum tertium comparando cum membro tertio seriei (a), emergit

$Y^{II} = \varphi y . \frac{d\varphi y}{dy} . \frac{d\psi y}{dy} + (\varphi y)^2 . \frac{d^2\psi y}{1.2.dy^2} = \frac{d((\varphi y)^2 . \frac{d\psi y}{dy})}{1.2.dy}$. Sic itaque coefficiens tertius seriei (a) definitus est.

6) Operationes, quibus coëfficientes Y^I et Y^{II} determinati sunt, simili omnino ratione sine negotio vlterius continuari possunt, indeque prodibunt coëfficientes sequentes $Y^{III}, Y^{IV}, Y^V, \ldots$ Etenim sumamus, inuentos iam esse coëfficientes n priores seriei (a), scilicet praeter primum ψy, hosce: $Y^I, Y^{II}, Y^{III}, \ldots Y^{N-1}$, tum coëfficiens sequens Y^N hunc in modum reperietur. Ex coëfficientibus per hypothesin notis seriei pro ψx, totidem coëfficientes serierum pro φx, $(\varphi x)^2$, $(\varphi x)^3, \ldots (\varphi x)^n$ innotescunt (2); quorum pro quauis serie vnum tantum hoc loco considerasse sufficit, scilicet seriei φx coefficientem n^{tum}, seriei $(\varphi x)^2$ coëfficientem $n-1^{tum}$, sicque porro serierum $(\varphi x)^3$, $(\varphi x)^4, \ldots (\varphi x)^n$ coëfficientes, $n-2^{tum}$, $n-3^{tum}, \ldots 1^{mum}$. Quibus coëfficientibus per $(\varphi x)kn$, $(\varphi x)^2k(n-1)$, $(\varphi x)^3k(n-2), \ldots (\varphi x)^nk1$ expressis (a*); ceteris autem earundem serierum coëfficientibus per puncta tantum exhibitis, erit

$$\varphi x = \varphi y + . z + . z^2 + \ldots + (\varphi x)kn . z^{n-1} + . z^n + \ldots$$

$$(\varphi x)^2 = (\varphi y)^2 + . z + . z^2 + \ldots + (\varphi x)^2 k(n-1) z^{n-2} + . z^{n-1} + \ldots$$

$$(\varphi x)^3 = (\varphi y)^3 + . z + . z^2 + \ldots + (\varphi x)^3 k(n-2) z^{n-3} + . z^{n-2} + \ldots$$

$$\ldots \quad \ldots \quad \ldots$$

$$(\varphi x)^{n-1} = (\varphi y)^{n-1} + (\varphi x)^{n-1} k2 . z + . z^2 + \ldots$$

$$(\varphi x)^n = (\varphi x)^n k1 + . z + \ldots$$

Gg 2 Hosce

a*) Hoc *signum coëfficientium* infra amplius illustrabitur atque in vsum vertetur.

Hosce valores in ferie (b) fupponendo, obtinetur:

$$\psi x = \psi y + z(\varphi y + . z^2 + \ldots + (\varphi x) k n . z^{n-1} + \ldots) \frac{d\psi y}{dy}$$

$$+ z^2((\varphi y)^2 + . z + . z^2 + \ldots + (\varphi x)^2 k(n-1) z^{n-2} + \ldots) \frac{d^2 \psi y}{1.2\, dy^2}$$

$$+ z^3((\varphi y)^3 + . z + . z^2 + \ldots + (\varphi x)^3 k(n-2) z^{n-3} + \ldots) \frac{d^3 \psi y}{1.2.3\, dy^3}$$

$$+ \ldots \ldots \ldots \ldots$$

$$+ z^n((\varphi x)^n k 1 + z + \ldots) \frac{d^n \psi y}{1.2 \ldots n\, dy^n}$$

$$+ z^{n+1}(\ldots \ldots)$$

Inde, collectis membris, quae z^n inuoluunt, fponte fluit $(\psi x) k(n+1) = \Upsilon^N = Y^N$ (3)

$$= (\varphi x) k n . \frac{d\psi y}{dy} + (\varphi x)^2 k(n-1) . \frac{d^2 \psi y}{1.2\, dy^2} + (\varphi x)^3 k(n-2) . \frac{d^3 \psi y}{1.2.3\, dy^3} + \ldots + (\varphi x)^n k 1 . \frac{d^n \psi y}{1.2 \ldots n\, dy^n}.$$

7) Haec formula oftendit, quomodo ex coëfficientibus n prioribus feriei (a), $n+1^{tus}$ fit eliciendus. Sic ex Y^I et Y^{II} (4. 5) prodit $Y^{III} = \dfrac{d^2((\varphi y)^3 \frac{d\psi y}{dy})}{1.2.3\, dy^2}$; vnde iam fatis clare lex apparet, quam coëfficientes obferuant, vi cuius erit, praefumtiue faltem,

$$Y^R = (\psi x) k(r+1) = \frac{d^{r-1}((\varphi y)^r \frac{d\psi y}{dy})}{1.2 \ldots r\, dy^{r-1}}.$$

8) Quam legem vniuerfalem effe demonftrandum reftat. Supponamus, eam valere pro $r = 1, 2, 3 \ldots$ vsque ad $r = n-1$: tum erit, pofito $(\varphi x)^s$ loco ψx, hincque loco $\frac{d\psi y}{dy}$, $s(\varphi y)^{s-1} \frac{d\varphi y}{dy}$, feriei qua $(\varphi x)^s$ exprimitur, coëfficiens $r+1^{tus} =$

$$(\varphi x)^s k(r+1) = \frac{d^{r-1}((\varphi y)^r . s(\varphi y)^{s-1} \frac{d\varphi y}{dy})}{1.2 \ldots r\, dy^{r-1}} = \frac{s}{r+s} \frac{d^n(\varphi y)^{r+s}}{1.2 \ldots r\, dy^r} =$$

$$s . \frac{(r+s-1)(r+s-2)\ldots(r+1)}{1.2 \ldots (r+s)} \frac{d^r(\varphi y)^{r+s}}{dy^r},$$ exiftente s numero integro. Sumto

to iam successiue $s=1$, $r=n-1$; $s=2$, $r=n-2$; $s=3$, $r=n-3$; .... $s=n$, $r=0$, vt summa $r+s$ maneat $=n$, erit $(\varphi' x)^s k(r+1)=\frac{1}{1.2\ldots n}$ $. s(n-1)(n-2)\ldots(r+1).\frac{d^r(\varphi y)^n}{dy^r}$; indeque ex formula (6) obtinetur

$$(\psi x)k(n+1)=Y^N=$$

$$\frac{1}{1.2.3\ldots n\, dy^n}.(d^{n-1}(\varphi y)^n.d\psi y+\frac{(n-1)}{1}d^{n-2}(\varphi y)^n d^2\psi y+$$

$$\frac{(n-1)(n-2)}{1.2}d^{n-3}(\varphi y)^n d^3\psi y+\ldots+\frac{(n-1)\ldots 1}{1.2\ldots n-1}.(\varphi y)^n d^n\psi y).$$

Quae expressio, per regulas notas pro differentialibus altioribus producti duarum variabilium, in hanc abit: $Y^N=\frac{1}{1.2\ldots n}.\frac{d^{n-1}((\varphi y)^n\frac{d\psi y}{dy})}{dy^{n-1}}$. Quare si lex assumta (7) pro n prioribus coëfficientibus obseruatur, eadem ad $n+1$tum, sicque ad omnes coëfficientes extenditur.

9) Haec igitur iam inuenta ac demonstrata est problematis solutio: Posito $y=x-z\varphi x$, erit

$$\psi x=\psi y+z\varphi y\frac{d\psi y}{dy}+z^2\frac{d(\varphi y^2\frac{d\psi y}{dy})}{1.2\,dy}+z^3\frac{d^2(\varphi y^3\frac{d\psi y}{dy})}{1.2.3\,dy^2}+\ldots$$

$$+z^n\frac{d^{n-1}(\varphi y^n\frac{d\psi y}{dy})}{1.2\ldots n\,dy^{n-1}}+\ldots$$

Corollarium.

§. II. 1) Posito $\varphi x=1$, erit $y=x-z$, porro $\varphi y=1$, hinc prodit

$$\psi x=\psi(y+z)=\psi y+z\frac{d\psi y}{dy}+z^2\frac{d^2\psi y}{1.2\,dy^2}+z^3\frac{d^3\psi y}{1.2.3\,dy^3}+\ldots,$$

i. e. theorema Taylorianum, quod igitur sub theoremate generali modo demonstrato (§. I. 9.), instar casus particularis, comprehensum est.

2) Po-

2) Ponatur $\psi x = (\phi x)^s$, tum erit $\psi y = (\phi y)^s$, $d\psi y = s\phi y^{s-1} d\phi y$,

$$\frac{d^{n-1}\left(\phi y^n \frac{d\psi y}{dy}\right)}{dy^{n-1}} = \frac{s d^{n-1}\left(\phi y^{s+n-1}\frac{d\psi y}{dy}\right)}{dy^{n-1}} = \frac{s}{s+n}\frac{d^n\left(\phi y^{s+n}\right)}{dy^n};$$

hinc sponte sequitur: $(\phi x)^s = \phi y^s + \frac{s}{s+1}\frac{d\left(\phi y^{s+1}\right)}{dy}z + \frac{s}{s+2}\frac{d^2\left(\phi y^{s+2}\right)}{1.2\,dy^2}.z^2$

$+ \ldots + \frac{s}{s+n}\frac{d^n\left(\phi y^{s+n}\right)}{1.2\ldots n\,dy^n} + \ldots$ dum relatio inter tres variabiles x, y, z, hac aequatione exprimitur: $y = x - z\phi x$.

3) Pro $\psi x = x^s$, simili ratione obtinetur:

$$x^s = y^s + sy^{s-1}\phi y.z + s\frac{d\left(y^{s-1}\phi y^2\right)}{1.2\,dy}.z^2 + s\frac{d^2\left(y^{s-1}\phi y^3\right)}{1.2.3\,dy^2}.z^3 + \ldots$$

Scholion.

La Grangii Analysis Problematis I. illustrata.

§. III. Theorema solutione modo inuenta expressum, idque simplici forma, lato ambitu, ac multiplici vsu eminens, primus exhibuit Ill. LA GRANGE (b), quanquam sub forma paulo diuersa, ponendo nimirum $z = 1$, ac considerando y tanquam quantitatem constantem α, et x ceu radicem aequationis algebraicae. Contra demonstrationem dubia mouerunt viri Clariss. TOEPFERUS et ROTHIUS (c), consentiente Celeb. HINDENBURGIO (d). Qua occasione equidem aliam demonstrationem inuestigaui, quam a dubiis omnino liberam iudicauit ac publici iuris fecit Analysta modo laudatus (e). Cum iam animus sit, problema grauissimum sane de reuersione serierum fusius pertractandi, quam tum consilium fuerat, ne sequentia obscuriora essent, istam demonstrationem hoc loco repetendam duxi, eidem tamen, quo distinctior fieret, nonnulla addidi.

Cete-

b) *Hist. de l'Acad. Roy. des Sciences etc. Tom. XXV. Année 1768. Berlin 1770.* Nouvelle methode pour resoudre les equations litterales par le moyen des series. §. II. 15. pag. 275. Cuius commentationis versio germanica legitur in Celeb. *Michelsen Theorie der Gleichungen, aus den Schriften der Herren Euler und de la Grange. Berlin 1791. 8. pp. 190-270.* (siue version. german. *Euleri* Introd. in Anal. infinit. P. III.). Cf. inprimis pp. 201-209.

c) *Combinatorische Analytik und Theorie der Dimensionszeichen, in Parallele gestellt von H. A. Toepfer. Leipz. 1793. 8. pp. 174. 175.*
Formulae de serierum reuersione demonstratio vniuersalis — auctore *M. H. A. Rothio.* Lipsiae 1793. 4. (Praef. pag. IV.).

d) *Archiv der reinen und angewandten Mathematik. 1. Heft. 1794. Leipz. p. 90.*

e) *Archiv I.* p. 81. cf. p. 90.

Ceterum cum operae pretium sit, viam nosse, quam inuentor huius theorematis ingressus est, eius Analysin commemorare haud superfluum habui, praesertim cum demonstratio sequenti ratione illustrari et firmari magis videatur.

1) Sit proposita aequatio $0 = a - bx + cx^2 - dx^3 + \ldots$ cuius radices designentur literis p, q, r, ...; tum erit

$$a - bx + cx^2 - dx^3 + \ldots = a\left(1 - \frac{x}{p}\right)\left(1 - \frac{x}{q}\right)\left(1 - \frac{x}{r}\right)\ldots$$

siue, diuidendo per bx, ac mutatis signis, $1 - \frac{a}{bx} - \frac{(cx - dx^2 + ex^3 - \ldots)}{b}$

$$= -\frac{a}{bx}\left(1 - \frac{x}{p}\right)\left(1 - \frac{x}{q}\right)\left(1 - \frac{x}{r}\right)\ldots = \frac{a}{bp}\left(1 - \frac{p}{x}\right)\left(1 - \frac{x}{q}\right)\left(1 - \frac{x}{r}\right)\ldots$$

Sumendo logarithmos fit $\log.\left(1 - \frac{a}{bx} - \xi\right) =$

$$\log.\frac{a}{bp} + \log.\left(1 - \frac{p}{x}\right) + \log.\left(1 - \frac{x}{q}\right) + \log.\left(1 - \frac{x}{r}\right) + \ldots,$$

posito $\xi = \frac{cx - dx^2 + ex^3 - \ldots}{b}$.

2) Quantitas ex parte dextra aequationis (1), ope expressionum logarithmicarum, facile in seriem secundum potestates τῆς x tam positiuas quam negatiuas progredientem conuerti potest. Iam in similem seriem logarithmus ex altera parte transformandus est. Quo facto coëfficientes membrorum easdem potestates τῆς x inuoluentium vtrinque inuicem aequantur (f). Cum vero id agitur, vt radix p, eiusque dignitates $p^2, p^3, \ldots p^m, \ldots$ inuestigentur, sufficit eos terminos vtriusque seriei considerasse, qui negatiuis potestatibus τῆς x affecti sunt. Sic ex parte dextra aequationis, omissis terminis cum $x, x^2, x^3, \ldots$ prodit

$$\log.\frac{a}{bp} - \frac{p}{x} - \frac{p^2}{2x^2} - \frac{p^3}{3x^3} \ldots - \frac{p^m}{mx^m} - \ldots$$ Ex altera parte habetur

$$\log.\left(1 - \frac{a}{bx} - \xi\right) = \log.\left(1 - \frac{a}{bx}\right) + \log.\left\{1 - \frac{\xi}{1 - \frac{a}{bx}}\right\},$$ vbi est $\log.\left(1 - \frac{a}{bx}\right)$

$$= -\frac{a}{bx} - \frac{a^2}{2b^2x^2} - \frac{a^3}{3b^3x^3} - \ldots - \frac{a^m}{mb^mx^m} - \ldots$$ Quare euoluendus restat

$$\log.\left\{1 - \frac{\xi}{1 - \frac{a}{bx}}\right\}.$$

3) Qui

f) Cf. *Rothe* p. IV.

3) Qui logarithmus eſt $=$

$$-\frac{\xi}{1-\frac{a}{bx}}-\frac{\xi^2}{2\left(1-\frac{a}{bx}\right)^2}-\frac{\xi^3}{3\left(1-\frac{a}{bx}\right)^3}-\ldots-\frac{\xi^r}{r\left(1-\frac{a}{bx}\right)^r}-\ldots$$

Quo iam modo generali coëfficiens τῶ $\frac{1}{x^m}$ ex hac ſerie reſultans aſſignari queat, ſeriem ξ eiusque poteſtates ſequenti ratione deſignare conuenit:

$$\xi = \pi(1)x + \pi(2)x^2 + \pi(3)x^3 + \ldots$$
$$\xi^2 = \pi^2(1)x^2 + \pi^2(2)x^3 + \pi^2(3)x^4 + \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\xi^r = \pi^r(1)x^r + \pi^r(2)x^{r+1} + \pi^r(3)x^{r+2} + \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

vbi index ſuperimpoſitus poteſtatem ſeriei, index adſcriptus locum coëfficientis in ſerie ſua exponit. Qui notandi modus alioquin etiam haud incommodus videtur. Cum porro ſit

$$\left(1-\frac{a}{bx}\right)^{-r} = 1 + r\frac{a}{bx} + \frac{r(r+1)}{1.2}\frac{a^2}{b^2x^2} + \ldots + \frac{r(r+1)\ldots(2r-1)}{1.2\ldots r}\frac{a^r}{b^r x^r} + \ldots,$$

hac ſerie in ſeriem ξ^r multiplicata, neglectis affirmatiuis poteſtatibus τῶ x (2), prodit

$$\frac{\xi^r}{\left(1-\frac{a}{bx}\right)^r} =$$

$$\pi^r(1)\frac{r(r+1)\ldots(2r-1)}{1.2\ldots r}\frac{a^r}{b^r} + \pi^r(1)\frac{r(r+1)\ldots 2r}{1.2\ldots(r+1)}\frac{a^{r+1}}{b^{r+1}}\frac{1}{x}\ldots$$

$$+\pi^r(1)\frac{r(r+1)\ldots(2r+m-1)}{1.2\ldots(r+m)}\frac{a^{r+m}}{b^{r+m}}\cdot\frac{1}{x^m}+\ldots$$

$$+\pi^r(2)\frac{r(r+1)\ldots 2r}{1.2\ldots(r+1)}\frac{a^{r+1}}{b^{r+1}} + \pi^r(2)\frac{r(r+1)\ldots(2r+1)}{1.2\ldots(r+2)}\frac{a^{r+2}}{b^{r+2}}\frac{1}{x}\ldots$$

$$+\pi^r(2)\frac{r\ldots(2r+m)}{1.2\ldots(r+m+1)}\frac{a^{r+m+1}}{b^{r+m+1}}\cdot\frac{1}{x^m}+\ldots$$

$+ \pi^r(3)$

$$+\pi^r(3)\frac{r(r+1)\ldots(2r+1)}{1.2\ldots(r+2)}\frac{a^{r+2}}{b^{r+2}}+\pi^r(3)\frac{r(r+1)\ldots(2r+2)}{1.2\ldots(r+3)}\frac{a^{r+3}}{b^{r+3}}\cdot\frac{1}{x}\ldots$$

$$+\pi^r(3)\frac{r\ldots(2r+m+1)}{1.2\ldots(r+m+2)}\frac{a^{r+m+2}}{b^{r+m+2}}\cdot\frac{1}{x^m}+\ldots$$

$$+\ldots$$

Quare feriei, in quam $\frac{\xi^r}{r\left(1-\frac{a}{bx}\right)^r}$ euoluitur, coëfficiens in $\frac{1}{x^m}$ ductus fit $=$

$$=\pi^r(1)\frac{(r+1)\ldots(2r+m-1)}{1.2\ldots(r+m)}\frac{a^{r+m}}{b^{r+m}}+\pi^r(2)\frac{(r+1)\ldots(2r+m)}{1.2\ldots(r+m+1)}\frac{a^{r+m+1}}{b^{r+m+1}}+$$

$$\pi^r(3)\frac{(r+1)\ldots(2r+m+1)}{1.2\ldots(r+m+2)}\frac{a^{r+m+2}}{b^{r+m+2}}+\ldots$$

$$=\frac{1}{1.2\ldots r}\Big(\pi^r(1)(2r+m-1)(2r+m-2)\ldots(r+m+1)\frac{a^{r+m}}{b^{r+m}}+$$

$$\pi^r(2)(2r+m)\ldots(r+m+2)\frac{a^{r+m+1}}{b^{r+m+1}}+\pi^r(3)(2r+m+1)\ldots(r+m+3)\frac{a^{r+m+2}}{b^{r+m+2}}+\ldots\Big)$$

Est autem $\xi^r x^{r+m-1}=\pi^r(1)x^{2r+m-1}+\pi^r(2)x^{2r+m}+\pi^r(3)x^{2r+m+1}+\ldots$, hinc $\frac{d^{r-1}(\xi^r x^{r+m-1})}{dx^{r-1}}=(2r+m-1)\ldots(r+m+1)\pi^r(1)x^{r+m}$

$$+(2r+m)\ldots(r+m+2)\pi^r(2)x^{r+m+1}+(2r+m+1)\ldots(r+m+3)\pi^r(3)x^{r+m+2}+\ldots$$

Exinde fponte fequitur, coëfficientem iftum τ8 $\frac{1}{x^m}$ ex $\frac{\xi^r}{r\left(1-\frac{a}{bx}\right)^r}$ oriundum, fore $=$

$$\frac{1}{1.2\ldots r}\cdot\frac{d^{r-1}(\xi^r x^{r+m-1})}{dx^{r-1}},$$ *dum poft differentiationem ponatur* $\frac{a}{b}$ *loco* x.

 4) Su-

4) Sumendo id, hac forma generali successiue $r = 1, 2, 3, 4$ ... obtinentur coëfficientes partiales $\tau\tilde{\varsigma}\ \frac{1}{x^m}$, quos singuli termini seriei pro log. $\left\{1 - \frac{x\,\xi\,\xi}{1 - \frac{a}{bx}}\right\}$ praebent: quibus coëfficientibus in vnam seriem collectis, additoque $-\frac{a^m}{m\,b^m}$ ob log. $\left(1 - \frac{a}{bx}\right)$ (a), emergit coëfficiens $\tau\tilde{\varsigma}\ \frac{1}{x^m}$ ex parte aequationis (1) sinistra, quem aequando coëfficienti ex altera aequationis parte (2) $= -\frac{p^m}{m}$, hanc nanciscimur expressionem:

$$\frac{p^m}{m} = \frac{x^m}{m} + \frac{\xi x^{m+1}}{1} + \frac{d\left(\xi^2 x^{m+1}\right)}{1.2\,dx} + \frac{d^2\left(\xi^3 x^{m+2}\right)}{1.2.3\,r\,dx^2} + \ldots + \frac{d^{r-1}\left(\xi^r x^{m+r-1}\right)}{1.2\ldots r\,dx^{r-1}} + \ldots,$$

in qua ex parte dextra pro x post differentiationes poni debet $\frac{a}{b}$. Sicque exhibita est pro aequatione $o = \frac{a}{b} - x + x\xi$ non tantum radix p, verum etiam quaeuis eius potestas (saltem cum exponente affirmatiuo integro) (g). Hinc, statuendo $x\xi = \varphi x$, $\frac{a}{b} = \alpha$, pro aequatione $\alpha = x - \varphi x$, fit $p^m =$

$$x^m + m x^{m-1}\varphi x + \frac{m\,d\left(x^{m-1}(\varphi x)^2\right)}{1.2\,dx} + \frac{m\,d^2\left(x^{m-1}(\varphi x)^3\right)}{1.2.3\,dx^2} + \ldots,$$

posito post differentiationem α pro x (h). Ex qua formula ad theorema generale progressu facto, quaeuis functio $\tau\tilde{\varsigma}$ p exprimitur: quod quidem breuiter attigisse sufficiat.

5) Iis, quae nr. 3. exposita sunt, vniuersaliter demonstratur id, quod in LA GRANGII demonstratione ex inductione magis inferri videtur: quippe (l. c. *vers. German.* pag. 208. 209.) exhibitis seriebus pro p, p^2 et p^3, ex his casibus particularibus statim ad p^m, pro quouis exponente m transitus fit. Cui desiderio satisfacturus, cum legem generalem quaererem, priusquam euolutio (3) sese mihi obtulisset, incidi primo in sequentem modum, quem breuiter commemorasse haud a re alienum esse videtur.

Con-

g) *Toepfer* l. c. p. 174.

h) Hac expositione dubio occurritur, quod in recensione scripti *Murhardiani* (not. i), *Ephemeridibus litterariis Goettingensibus* inserta, a *Kaestnero* motum esse memini. Dum nimirum differentiatio suscipitur, quantitas x ceu *variabilis* spectatur, ac demum *post* differentiationem illi tribuitur valor *constans* α.

Considerari nimirum potest ξ et quaevis potestas $\tau\tilde{\upsilon}\ \xi$ tanquam functio $\tau\tilde{\upsilon}\ x = \frac{a}{b} + x\left(1 - \frac{a}{bx}\right)$, qua functione secundum theorema *Taylorianum* euoluta, fit $\xi^r =$

$$\xi^r + x\left(1 - \frac{a}{bx}\right)\frac{d\xi^r}{dx} + x^2\left(1 - \frac{a}{bx}\right)^2 \frac{d^2\xi^r}{1.2\,dx^2} + x^3\left(1 - \frac{a}{bx}\right)^3 \frac{d^3\xi^r}{1.2.3.dx^3}$$

$+ \ldots$ posito ex parte dextra huius aequationis in functione ξ^r eiusque differentialibus $\frac{a}{b}$ loco x. Hinc, neglectis terminis affirmatiuas potestates $\tau\tilde{\upsilon}$ x inuoluentibus, poni poterit

$$\frac{\xi^r}{\left(1 - \frac{a}{bx}\right)^r} = \frac{\xi^r}{\left(1 - \frac{a}{bx}\right)^r} + \frac{x}{\left(1 - \frac{a}{bx}\right)^{r-1}} \cdot \frac{d\xi^r}{dx} + \frac{x^2}{\left(1 - \frac{a}{bx}\right)^{r-2}} \cdot \frac{d^2\xi^r}{1.2\,dx^2}$$

$$+ \ldots + \frac{x^{r-1}}{\left(1 - \frac{a}{bx}\right)^1} \cdot \frac{d^{r-1}\xi^r}{1.2\ldots(r-1)\,dx^{r-1}}.$$

Quae series cum numero terminorum finito constet, facile colligere licet coëfficientem $\tau\tilde{\upsilon}\ \frac{1}{x^m}$, qui quippe componitur ex coëfficientibus respondentibus, quos singula seriei membra seorsim praebent. Est autem coëfficiens $\tau\tilde{\upsilon}\ \frac{1}{x^p}$ ex serie binomiali pro

$$\left(1 - \frac{a}{bx}\right)^{-q} \text{ siue } \tau\tilde{\upsilon}\ \frac{1}{x^{p-\pi}} \text{ ex } \frac{x^\pi}{\left(1 - \frac{a}{bx}\right)^q}, = \frac{q(q+1)\ldots(q+p-1)}{1.2\ldots p}\,\frac{a^p}{b^p},$$

siue (statuendo $p = \pi + m$, $q = r - \pi$) $= \frac{(r-\pi)(r-\pi+1)\ldots(r+m-1)}{1.2\ldots(\pi+m)}\,\frac{a^{\pi+m}}{b^{\pi+m}}$

$$= (r-\pi)(r-\pi+1)\ldots r.\frac{(r+1)(r+2)\ldots(r+m-1)}{1.2\ldots\pi+m}\,\frac{a^{\pi+m}}{b^{\pi+m}} =$$

$$r(r-1)\ldots(r-\pi).\frac{(\pi+m+1)(\pi+m+2)\ldots(r+m-1)}{1.2\ldots r}\,\frac{a^{\pi+m}}{b^{\pi+m}}$$

$\frac{r^{-1}}{1.2...r} r(r-1)...(r-\pi) \frac{d^{r-\pi-1}(x^{r+m-\pi})}{dx^{r-\pi-1}}$, posito post differentiationem $\frac{a}{b}$ pro x. Quare sumendo successiue $\pi = 0, 1, 2, \ldots r-1$, prodit coëfficiens τῦ $\frac{1}{x^m}$ ex euolutione τῦ $\frac{\xi^r}{r\left(1-\frac{a}{bx}\right)^r}$ in seriem,

$$= \frac{1}{1.2...r}\left\{\xi^r \frac{d^{r-1}(x^{r+m-1})}{dx^{r-1}} + (r-1)\frac{d\xi^r}{dx}\frac{d^{r-2}(x^{r+m-1})}{dx^{r-2}} + \frac{(r-1)(r-2)}{1.2}\frac{d^2\xi^r}{dx^2}\frac{d^{r-3}(x^{r+m-1})}{dx^{r-3}} + \ldots\right\}$$

$= \frac{1}{1.2...r}\frac{d^{r-1}(\xi^r x^{r+m-1})}{dx^{r-1}}$. Qui valor cum supra (nr. 3.) inuento conspirat. Reliqua, vti prius, peraguntur (i).

Scho-

i) Posteaquam haec iam elaborassem, peruenit ad manus programma Clariss. *Murhardi*, sic inscriptum: "*Ueber die Methode des Herrn de la Grange, alle Gleichungen durch Näherung vermittelst der Reihen aufzulösen, von F. W. A. Murhard. Nebst Anzeige seiner Vorlesungen im Winterhalbenjahre auf der Georg-Augusts-Universität zu Göttingen. Göttingen, gedruckt bey Johann Georg Rosenbusch, 1796.*" Cum vero auctor *La Grangii* mentem haud ex omni parte assecutus esse, certe non distincte satis expressisse videatur, parumque ad illustrandam ac rigorosius firmandam solutionem La Grangianam contulerit, meam expositionem haud prorsus superfluam, eamque lectoribus in Analysi nondum satis exercitatis haud ingratam fore arbitratus sum. Commentationem P. *Canovai*, insertam Actis Academiae *Sienensis* (*Atti dell' Academia delle Scienze di Siena detta de' Fisico critici, Tomo VII. 1794. n. III.*), in qua, referente Murhardo l. c. p. 15, theorema *La Grangianum* ex propositionibus haud ignotis analyticis deducitur, problematis etiam historia enarratur, addita tamen querela de operationum prolixitate, hanc, inquam, commentationem equidem nondum vidi: quanquam Bibliotheca publica Vniuersitatis *Iuliae Carolinae*, (rara, vel in Vniuersitatibus confluxu studiosorum celebrioribus, felicitate), instructa sit ac ornata collectione fere completa scriptorum ab Academiis et Societatibus scientiarum editorum, quae quam vtilia ac pene necessaria sint ad studium disciplinarum physicarum praesertim et mathematicarum amplius atque profundius, apud harum rerum peritos satis constat (cf. quae alia occasione de valore Academiarum rite aestimando monuit Celeberr. *E. A. W. Zimmermann*, cui quippe dicta Vniuersitas istam collectionem maxime debet, in *Annalibus Geographiae et Statisticae*. Anni 1790. Fasc. I. pag. 5 sq.). — Quo enumeratio scriptorum theorema *La Grangianum* concernentium sit completa, haec duo insuper commemorasse iuuat: 1) Observations analytiques par Mr. *Lambert Nouv. Memoires de l'Acad. Royale des Sciences et Belles-Lettres. Année MDCCLXX. à Berlin MDCCLXXII.*

Scholion.

LA PLACII Analysis Probl. I.

§. IV. Aliam problematis (§. I.) solutionem, seu theorematis (§. I. 9.) demonstrationem, eamque concinnam ac ingeniosam exhibuit Illustris LA PLACE (k). Qua deinceps vsus etiam est Celeberr. COUSIN (l). Noua, (idque proprium est huic demonstrationi) introducitur quantitas variabilis z (m), ita vt aequatio *La Grangiana* $y = x - \phi x$ hanc formam generaliorem recipiat: $y = x - z \cdot \phi x$; tum considerando x tanquam functionem τῶν y et z, et differentiando ψx, quamuis functionem τῦ x hincque etiam τῶν y et z, secundum z et y, probatur esse: $\left(\frac{d\psi x}{dz}\right) = \phi x \cdot \left(\frac{d\psi x}{dy}\right)$. Iam porro (et in hoc maxime cernitur vis atque neruus huius demonstrationis) ex hac aequatione inter differentialia prima τῦ ψx secundum z et y, deducitur inter differentialia quaeuis altiora eiusdem functionis, eodem sensu accepta, relatio haec:

$$\left\{\frac{d^m \psi x}{dz^m}\right\} = \left\{\frac{d^{m-1}\left((\phi x)^m \left(\frac{d\psi x}{dy}\right)\right)}{dy^{m-1}}\right\}.$$

Quae aequatio ab Analystis laudatis pro m = 2 et = 3 inuestigata, nec tamen pro quouis m vniuersaliter demonstrata est. Ac is quidem modus, quo COUSINIUS differentiale alterum et tertium τῦ ψx secundum z euoluit, ita comparatus est, vt inuestigatio formae differentialium altiorum haud sine prolixis calculi ambagibus perfici posse videatur (o). Quod

MDCCLXXIII. p. 231 sq. 2) Memoire sur differentes questions d'Analyse. Par Mr. *Le Marquis de Condorcet:* Article II. Demonstration d'un Theorème de Mr. de la Grange . . . p. 7 et 8. *Miscellan. Taurinens. Tom. V.*, aut, vti altera inscriptio se habet, *Melanges de Philosophie et de Mathematique. De la Societé Royale de Turin. Pour les Années 1770-1773. A Turin, de l'Imprimerie Royale. Avec Permission.* Quarum demonstrationum neutra satis est rigorosa ac absoluta: *Condorcetiana* tamen perfectior esse videtur, inter quam etiam ac nostram (§. I.) aliqua intercedit similitudo, differentia maxime prouenientे ex assumtione quantitatis variabilis z (cf. §. IV. not. m) — De demonstratione *Petri Paoli*, ex meo quidem iudicio summi nunc Italorum Analystae, infra sermo erit.

k) *Theorie du mouvement et de la figure elliptique des Planetes;* par Mr. *de la Place etc. à Paris, MDCCLXXXIV.* pp. 15-18. cf. *Memoires de l'Academie des Sciences, année 1777, 4. pag. 99 sq.*

l) *Introduction à l'étude de l'astronomie physique, par Mr. Cousin etc. à Paris MDCCLXXXVII. 4. p. 15 sq.*

m) Nouae variabilis z introductio, quae *La Placio* debetur, perquam vtilis est: hac forma aequationis generaliore nostra etiam solutio (§. I.) nititur, ex eademque infra alia insuper problematis analysis satis concinna deducetur.

o) Tum temporis, cum in solutionem §. I. expositam inciderem, demonstrationem hoc §pho commemoratam ex *Cousinii* tantum libro noueram, huncque igitur Analystam auctorem demonstrationis praedicaui: in qua opinione etiam versatus est Celeberr. *Hindenburgius* (l. c.). *Cousinius* a *la Placii* expositione

Quod desiderium cum equidem sentirem, idque expleri cuperem, perueni ad Lemma §phi sequentis, cuius addere demonstrationem haud superfluum videbatur, cum illud ab Analysis, qui theorema speciale satis notum: $\left(\frac{d\left(\frac{dW}{dx}\right)}{dy}\right) = \left(\frac{d\left(\frac{dW}{dy}\right)}{dx}\right)$ demonstrant, praetermitti soleat (o). Ex hoc deinceps Lemmate petere licebit rigorosam problematis (I) solutionem.

Theorema.

§. V. Sumendo functionis τῶν z et y (= W) differentiale rtum *secundum z*, deinde differentiale ϱtum secundum y, idem obtinetur, ac si eadem functio primo ϱies secundum y, deinde ries secundum z differentietur: siue est $\overset{y}{d^{\varrho}}\overset{z}{d^{r}} W = \overset{z}{d^{r}}\overset{y}{d^{\varrho}} W$.

Demonstratio.

1) Ponamus primo $\varrho = 1$, tum demonstrandum est, esse $\overset{yz}{d\,d^{r}} W = \overset{zy}{d^{r}\,d} W$. Pro $r = 1$ haec aequatio ex theoremate satis noto consequitur. Iam assumta aequatione pro r, eandem ad $r + 1$ extendi, sequenti ratiocinio colligi potest. Habetur nimirum, ponendo $r + 1$ pro r, $\overset{yz}{d\,d^{r+1}} W = \overset{yzz}{d\,d\,d^{r}} W$

$= \overset{zyz}{d\,d\,d^{r}} W$, (quoniam $\overset{z}{d^{r}} W$ est functio τῶν z et y, quam secundum y et z vel secundum z et y differentiare perinde est,)

=

stione in eo discedit, quod ille, cui alterius breuitas obscura forte videbatur, calculos pro $\frac{d^2 \psi x}{dz^2}$ et $\frac{d^3 \psi x}{dz^3}$ aliter atque fusius euoluerit: nec tamen dubito, principiorum, quibus *La Placiana* demonstratio nititur, quamquam ab auctore haud diserte expressorum, distinctiori et generaliori euolutione, prolixiores calculos euitari, ac breuitatem operationum cum rigore demonstrationis conciliari potuisse. Equidem ad Lemma (§. IV.), et demonstrationem inde deductam peruenî, ante compertam *La Placii* expositionem, quantum nunc quidem memini.

o) Cf. *Euleri* Inst. Calc. Diff. P. I. Cap. VII. *Kaestner Analys. des Unendl.* §. 481 sqq. Lemmatis, de quo hic loquor, demonstrationem a nostra diuersam ex theoremate *Tayloriano* deduxit Cel. *L'Huilier* (*Principior. Calc. Diff. et Integr. Expos. elem.* §. 161. pag. 239.), quae tum quidem mihi nondum erat nota, cum in schediasmate supra (§. II.) allegato Lemmatis mentionem inicerem (*Archiv der Mathem. III. Heft, S. 349.*). Characterem pro designandis differentiis *partialibus* ab eodem auctore commendatum, ipse etiam tanquam commodiorem hac data occasione adhibueram: eundemque igitur, leui facta immutatione, in demonstratione Lemmatis vsitato praeferendum, hac auctoritate confirmatus, censui. — Monendum insuper est, in differentialibus altioribus eius quantitatis, quae sola ceu variabilis tractatur, differentiale primum pro constanti habendum, seu post quamuis differentiationem simplicem differentiale functionis per differentiale variabilis diuidendum esse.

$= \overset{z}{d}\,\overset{z}{d}{}^{r}\,\overset{y}{d}W$, (quod ex hypothesi aequatio pro r obtinet,)

$= \overset{z}{d}{}^{r+1}\,\overset{y}{d}W$, i. e. aequatio pro r assumta etiam pro r + 1 vera est.

2) Demonstrato itaque theoremate pro ρ = 1 et quouis r, vlterius a ρ ad ρ + 1 simili omnino ratione concludere licet. Est nimirum $\overset{z}{d}{}^{r}\,\overset{y}{d}{}^{\rho+1}W = \overset{z}{d}{}^{r}\,\overset{y}{d}\,\overset{y}{d}{}^{\rho}W$

$= \overset{y}{d}\,\overset{z}{d}{}^{r}\,\overset{y}{d}{}^{\rho}W$, (quoniam $\overset{y}{d}{}^{\rho}W$ est functio τῶν y et z, ad quam theorema modo demonstratum (1) applicari potest,)

$= \overset{y}{d}\,\overset{y}{d}{}^{\rho}\,\overset{z}{d}{}^{r}W$, quia ex hypothesi aequatio theorematis pro r et ρ locum habet,)

$= \overset{y}{d}{}^{\rho+1}\,\overset{z}{d}{}^{r}W$; hincque manifestum est, eandem aequationem etiam ad ρ + 1 extendi, i. e. pro quouis valore τῶν ρ et r (1) valere.

3) Nondum supposito theoremate speciali, quod sit $\overset{z}{d}\,\overset{y}{d}W = \overset{y}{d}\,\overset{z}{d}W$, statim theorema generale sequentem in modum demonstrare licet.

Cum W sit functio τῶν z et y, ea tanquam functio τοῦ z spectata exprimi poterit per seriem secundum potestates τοῦ z progredientem, cuius vero seriei coëfficientes erunt functiones τοῦ y. Sit igitur

$$W = Y^{I}z^{\alpha} + Y^{II}z^{\beta} + Y^{III}z^{\gamma} + Y^{IV}z^{\delta} + \ldots$$

erit $\overset{z}{d}{}^{r}W = Y^{I}\alpha(\alpha-1)\ldots(\alpha-r+1)z^{\alpha-r} + Y^{II}\beta(\beta-1)\ldots(\beta-r+1)z^{\beta-r} + \ldots$, dum z tantum pro variabili habetur; porro differentiando secundum y, vt z instar constantis sit, habetur

$$\overset{y}{d}{}^{\rho}\,\overset{z}{d}{}^{r}W = d^{\rho}Y^{I}.\alpha(\alpha-1)\ldots(\alpha-r+1)z^{\alpha-r} + d^{\rho}Y^{II}.\beta(\beta-1)\ldots(\beta-r+1)z^{\beta-r} + \ldots$$

Easdem differentiationes inuerso ordine suscipiendo obtinetur:

$$\overset{y}{d}{}^{\rho}W = d^{\rho}Y^{I}.z^{\alpha} + d^{\rho}Y^{II}.z^{\beta} + d^{\rho}Y^{III}.z^{\gamma} + \ldots$$

et $$\overset{z}{d}{}^{r}\,\overset{y}{d}{}^{\rho}W = d^{\rho}Y^{I}.\alpha(\alpha-1)\ldots(\alpha-r+1)z^{\alpha-r} + d^{\rho}Y^{II}.\beta(\beta-1)\ldots(\beta-r+1)z^{\beta-r} + \ldots$$

i. e. idem prodit, quod modo prodibat ex ordine directo.

Coroll.

Corollarium.

§. VI. 1) Theorema modo demonstratum non tantum de *differentialibus*, verum etiam de *differentiis finitis* valet, id quod ex demonstratione priori (§. V. 1. 2.) facile apparet, quippe theorema particulare notum, cui illa superstructa est, ad differentias finitas patet (p), ad quas etiam alteram demonstrationem (§. IV. 3.) sine negotio extendere licet.

2) Quodsi functio τῶν z et y, r^{ies} secundum z, ϱ^{ies} secundum y, deinde rursus $r^{I ies}$ secundum z, $\varrho^{I ies}$ secundum y, porro $r^{II ies}$ secundum z, $\varrho^{II ies}$ secundum y, differentietur; tum, hisce differentiationibus pro lubitu continuatis, idem semper prodibit, quocunque ordine eae suscipiantur. Nec non quemuis exponentem $r, \varrho, r^I, \varrho^I, \dots$, repetitam differentiationem secundum *vnam* variabilem indicantem, in plures discerpere, ac loco vnius differentialis altioris plura minus composita substituere, haecque quolibet ordine collocare licet.

3) Similia omnino de functionibus *plurium variabilium* valent: ita quidem, vt si quotuis differentiationes secundum z, y, x, u, ... instituantur, eaeque vel *simplices*, vel *repetitae*, idem semper resultaturum sit, quocunque demum ordine differentiationes *simplices* sibi inuicem succedant. Quae tamen cum ad nostrum scopum minus pertineant, commemorasse hoc loco sufficiat: quanquam demonstrationes de differentiationibus secundum plures variabiles vulgo haud satis generaliter proponi soleant (q).

Alia problematis (I) solutio.

§. VII. Ex aequatione proposita: $y = x - z\varphi x$ quaeritur expressio cuiusuis functionis τῆ x, $= \psi x$, per seriem secundum potestates τῆ z progredientem. Ponatur igitur, vti iam §. I. factum est, $\psi x = \psi y + Y^I z^1 + Y^{II} z^2 + Y^{III} z^3 + \dots + Y^N z^n + \dots$, tum erit, sumto differentiali n^{to} huius seriei, ita quidem, vt z tantum ceu variabilis tractetur, $\overset{z}{d^n}\psi x = n(n-1)\dots 1 . Y^N + (n+1)\dots 2 . Y^{N+1} z + \dots$

Hinc pro $z = 0$, habetur

$$\overset{z}{d^n}\psi x = n(n-1)\dots 1 . Y^N, \text{ siue } Y^N = \frac{1}{1.2.3\dots n} . \overset{z}{d^n}\psi x.$$

Quare ad definiendos coefficientes assumtos $Y^I, Y^{II}, \dots Y^N, \dots$ opus tantum est nosse valores, quos differentialia, primum, alterum, tertium, ... n^{tum} ... functionis ψx recipiunt, supposita in differentiationibus quantitate z tantum variabili, y constanti, ac post differentiationes $z = 0$.

1) Dif-

p) Cf. *L. Euler* l. c.; *W. I. G. Karsten Mathesis theoretica elementaris atque sublimior.* (*Rostochii et Gryphiswaldiae, apud Ant. Ferd. Röseum 1760. 8.*) *Sect. XXVI. p. 780 sq.*

q) Cf. tamen *Karstenii* libri modo laudati Sect. XXVI.

2) Differentiando aequationem propositam $y = x - z\varphi x$, et statuendo $d\varphi x = dx.\varphi^{I}x$, obtinetur $dy = dx - zdx.\varphi^{I}x$, siue $dx = \frac{dy + dz.\varphi x}{1 - z\varphi^{I}x}$; hinc considerando x tanquam functionem τῶν z et y, et accipiendo eius differentiale primo secundum z, tum idem secundum y, erit $\overset{z}{d}x = \frac{\varphi x}{1 - z\varphi^{I}x}$, $\overset{y}{d}x = \frac{1}{1 - z\varphi^{I}x}$, vnde fit $\overset{z}{d}x = \varphi x.\overset{y}{d}x$, et multiplicando vtrinque per $\psi^{I}x$, existente $d\psi x = dx.\psi^{I}x$, prodit $\psi^{I}x.\overset{z}{d}x = \varphi x.\psi^{I}x.\overset{y}{d}x$, siue $\overset{z}{d}\psi x = \varphi x.\overset{y}{d}\psi x$. (a)

3) Aequationem (a) vlterius differentiando, quantitate z tantum pro variabili habita, prodit $\overset{z}{d}{}^{2}\psi x = \overset{z}{d}(\varphi x.\overset{y}{d}\psi x)$. Iam vero formula differentialis $\varphi x.d\psi x = dx.\psi^{I}x.\varphi x$ semper refert differentiale certae cuiuspiam functionis τοῦ x, quae igitur sit $= W$, siue $\varphi x.d\psi x = dW$, et $\varphi x.\overset{y}{d}\psi x = \overset{y}{d}W$. Cum porro x hoc loco consideretur tanquam functio τῶν y et z, etiam W erit eiusmodi functio, vnde ex theoremate noto fit $\overset{zy}{dd}W = \overset{yz}{dd}W$. Hinc prodit

$$\overset{z}{d}{}^{2}\psi x = \overset{y}{d}(\varphi x.\overset{z}{d}\psi x) = \overset{y}{d}((\varphi x)^{2}\overset{y}{d}\psi x),$$

supposito nimirum loco $\overset{z}{d}\psi x$, valore $\varphi x.\overset{y}{d}\psi x$ (2)

4) Aequationem modo inuentam rursus secundum z differentiando, hasque differentiationes vlterius continuando, simili ratione prodeunt valores τῶν $\overset{z}{d}{}^{3}\psi x$, $\overset{z}{d}{}^{4}\psi x$, ... Quorum lex generalis breuiter et rigorose sequentem in modum demonstratur. Supponatur nimirum, esse ex iam inuentis pro certo exponente m, $\overset{z}{d}{}^{m}\psi x = \overset{y}{d}{}^{m-1}(\varphi x^{m}\overset{y}{d}\psi x)$; tum erit iterum differentiando secundum z, $\overset{z}{d}{}^{m+1}\psi x = \overset{zy}{dd}{}^{m-1}(\varphi x^{m}\overset{y}{d}\psi x)$. At vero $(\Phi x)^{m}d\psi x = (\varphi x)^{m}\psi^{I}x.dx$ considerari potest tanquam differentiale certae functionis τοῦ x, quae porro erit, vti ipsa variabilis x, functio τῶν z et y; quaque igitur posita $= W$, fit $\overset{z}{d}{}^{m+1}\psi x = \overset{zy}{dd}{}^{m-1}\overset{y}{d}W = \overset{zy}{dd}{}^{m}W$.

 Nunc

Nunc ex Lemmate (§. IV. 1.) habetur pro quauis functione τῶν z et y, $\overset{z}{d}\overset{y}{d}{}^m W = \overset{y}{d}{}^m \overset{z}{d} W$; quare erit $\overset{z}{d}{}^{m+1}\psi x = \overset{y}{d}{}^m \overset{z}{d} W = \overset{y}{d}{}^m(\varphi x^m \overset{z}{d}\psi x) = \overset{y}{d}{}^m(\varphi x^{m+1} \overset{y}{d}\psi x)$, ob $\overset{z}{d}\psi x = \varphi x . \overset{y}{d}\psi x$ (2). Exinde manifestum est, formulam assumtam pro $\overset{z}{d}{}^m\psi x$ etiam, posito $m+1$ pro m, ad $\overset{z}{d}{}^{m+1}\psi x$ extendi, i. e. eandem, vti pro $m = 1$ et $= 2$ (2. 3.), sic pro quouis exponente m locum habere.

5) Ex formula modo demonstrata sponte iam deducere licet expressionem generalem coëfficientium seriei assumtae pro ψx. Est nimirum (1) $Y^N = \frac{1}{1.2..n}\overset{z}{d}{}^n\psi x = \frac{1}{1.2..n}\overset{y}{d}{}^{n-1}(\varphi x^n \overset{y}{d}\psi x)$, posito post differentiationem $z = 0$. Iam vero in differentialibus, quae inuoluit nouus valor τοῦ Y^N, quantitas z pro constante habetur, y tantum pro variabili; hinc, cnm x sit functio τοῦ y, quae ex aequatione proposita pro $z = 0$ abit in y, $(\varphi x)^n$ et $\overset{y}{d}\psi x$, ergo etiam $\overset{y}{d}{}^{n-1}(\varphi x^n \overset{y}{d}\psi x)$ different a φy^n, $\overset{y}{d}\psi y$, et $\overset{y}{d}{}^{n-1}(\varphi y^n \overset{y}{d}\psi y)$, in eo tantum, quod illic quantitas constans (z) occurrat, quae hic abit in o. Exinde, cum expressione τοῦ Y^N poni debeat $z = 0$, manifestum est, fore

$$Y^N = \frac{1}{1.2..n}\,\frac{d^{n-1}\left(\varphi y^n \frac{d\psi y}{dy}\right)}{dy^{n-1}}.$$

Idem sequenti etiam ratione apparet. Cum pro $z = 0$, sit $x = y$, quauis functione τοῦ x exhibita per seriem secundum potestates τοῦ z progredientem, primum talis seriei membrum erit functio similis τοῦ y, reliqua membra afficientur factore z; hinc erit

$$(\varphi x)^n = (\varphi y)^n + zM$$
$$\psi x = \psi y + zN,$$

vnde fit $\overset{y}{d}\psi x = \frac{d\psi y}{dy} + z\overset{y}{d}N$, et $\varphi x^n . \overset{y}{d}\psi x = (\varphi y)^n \frac{d\psi y}{dy} + zR$,

$$\overset{y}{d}{}^{n-1}(\varphi x^n \overset{y}{d}\psi x) = \frac{d^{n-1}\left(\varphi y^n \frac{d\psi y}{dy}\right)}{dy^{n-1}} + z\overset{y}{d}{}^{n-1}R,$$

quod pro $z = 0$ sponte

abit

abit in $\frac{d^{n-1}\left(\varphi x^n \frac{d\psi y}{dy}\right)}{dy^{n-1}}$. Sic itaque pro ψx eadem feries, ac §. 1. inuenta et rite demonſtrata eſt.

Continuatio.

§. VIII. Solutio praecedenti §pho illuſtrata exinde petita fuit, quod ex aequatione $\overset{z}{d}\psi x = \varphi x \, . \, \overset{y}{d}\psi x$ (VI. 2.) per iteratas differentiationes, altiora differentialia τȣ̃ ψx, ſecundum z, deducta fuerint. Quare haud ſuperfluum mihi videtur, oſtendere, quomodo ex ſimplici iſta aequatione, absque auxilio differentialium altiorum, idem problema reſolui, ſiue feries pro ψx erui queat.

1) Seriei pro ψx coëfficientes aſſumti Y^{I}, Y^{II}, Y^{III}, ... Y^{N}, ... deſignentur per $\psi x k 2$, $\psi x k 3$, $\psi x k 4$, ... $\psi x k (n+1)$; vt ſit $\psi x = \psi x k 1 + \psi x k 2 . z + \psi x k 3 . z^2 + \ldots + \psi x k (n+1) . z^n + \ldots$ Simili modo ponatur

$\varphi x = \varphi x k 1 + \varphi x k 2 . z + \varphi x k 3 . z^2 + \ldots + \varphi x k (n+1) z^n + \ldots$; tum erit

$$\overset{y}{d}\psi x = \overset{y}{d}\psi x k 1 + \overset{y}{d}\psi x k 2 . z + \overset{y}{d}\psi x k 3 . z^2 + \ldots + \overset{y}{d}\psi x k (n+1) . z^n + \ldots;$$

$$\overset{z}{d}\psi x = \psi x k 2 + 2 \psi x k 3 . z + 3 \psi x k 4 . z^2 + \ldots + n \psi x k (n+1) . z^{n-1} + \ldots$$

2) Quibus ſeriebus ſuppoſitis in aequatione $\overset{z}{d}\psi x = \varphi x . \overset{y}{d}\psi x$, ac aequando inuicem coëfficientes τȣ̃ z^{n-1}, ſponte prodit haec aequatio: $n \psi x k (n+1) =$

$$\varphi x k 1 . \overset{y}{d}\psi x k n + \varphi x k 2 . \overset{y}{d}\psi x k (n-1) + \varphi x k 3 . \overset{y}{d}\psi x k (n-2) + \ldots + \varphi x k n . \overset{y}{d}\psi x k 1.$$

Iam ſupponamus, cognitos eſſe feriei pro ψx coëfficientes n priores, ſeu $\psi x k n$, $\psi x k (n-1)$, ... $\psi x k 1$, inde ſponte etiam innoteſcent totidem coëfficientes feriei pro φx, permutando tantum ſignum functionale ψ cum φ; hincque manifeſtum eſt, ex aequatione pro $\psi x k (n+1)$, per datos n coëfficientes determinari $n+1$tum. Quare cum coëfficiens primus $\psi x k 1$ ſit $= \psi y$, ceteros ſucceſſiue definire licet.

3) Quod ſi nunc formula $\psi x k r = \frac{d^{r-2}\left(\varphi y^{r-1} \frac{d\psi y}{dy}\right)}{1 . 2 \ldots r-1 . dy^{r-2}}$ valeat vsque ad

$r = n$, erit etiam eousque $\phi x k r = \dfrac{d^{r-2}\left(\phi y^{r-1}\dfrac{d\phi y}{dy}\right)}{1.2\ldots r-1.dy^{r-2}}$, hinc fit

$$n\psi x k(n+1) = \phi y \cdot \frac{d^{n-1}\left(\phi y^{n-1}\dfrac{d\psi y}{dy}\right)}{1.2\ldots n-1.dy^{n-2}} + \frac{\phi y\, d\phi y}{dy}\cdot\frac{d^{n-2}\left(\phi y^{n-2}\dfrac{d\psi y}{dy}\right)}{1.2\ldots n-2.dy^{n-3}}$$

$$+\frac{d\left(\phi y^2\dfrac{d\phi y}{dy}\right)}{1.2\,dy}\cdot\frac{d^{n-3}\left(\phi y^{n-3}\dfrac{d\psi y}{dy}\right)}{1.2\ldots n-3\,dy^{n-4}}+\ldots;$$ siue $\psi x k(n+1) =$

$$\frac{1}{1.2\ldots n.dy^{n}}\left\{\begin{array}{l}\phi y.d^{n-1}\left(\phi y^{n-1}\dfrac{d\psi y}{dy}\right)+(n-1)\phi y\, d\phi y.d^{n-2}\left(\phi y^{n-2}\dfrac{d\psi y}{dy}\right)\\ +\dfrac{(n-1)(n-2)}{1.2}d(\phi y^2\, d\phi y)d^{n-3}\left(\phi y^{n-3}\dfrac{d\psi y}{dy}\right)+\ldots\end{array}\right\}$$

4) Termini, quibus haec expreſſio conſtat, in ſummam rediguntur ope ſummationis longe generalioris, ad quam hac data occaſione perueni, quamque ſequens complectitur

THEOREMA.

Summa terminorum differentialium

$$p\,d^{n}q + n u d p.\,d^{n-1}\left(\frac{q}{u}\right) + \frac{n(n-1)}{1.2}d(u^2 dp).\,d^{n-2}\left(\frac{q}{u^2}\right) +$$
$$\frac{n(n-1)(n-2)}{1.2.3}d^2(u^3 dp).\,d^{n-3}\left(\frac{q}{u^3}\right) + \ldots + n d^{n-2}(u^{n-1}dp).\,d\left(\frac{q}{u^{n-1}}\right)$$

$+ d^{n-1}(u^{n}dp).\dfrac{q}{u^{n}}$ eſt $= d^{n}(pq)$; vbi p, q, u denotant quasuis quantitates variabiles, nec in differentiationibus vllo differentialis conſtantis reſpectu opus eſt (r).

5) Po-

(r) Pro $u = 1$, vel $=$ cuiuis *conſtanti* a, ex hac ſummatione fluit formula ſatis nota pro $d^{n}(pq)$. Obſeruatione dignum videtur, ſummam ſeriei eandem manere, quicunque valor quantitati *variabili* u tribuatur. Demonſtrationem vniuerſalem huius Theorematis exhibui, cum variis inde deductis corollariis, (*Hindenburgs Archiv der Mathematik III. Heft, S. 357. ſq.* cf. *V. Heft, S. 67. ſq.*)

5) Posito in hac summatione $p = \varphi y$, $q = \varphi y^{n-1}\frac{d\psi y}{dy}$, $u = \varphi y$, prodit

$$(3)\ \psi x k(n+1) = \frac{1}{1.2\ldots n}\,\frac{d^{n-1}\left(\varphi y^{n}\frac{d\psi y}{dy}\right)}{dy^{n-1}},$$ i. e. formula (3) pro coëfficientibus n prioribus assumta etiam ad $n+1^{tum}$ extenditur, hincque eadem generalis est.

Scholion.

A. I. Lexellii Demonstratio.

§. VIIII. Demonstrationem ab hactenus commemoratis diuersam, theorematis a La Grangio inuenti (§. I. 9.), exhibuit Lexell (s). Quae demonstratio ita procedit, vt seriei $\psi y + \varphi y\frac{d\psi y}{dy} + \frac{d\left(\varphi y^{2}\frac{d\psi y}{dy}\right)}{1.2\,dy} + \frac{d^{2}\left(\varphi y^{3}\frac{d\psi y}{dy}\right)}{1.2.3\,dy^{2}} + \ldots$ (posito sc. §. I. 9. $z = 1$) singuli termini per x exprimantur, ope theorematis *Tayloriani* ad aequationem datam $y = x - \varphi x$ applicati; tum ostendatur, in serie sic transformata omnia membra se mutuo destruere, praeter primum ψx. Ad quod ostendendum ab auctore praemittitur Lemma, ex quo sequitur, esse pro quocunque numero integro m,

$$u^{m} d^{m-1} w - m u^{m-1} d^{m-1}(uw) + \frac{m(m-1)}{1.2} u^{m-2} d^{m-1}(u^{2} w) - \ldots \pm d^{m-1}(u^{m} w) = 0.$$

Liceat mihi obseruationes nonnullas addere, quae super hac demonstratione sese mihi obtulerunt.

Primo Lemmate memorato perductus sum ad theorema generalius (§. IX.), quod, quippe nondum obseruatum, hoc loco demonstrare haud superfluum erit. *Deinde* Lexellii demonstratio theorematis principalis (§. I.), etiamsi rigorosa, hoc tamen defectu laborat, quod *ea* prorsus *synthetica* sit, ac theorema demonstrandum iam *cognitum* esse supponat, nec inde perspiciatur, quo pacto series theorematis *inueniri*, seu problema (§. I.) *analytice* solui queat. Quem defectum Lexellius ipse haud diffessus est (t): ea autem, quae medelae instar vlterius addidit, ratiocinio minus perspicuo innituntur, nec desiderio plane satisfacere videntur. Quare processum demonstrationis paullo aliter dirigendo, introductaque quantitate variabili z, huic fini aptissima, ostendere e re esse duxi, quo pacto ope Lemmatis praedicti analytica problematis (§. I.) solutio, eaque satis concinna, exhiberi queat.

Theo-

(s) Nou. Comment. Acad. Petr. Tom. XVI. pp. 230-254.

(t) l. c. §. VII. p. 238.

THEOREMA.

§. X. Pro quibusuis variabilibus W, w, u, et numeris integris n, r, ϱ, dummodo sit $m > r + \varrho$, euanescet summa

$$S = d^r W . d^\varrho w - m d^r\left(\frac{W}{u}\right) . d^\varrho (wu) + \frac{m(m-1)}{1.2} d^r\left(\frac{W}{u^2}\right) . d^\varrho (wu^2)$$

$$- \frac{m(m-1)(m-2)}{1.2.3} d^r\left(\frac{W}{u^3}\right) . d^\varrho (wu^3) + \ldots \pm d^r\left(\frac{W}{u^m}\right) . d^\varrho (wu^m)$$ (u).

Demonstratio.

1) Ponamus, theorema locum habere pro certo valore τȣ ϱ, ita vt τῷ m quemcunque valorem tribuendo, qui sit $> r + \varrho$, summa S semper prodeat $= 0$. Iam dum ϱ crescit vnitate, abit S in $S^I =$

$$d^r W . d^\varrho d w - m d^r\left(\frac{W}{u}\right) d^\varrho d(wu) + \frac{m(m-1)}{1.2} d^r\left(\frac{W}{u^2}\right) d^\varrho d(wu^2) - \text{etc.}$$

$$= d^r W . d^\varrho d w - m d^r\left(\frac{W}{u}\right) d^\varrho (u d w) + \frac{m(m-1)}{1.2} d^r\left(\frac{W}{u^2}\right) d^\varrho (u^2 d w) - \text{etc.}$$

$$- m\left(d^r\left(\frac{W}{u}\right) d^\varrho (w d u) + (m-1) d^r\left(\frac{W}{u^2}\right) d^\varrho (w u d u)\right.$$

$$\left. - \frac{(m-1)(m-2)}{1.2} d^r\left(\frac{W}{u^3}\right) d^\varrho (w u^2 d u) + \text{etc.}\right);$$

sicque S^I duabus seriebus constat, quarum prima ex hypothesi euanescit, posito nimirum in serie assumta S loco w, dw; altera series etiam euanescet, posito in S loco W, $\frac{W}{u}$, loco w, wdu, et pro m, m − 1, dum fuerit $m - 1 > r + \varrho$, i. e. $m > r + \varrho + 1$. Hinc manifestum est, ex assumta hypothesi $S = 0$, pro $m > r + \varrho$, sequi etiam, posito $\varrho + 1$ loco ϱ, $S^I = 0$, pro quouis $m > r + \varrho + 1$.

2) Cum quantitates W, w; u, $\frac{1}{u}$; r, ϱ; inuicem permutare liceat, ex (1) sponte sequitur, quod si S pro certo valore τȣ r et $m > r + \varrho$ euanescens ponatur, etiam S^I pro $r + 1$ et $m > r + \varrho + 1$ fore $= 0$.

3) Conclusiones (1) et (2) combinando, ac ab $\varrho + 1$, $r + 1$ ad $\varrho + 2$, $r + 2$, et sic porro continuando, manifestum sit, ex $S = 0$ pro certis valoribus τῶν r et ϱ et quouis $m > r + \varrho$, sequi $S = 0$, posito $r + l$, $r + \lambda$ loco r et ϱ, ac assumto $m > r + \varrho + l + \lambda$, denotantibus l et λ numeros quosuis integros.

4) Iam

(u) Inter hoc theorema, ac alterum prius commemoratum (§. VII. 4.) aliqua similitudo intercedit, differentia autem in eo maxime cernitur, quod in singulis terminis seriei hac §pho propositae exponentes differentiales iidem maneant, in priori contra ii varient, summa tantum manente constante; vnde prius theorema altioris indaginis esse videtur.

4) Iam pro r = 1 et $\varrho = 0$, $S = Ww - mWw + \frac{m(m-1)}{1.2} Ww - \ldots$
$= Ww(1-1)^m = 0$, dum sit $m > 0$. Hinc ope conclusionis (3) sequitur, fore etiam $S = 0$, si pro r et ϱ ponantur quicunque valores l et λ, dum fuerit $m > l + \gamma$.

Corollarium.

§. XI. Ponatur 1) $r = 0$; 2) $W = u^m$; et 3) $\varrho = m - 1$; tum habetur:
$u^m d^{m-1} w - m u^{m-1} d^{m-1}(wu) + \frac{m(m-1)}{1.2} u^{m-2} d^{m-2}(wu^2) - \text{etc.}$
$= 0$, quod est theorema Lexellii. Idem auctor obseruat (v), esse $u^m d^\varrho w - m u^{m-1} d^\varrho (wu) + \frac{m(m-1)}{1.2} u^{m-2} d^\varrho (wu^2) - \text{etc.} = 0$, "quicunque demum fuerit valor numeri m", vbi tamen conditio limitans adiicienda est, quod m debeat esse $> \varrho$; sicque haec ipsa aequatio generalior sistit casum tantum specialem theorematis praecedentis (§. IX.), posito rursus $r = 0$, $W = u^m$, et $\varrho < m$.

Continuatio.

§. XII. Coëfficientes seriei pro ψx assumtae (§. I. 1.) Y^{I}, Y^{II}, Y^{III}, ... Y^{N}, ..., certae functiones τȣ y, designentur per $\psi^{I} y$, $\psi^{II} y$, $\psi^{III} y$, ... ψ^{N}, ..., vt habeatur $\psi x = \psi y + z\psi^{I} y + z^2 \psi^{II} y + z^3 \psi^{III} y + \ldots + z^n \psi^{N} y + \ldots$, pro data aequatione inter tres variabiles z, x, y, hac: $y = x - z\varphi x$.

2) Priusquam determinationem functionum signis ψ^{I}, ψ^{II}, ψ^{III}, ... expressarum aggrediamur, iuuat in memoriam reuocare methodum, quae in solutione problematis de reuersione serierum Analystis haud inusitata est. Data nimirum serie quantitatem y per x exprimente, veluti hac:

$$y = a^{I} x + a^{II} x^2 + a^{III} x^3 + \ldots$$

assumitur series pro x, secundum y procedens,

$$x = \alpha^{I} y + \alpha^{II} y^2 + \alpha^{III} y^3 + \ldots,$$

tum in hac serie *assumta* substituitur series *data*, siue y, y^2, y^3, ... exprimuntur per x ope aequationis datae: quo facto obtinetur aequatio solam variabilem x inuoluens, quae identica esse debet, quaeque igitur determinationi coefficientium assumtorum inseruit.

3) Iam *ad similitudinem huius methodi* etiam in solutione problematis, de quo nunc agitur, felici successu operari licet, idque sequentem in modum. Cum sit ex aequatione *data* $y = x - z\varphi x$, id primo efficiendum est, vt singuli termini aequationis siue seriei *assumtae*

(v) l. c. §. IV. p. 234.

assumtae (1), variabilem y inuoluentes, per x exprimantur, quod quidem ope theorematis *Tayloriani* peragitur. Est nimirum quaeuis functio $\psi^N y = \psi^N (x - z\varphi x)$

$= \psi^N x - z\varphi x \cdot \frac{d\psi^N x}{dx} + z^2 \varphi x^2 \cdot \frac{d^2\psi^N x}{1.2\, dx^2} - \ldots$ Expressis ita functionibus $\psi^I y, \psi^{II} y, \ldots$ per x, iisque substitutis in serie assumta, haec prodit aequatio, secundum potestates variabilis z ordinata:

$$\psi x = \psi x - z\varphi x \cdot \frac{d\psi x}{dx} + z^2 \varphi x^2 \cdot \frac{d^2\psi x}{1.2\, dx^2} - z^3 \varphi x^3 \cdot \frac{d^3 \psi x}{1.2.3\, dx^3} \ldots$$

$$\pm z^n \varphi x^n \cdot \frac{d^n \psi x}{1.2 \ldots n\, dx^n} \mp \ldots$$

$$+ z\psi^I x \qquad - z^2 \varphi x \cdot \frac{d\psi^I x}{dx} + z^3 \varphi x^2 \cdot \frac{d^2 \psi^I x}{1.2\, dx^2} \ldots$$

$$\pm z^n \varphi x^{n-1} \cdot \frac{d^{n-1} \psi^I x}{1.2 \ldots n-1\, dx^{n-1}} \pm \ldots$$

$$+ z^2 \psi^{II} x \qquad - z^3 \varphi x \cdot \frac{d\psi^{II} x}{dx} \ldots$$

$$\pm z^n \varphi x^{n-2} \cdot \frac{d^{n-2} \psi^{II} x}{1.2 \ldots n-2\, dx^{n-2}} \mp \ldots$$

$$+ z^3 \psi^{III} x \quad . \quad . \quad . \quad . \quad .$$

$$. \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\ldots - z^n \varphi x \cdot \frac{d\psi^{N-1} x}{dx}$$

$$+ z^n \psi^N x.$$

4) Quae aequatio cum identica esse debeat, quippe x et z haud a se inuicem pendent, factores cuiusuis potestatis τῆς z seorsim $= 0$ ponendi sunt. Sic prodit $\psi^I x = \varphi x \cdot \frac{d\psi x}{dx}$;

$\psi^{II} x = \varphi x \cdot \frac{d\psi^I x}{dx} - (\varphi x)^2 \cdot \frac{d^2 \psi x}{1.2\, dx^2} = d \frac{\left(\varphi x^2 \cdot \frac{d\psi x}{dx}\right)}{1.2\, dx}$. Simili modo reperiuntur

$\psi^{III} x$,

$\psi^{III} x$, $\psi^{IV} x, \ldots$; et coëfficientem τῦ z^n ponendo $= 0$, manifestum est, $\psi^N x$ determinari per functiones praecedentes $\psi^{N-1} x$, $\psi^{N-2} x, \ldots \psi x$.

5) Sit iam $\psi^R x = \dfrac{d^{r-1}\left(\varphi x^r \cdot \dfrac{d\psi x}{dx}\right)}{1.2\ldots r.dx^{r-1}}$, vsque ad $r = n - 1$, erit $\psi^N x$

$$= \varphi x . \frac{d\psi^{N-1} x}{dx} - (\varphi x)^2 . \frac{d^2 \psi^{N-2} x}{1.2\, dx^2} + (\varphi x)^3 \frac{d^3 \psi^{N-3} x}{1.2.3\, dx^3} \ldots$$

$$\pm (\varphi x)^n \frac{d^n \psi x}{1.2\ldots n\, dx^n}$$

$$= \frac{1}{1.2..n\, dx^{n-1}} \left\{ \begin{array}{l} n\varphi x . d^{n-1}\left(\varphi x^{n-1} \dfrac{d\psi x}{dx}\right) - \dfrac{n(n-1)}{1.2} \varphi x^2 . d^{n-1}\left(\varphi x^{n-2} \dfrac{d\psi x}{dx}\right) \\ + \dfrac{n(n-1)(n-2)}{1.2.3} \varphi x^3 . d^{n-1}\left(\varphi x^{n-3} \dfrac{d\psi x}{dx}\right) \ldots \pm \varphi x^n . d^{n-1} \dfrac{d\psi x}{dx} \end{array} \right\}$$

Posito nunc in Coroll. §. X. $u = \dfrac{1}{\varphi x}$, $w = (\varphi x)^n \dfrac{d\psi x}{dx}$, diuidendo per u^m, aequatio:

$$d^{n-1} w = n . \frac{1}{u} d^{n-1} (wu) - \frac{n(n-1)}{1.2} . \frac{1}{u^2} d^{n-1} (wu^2) + \text{etc.}$$

in hanc abit:

$$d^{n-1}\left(\varphi x^n \frac{d\psi x}{dx}\right) = n\varphi x . d^{n-1}\left(\varphi x^{n-1} . \frac{d\psi x}{dx}\right) - \frac{n(n-1)}{1.2} \varphi x^2$$

$$. d^{n-1}\left(\varphi x^{n-2} \frac{d\psi x}{dx}\right) + \ldots$$

Hinc prouenit $\psi^N x = \dfrac{1}{1.2..n} \dfrac{d^{n-1}\left(\varphi x^n \dfrac{d\psi x}{dx}\right)}{dx^{n-1}}$.

6) Quanquam in formulis hactenus per processum reuersioni vsitatae analogum inuentis, coëfficientium quaesitorum nulla ratio habita sit, quippe qui sunt functiones variabilis y; hos tamen ipsos coëfficientes sponte iam definire licet, dummodo obseruetur, esse $\psi^I y$, $\psi^{II} y, \ldots \psi^N y$ similes functiones τῦ y, ac $\psi^I x$, $\psi^{II} x, \ldots \psi x^N$, τῦ x.

 Hinc

Hinc nimirum ſtatim prodit $Y^N = \psi^N y = \frac{d^{n-1}\left(\varphi y^n \cdot \frac{d\psi y}{dy}\right)}{1.2\ldots n\, dy^n}$; quod cum ſerie ſupra aliunde inuenta conſentit.

PROBLEMA.

§. 13. Propoſita ſerie haec:

$$z = a^{I}x^{p} + a^{II}x^{p+d} + a^{III}x^{p+2d} + \ldots a^{N}x^{p+nd} + \ldots,$$

exprimere quamuis poteſtatem τ8 x (x^s) per ſeriem ſecundum z progredientem.

Solutio.

1) Ponatur primo $p = d = 1$, vt fit

$$z = a^{I}x + a^{II}x^2 + a^{III}x^3 + a^{IV}x^4 + \ldots + a^{N}x^{n} + \ldots$$

Quae ſeries data, tanquam functio τ8 x deſignetur per Fx, atque aequationi $z = Fx$ haec tribuatur forma: $o = x - z\left(\frac{x}{Fx}\right)$.

2) Ad hanc aequationis propoſitae formam ſtatim applicare licet expreſſionem ſupra §. II. inuentam, poſito illic $y = o$, $\varphi x = \frac{x}{Fx}$. Hinc nimirum fit $(\varphi x)^s = \frac{x^s}{Fx^s} = \frac{x^s}{z^s}$

$$= y^s (Fy)^{-s} + \frac{s}{s+1}\frac{d(y^{s+1}.Fy^{-s-1})}{dy}.z + \frac{s}{s+2}\frac{d^2(y^{s+2}.Fy^{-s-2})}{1.2\,dy^2}.z^2$$

$$+ \ldots + \frac{s}{s+n}\frac{d^n.(y^{s+n}Fy^{-s-n})}{1.2.3\ldots n\,dy^n}.z^n + \ldots$$

Cum vero in hac ſerie y poni debeat $= o$, hanc variabilem cum x, indeque Fy cum Fx ſiue z permutare licet, dummodo notetur, in primo membro ſeriei atque in differentialibus ſequentibus poſt differentiationem ponendum eſſe $x = o$. Quo ſenſu obtinetur pro $\frac{x^s}{z^s}$ haec ſeries: $\frac{x^s}{z^s} = x^s.z^{-s} + \frac{s}{s+1}\frac{d(x^{s+1}.z^{-s-1})}{dx}.z$

$$+ \frac{s}{s+2}\frac{d^2(x^{s+2}.z^{-s-2})}{1.2\,dx^2}.z^2 + \ldots + \frac{s}{s+n}\frac{d^n(x^{s+n}.z^{-s-n})}{1.2\ldots n\,dx^n}.z^n + \ldots$$

3) Pri-

3) Primum membrum huius feriei fponte prodit $= (a^I)^{-s}$, quippe $z^{-s} = (a^I)^{-s} x^{-s} + . x^{-s+1} + . x^{-s+2} + \ldots$ (coëfficientes post primum, breuitatis gratia, per puncta notando), et

$$x^s z^{-s} = (a^I)^{-s} + . x + . x^2 + . x^3 + \ldots$$
$$= (a^I)^{-s}, \text{ pro } x = 0.$$

Ad reliquos terminos feu coëfficientes poteftatum τοῦ z formula generali exprimendos, euoluatur feriei datae $= z$ poteftas $-(s+n)^{ta}$, quae erit formae fequentis:

$$z^{-s-n} = \mathfrak{A}^I x^{-s-n} + \mathfrak{A}^{II} x^{-s-n+1} + \mathfrak{A}^{III} x^{-s-n+2} + \ldots$$
$$+ \mathfrak{A}^N x^{-s-1} + \mathfrak{A}^{N+1} x^{-s} + \mathfrak{A}^{N+2} x^{-s+1} + \ldots;$$

tum habetur $x^{s+n} z^{-s-n} =$

$$\mathfrak{A}^I + \mathfrak{A}^{II} x + \mathfrak{A}^{III} x^2 + \ldots + \mathfrak{A}^N x^{n-1} + \mathfrak{A}^{N+1} x^n + \mathfrak{A}^{N+2} x^{n+1} + \ldots;$$

atque n^{ies} differentiando,

$$\frac{d^n (x^{s+n} z^{-s-n})}{d x^n} = n(n-1)\ldots 1 . \mathfrak{A}^{N+1} + (n+1) n \ldots 2 . \mathfrak{A}^{N+2} . x + \ldots,$$

quod pro $x = 0$ fponte abit in $1 . 2 . 3 \ldots n . \mathfrak{A}^{N+1}$.

4) Litera $\mathfrak{A}^{N+1}$ denotat coëfficientem $n+1^{tum}$ τοῦ z^{-s-n}, qui pro cognito habendus eft, cum feries $z = a^I x + a^{II} x^2 + a^{III} x^3 + \ldots$ data, indeque etiam eius poteftas $-s-n^{ta}$ determinata fit. Defignando igitur $\mathfrak{A}^{N+1}$ per $z^{-s-n} \overset{x}{k}(n+1)$, feries pro x^s defiderata (ex 2.) haec eft:

$$x^s = \frac{s}{s} z^{-s} k1 . z^s + \frac{s}{s+1} z^{-s-1} k2 . z^{s+1} + \frac{s}{s+2} z^{-s-2} k3 . z^{s+2} + \ldots$$
$$+ \frac{s}{s+n} z^{-s-n} k(n+1) . z^{s+n} + \ldots$$

Cuius itaque feriei coëfficiens quilibet $n+1^{tus}$ hac aequatione definitur:

$$x^s k(n+1) = \frac{s}{s+n} z^{-s-n} k(n+1).$$

5) Expedito iam cafu fimpliciori (1), vbi $p = d = 1$, fponte patet tranfitus ad feriem generaliorem:

Kk 2 $\qquad z =$

$$z = a^{\text{I}} x^p + a^{\text{II}} x^{p+d} + a^{\text{III}} x^{p+2d} + \ldots$$

Hac nimirum ferie ad poteftatem exponentis $\frac{d}{p}$ euecta, prodit

$$z^{\frac{d}{p}} = \left(x^p (a^{\text{I}} + a^{\text{II}} x^d + a^{\text{III}} x^{2d} + \ldots)\right)^{\frac{d}{p}}$$

$$= x^d (a^{\text{I}} + a^{\text{II}} x^d + a^{\text{III}} x^{2d} + \ldots)^{\frac{d}{p}}$$

$$= x^d (A^{\text{I}} + A^{\text{II}} x^d + A^{\text{III}} x^{2d} + \ldots),$$

vbi coëfficientes A^{I}, A^{II}, $A^{\text{III}}, \ldots$ per datos a^{I}, a^{II}, $a^{\text{III}}, \ldots$ etiam datos effe liquet.

Pofito iam $z^{\frac{d}{p}} = \xi$, $x^d = \chi$, haec habetur aequatio:

$$\xi = A^{\text{I}} \chi + A^{\text{II}} \chi^2 + A^{\text{III}} \chi^3 + A^{\text{IV}} \chi^4 + \ldots,$$

quae eft formae fimplicis (1), indeque ex prius inuentis (4) refolui poterit. Erit nimirum pro ferie, qua poteftas quaeuis σ^{ta} τῦ χ per ξ exprimitur, $\chi^\sigma k(n+1) = \frac{\sigma}{\sigma+n} \xi^{-\sigma-n} k(n+1)$, ex (4), qui coëfficiens in $\xi^{\sigma+n}$ ducendus eft. Hinc ob $\chi^\sigma = x^{\sigma d}$, $\xi^{-\sigma-n} = z^{-(\sigma+n)\frac{d}{p}}$, pofito $\sigma d = s$, prodit $x^s k(n+1) = \frac{s}{s+nd} . z^{-\frac{s-nd}{p}} k(n+1)$; fiue pro x^s haec habetur feries:

$$x^s = \frac{s}{s} z^{-\frac{s}{p}} k\,1 . z^{\frac{s}{p}} + \frac{s}{s+d} z^{-\frac{s-d}{p}} k\,2 . z^{\frac{s+d}{p}} + \frac{s}{s+2d} z^{-\frac{s-2d}{p}} k\,3 . z^{\frac{s+2d}{p}} + \ldots$$

$$+ \frac{s}{s+nd} z^{-\frac{s-nd}{p}} k(n+1) . z^{\frac{s+nd}{p}} + \ldots$$

Haec igitur problematis propofiti folutio fequens fuppeditat:

Theorema.

§. XIV. Quodfi ex data ferie (*reuertenda*): $z = a^{\text{I}} x^p + a^{\text{II}} x^{p+d} + a^{\text{III}} x^{p+2d} + \ldots$ per reuerfionem valor cuiusuis poteftatis τῦ x, $= x^s$, per feriem fecundum z progredientem exprimendus eft: tum ferie data ad poteftatem exponentis $-\frac{(s+nd)}{p}$ euecta, huius poteftatis coëfficiens $n+1^{tus}$ in $\frac{s}{s+nd} . z^{\frac{s+nd}{p}}$ ductus praebebit feriei quae-

fitae

ſitae (reuerſae) terminum $n+1^{tum}$. Quod theorema, Analyſtarum attentione omnino dignum, concinne ac commode his ſignis exprimitur:

$$x^s k(n+1) = \frac{s}{s+nd} . z^{-\frac{(s+nd)}{p}} k(n+1),$$

$$\text{ſiue } x^s 7 (n+1) = \frac{s}{s+nd} . z^{-\frac{(s+nd)}{p}} k(n+1) . z^{\frac{s+nd}{p}} .$$

Scholion.

§. XV. Priusquam in ſolutionem §. XIII. expoſitam incidiſſem, ex qua apparet, theorema §. XIV. abſque calculi ambagibus tanquam corollarium ex theoremate generali *LaGrangiano* (§. I. 9.) deduci poſſe, nexum vtriusque theorematis ſequenti ratione inueſtigaui.

1) In ſerie data ſumamus, quod ſemper conceſſum eſt, coëfficientem primum $= 1$, $p = d = 1$, loco z ſcribatur y, vt ſit $y = x + \alpha(1)x^2 + \alpha(2)x^3 + \alpha(3)x^4 + \ldots$, quae aequatio cum generali (§. I.) comparata praebet: $z = -1$, $\varphi x = \alpha(1)x^2 + \alpha(2)x^3 + \alpha(3)x^4 + \ldots$ Denotentur nimirum, more iam ſupra §. III. 3. obſeruato, eoque interdum haud incommodo, coëfficientes ſeriei φx ex ordine per $\alpha(1)$, $\alpha(2)$, $\alpha(3)$, $\alpha(4)$, ...; porro coëfficientes poteſtatum huius ſeriei:

ſecundae, per $\overset{2}{\alpha}(1)$, $\overset{2}{\alpha}(2)$, $\overset{2}{\alpha}(3)$, $\overset{2}{\alpha}(4)$, ...

tertiae; $\overset{3}{\alpha}(1)$, $\overset{3}{\alpha}(2)$, $\overset{3}{\alpha}(3)$, $\overset{3}{\alpha}(4)$, ...

. . . . . .

n^{tae}; $\overset{n}{\alpha}(1)$, $\overset{n}{\alpha}(2)$, $\overset{n}{\alpha}(3)$, $\overset{n}{\alpha}(4)$, ...

vt ſit

$$(\varphi x)^n = \overset{n}{\alpha}(1)x^{2n} + \overset{n}{\alpha}(2)x^{2n+1} + \overset{n}{\alpha}(3)x^{2n+2} + \ldots + \overset{n}{\alpha}(r)x^{2n+r-1} + \ldots$$

2) Iam ex ſupra demonſtratis (§. II. 3.) reſoluta aequatione $y = x + \varphi x$, habetur poteſtas quaeuis τ8 x,

$$x^s = y^s - sy^{s-1}\varphi y + \frac{s d (y^{s-1} \varphi y^2)}{1.2\, dy} - \frac{s d^2 (y^{s-1} \varphi y^3)}{1.2.3\, dy^2} + \ldots$$
$$\pm \frac{s d^{n-1} (y^{s-1} \varphi y^n)}{1.2\ldots n\, dy^{n-1}} \mp \ldots$$

Cuius ſeriei termini ſecundum poteſtates variabilis y ordinandi ſunt. Quod quidem ſequenti ratione obtinetur. Functionem φy, eiusque poteſtates ex ſeriebus (1) exprimere licet, dum pro x ponatur y; ſicque fit

$(\varphi y)^n$

$$(\varphi y)^n = \overset{n}{\alpha}(1)\, y^{2n} + \overset{n}{\alpha}(2)\, y^{2n+1} + \overset{n}{\alpha}(3)\, y^{2n+2} + \ldots + \overset{n}{\alpha}(r)\, y^{2n+r-1} + \text{etc.};$$

porro $y^{s-1}(\varphi y)^n =$

$$\overset{n}{\alpha}(1)\, y^{2n+s-1} + \overset{n}{\alpha}(2)\, y^{2n+s} + \overset{n}{\alpha}(3)\, y^{2n+s+1} + \ldots + \overset{n}{\alpha}(r)\, y^{2n+s+r-2} + \ldots;$$

ac $n-1$ vicies differentiando,

$$\begin{aligned} d^{n-1}(y^{s-1}\varphi y^n) = {} & (2n+s-1)(2n+s-2)\ldots(n+s+1)\overset{n}{\alpha}(1).y^{n+s} \\ & + (2n+s)(2n+s-1)\ldots(n+s+2)\overset{n}{\alpha}(2).y^{n+s+1} \\ & + (2n+s+1)(2n+s)\ldots(n+s+3)\overset{n}{\alpha}(3).y^{n+s+2} \\ & + \ldots \\ & + (2n+s+r-2)(2n+s+r-3)\ldots(n+s+r)\overset{n}{\alpha}(r).y^{n+s+r-1} \\ & + \ldots \end{aligned}$$

Quem valorem substituendo, posito successiue $n = 1, 2, 3, \ldots$ fit

$$\begin{aligned} x^s = y^s & - s\overset{1}{\alpha}(1)\, y^{s+1} - s\overset{1}{\alpha}(2)\, y^{s+2} - s\overset{1}{\alpha}(3)\, y^{s+3} - s\overset{1}{\alpha}(4)\, y^{s+4} - \ldots \\ & + \frac{s(s+3)}{1.2}\overset{2}{\alpha}(1) + \frac{s(s+4)}{1.2}\overset{2}{\alpha}(2) + \frac{s(s+5)}{1.2}\overset{2}{\alpha}(3) + \ldots \\ & - \frac{s(s+5)(s+4)}{1.2.3}\overset{3}{\alpha}(1) - \frac{s(s+6)(s+5)}{1.2.3}\overset{3}{\alpha}(2) - \ldots \\ & + \frac{s(s+7)(s+6)(s+5)}{1.2.3.4}\overset{4}{\alpha}(1) + \ldots \end{aligned}$$

3) Quo nunc huius seriei secundum y ordinatae coëfficiens generalis $\lambda + 1^{tus}$, siue factor τῦ $y^{s+\lambda}$ rite determinetur, obseruandum est, hunc coëfficientem ex pluribus partibus constare, quas singuli termini seriei nondum secundum y ordinatae (2), sub forma $\pm \dfrac{s\, d^{n-1}(y^{s-1}\varphi y^n)}{1.2\ldots n\, dy^{n-1}}$ comprehensi, suppeditant, quarumque quaelibet, sumto $n+s+r-1 = s+\lambda$, siue $n+r = \lambda+1$, prodit $=$

$$\pm \frac{s}{1.2\ldots n}(2n+s+r-2)(2n+s+r-3)\ldots(n+s+r).\overset{n}{\alpha}(r) =$$

$$\pm \frac{s}{1.2\ldots n}(s+\lambda+n-1)(s+\lambda+n-2)\ldots(s+\lambda+1).\overset{n}{\alpha}(\lambda+1-n).$$

Hinc sumto n ex ordine $= 1, 2, 3, \ldots, \lambda$, vel $r = \lambda, \lambda-1, \ldots 1$, coniunctim obtinetur iste coëfficiens $=$

$$= -s\overset{1}{\alpha}(\lambda) + \frac{s(s+\lambda+1)}{1.2}\cdot\overset{2}{\alpha}(\lambda-1) - \frac{s(s+\lambda+2)(s+\lambda+1)}{1.2.3}\cdot\overset{3}{\alpha}(\lambda-2)$$
$$+\frac{s(s+\lambda+3)(s+\lambda+2)(s+\lambda+1)}{1.2.3.4}\cdot\overset{4}{\alpha}(\lambda-3) - \ldots \pm \frac{s(s+2\lambda-1)(s+2\lambda-2)\ldots(s+\lambda+1)}{1.2\ldots\lambda}\cdot\overset{\lambda}{\alpha}(1).$$

4) Quam expressionem sequenti modo ad formam simpliciorem reuocare licet. Euehendo $y = x + \varphi x$ ad potestatem $-s-\lambda$, fit $y^{-s-\lambda} = (x+\varphi x)^{-s-\lambda} =$
$x^{-s-\lambda} - (s+\lambda)x^{-s-\lambda-1}\varphi x + \frac{(s+\lambda)(s+\lambda+1)}{1.2}x^{-s-\lambda-2}\varphi x^2$
$-\frac{(s+\lambda)(s+\lambda+1)(s+\lambda+2)}{1.2.3}x^{-s-\lambda-3}\varphi x^3 + \ldots$, vel substituendo pro φx, φx^2, φx^3, ... series (1), ac ordinando secundum x,

$$\begin{aligned} y^{-s-\lambda} = x^{-s-\lambda} &- (s+\lambda)\overset{1}{\alpha}(1)x^{-s-\lambda+1} \\ &- (s+\lambda)\overset{1}{\alpha}(2)x^{-s-\lambda+2} - (s+\lambda)\overset{1}{\alpha}(3)x^{-s-\lambda+3} - \ldots \\ &+ \frac{(s+\lambda)(s+\lambda+1)}{1.2}\overset{2}{\alpha}(1) + \frac{(s+\lambda)(s+\lambda+1)}{1.2}\overset{2}{\alpha}(2) + \ldots \\ &- \frac{(s+\lambda)(s+\lambda+1)(s+\lambda+2)}{1.2.3}\overset{3}{\alpha}(1) - \ldots \\ &+ \ldots \end{aligned}$$

Quae series quomodo progrediatur, satis manifestum est; vnde apparet, eius coëfficientem $\lambda+1^{tum}$ in $\frac{s}{s+\lambda}$ ductum aequari coëfficienti (3), i. e. coëfficienti $\lambda+1^{to}$ seriei reuersae; id quod cum formula prius inuenta consentit (w).

Scholion.

§. XVI. Accuratius considerando analysin, qua LAGRANGIVS, data aequatione: $o = a - bx + cx^2 - dx^3 + \ldots$ radicis potestatem m^{tam} per seriem expressit, animaduerti, simili processu seriem §. XIII. 5, vel theorema §. XIV. elici posse: dummodo euolutio vnius termini aequationis, ope logarithmorum transformatae, in seriem secundum x progredientem, alia ratione suscipiatur. *Ab iisdem nimirum principiis exeundo, prouti deinceps vlterior progressus dirigatur, vel ad seriem LaGrangii, vel ad alteram* (§. XIII.) *pertingere licet.* Quod strictius ostendere operae pretium videtur, cum ita consensus atque nexus inter vtrumque theorema clarius elucescet.

1) Or-

(w) cf. quae monet *Toepferus* de transitu a theoremate *LaGrangii* ad formulam reuersoriam theorematis §. XIV. (l. c. p. 110.).

1) Ordiundum est ab aequatione hac, supra (§. III. 2.) in expositione analyseos *LaGrangianae* demonstrata: (x) (A) $\log.\left(1-\frac{a}{x}-\xi\right)=$

$$\log.\frac{a}{p}+\log.\left(1-\frac{p}{x}\right)+\log.\left(1-\frac{x}{q}\right)+\log.\left(1-\frac{x}{r}\right)+\ldots$$

posito $\xi = cx - dx^2 + ex^3 - \ldots$ et denotante p vnam ex radicibus aequationis: $o = a - x + cx^2 - dx^3 + ex^4 - \ldots,$ reliquis existentibus $= q, r, \ldots$ Iam quomodo pars dextra aequationis (A) in seriem secundum potestates τῦ x progredientem euoluenda sit, satis manifestum est: ad cuius seriei terminum $\frac{-p^m}{mx^m}$, ex $\log.\left(1-\frac{p}{x}\right)$ oriundum, pro nostro scopo maxime respicere oportet.

2) Altera autem pars aequationis (A), siue $\log.\left(1-\frac{a}{x}-\xi\right)$, praeter euolutionem a LAGRANGIO adhibitam, alia insuper ratione in seriem conuerti potest, cuius deinceps coëfficiens in $\frac{1}{x^m}$ ductus (signis mutatis) erit $=\frac{p^m}{m}$. Posito nimirum $\frac{a}{x}+\xi = X$, est $-\log.\left(1-\frac{a}{x}-\xi\right)=X+\frac{X^2}{2}+\frac{X^3}{3}+\ldots+\frac{X^r}{r}+\ldots$ Ad eruendas potestates τῦ X in auxilium vocandae sunt potestates τῦ ξ, quarum quamlibet ϱ^{tam} sic denotare sufficit:

$$\xi^\varrho = \xi^\varrho k1.x^\varrho + \xi^\varrho k2.x^{\varrho+1} + \xi^\varrho k3.x^{\varrho+2} + \ldots$$

Hinc fit $\frac{X^r}{r}=\frac{1}{r}\left(\frac{a}{x}+\xi\right)^r=\frac{1}{r}a^r x^{-r}$

$$+a^{r-1}(\xi k1.x^{-r+2}+\xi k2.x^{-r+3}+\xi k3.x^{-r+4}+\ldots)$$
$$+\frac{(r-1)}{1.2}a^{r-2}(\xi^2 k1.x^{-r+4}+\xi^2 k2.x^{-r+5}+\xi^2 k3.x^{-r+6}+\ldots)$$
$$+\quad.\quad.\quad.\quad.\quad.\quad.\quad.$$
$$+\frac{(r-1)\ldots(r-\varrho+1)}{1.2\ldots\varrho}a^{r-\varrho}(\xi^\varrho k1.x^{-r+2\varrho}+\xi^\varrho k2.x^{-r+2\varrho+1}$$
$$+\xi^\varrho k3.x^{-r+2\varrho+2}+\ldots)$$
$$+\quad.\quad.\quad.\quad.\quad.\quad.\quad.$$

Haec

(x) Litera b (l. c.) hic breuitatis gratia = 1 sumitur.

Haec pars feriei logarithmicae tum demum inuoluet terminos in $\frac{1}{x^m}$ ductos, cum fuerit $r = m$ vel $> m + 1$. Sit igitur $r = m + \lambda$, et colligendo terminos feriei $\frac{x^r}{r}$ factore x^{-m} affectos, coëfficiens huius poteftatis τῦ x orietur =

$$a^{m+\lambda-1}.\xi k(\lambda-1) + \frac{(m+\lambda-1)}{1.2}a^{m+\lambda-2}.\xi^2 k(\lambda-3) + \frac{(m+\lambda-1)(m+\lambda-2)}{1.2.3}a^{m+\lambda-3}.\xi^3 k(\lambda-5) + \ldots,$$

continuata hac progreffione, vsque dum ad $k1$ vel $k2$ perueniatur. Pofito nunc fucceffiue $\lambda = 2, 3, 4, \ldots$, obtinetur coëfficiens τῦ x^{-m} in ferie logarithmica, fiue $\frac{p^m}{m} = \frac{a^m}{m}$

$$+ a^{m+1}.\xi k1$$
$$+ a^{m+2}.\xi k2$$
$$+ a^{m+3}.\xi k3 + \frac{(m+3)}{1.2}a^{m+2}.\xi^2 k1$$
$$+ a^{m+4}.\xi k4 + \frac{(m+4)}{1.2}a^{m+3}.\xi^2 k2$$
$$+ a^{m+5}.\xi k5 + \frac{(m+5)}{1.2}a^{m+4}.\xi^2 k3 + \frac{(m+5)(m+4)}{1.2.3}a^{m+3}.\xi^3 k1$$
$$+ a^{m+6}.\xi k6 + \frac{(m+6)}{1.2}a^{m+5}.\xi^2 k4 + \frac{(m+6)(m+5)}{1.2.3}a^{m+4}.\xi^3 k2$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$+ a^{m+\lambda-1}.\xi k(\lambda-1) + \frac{(m+\lambda-1)}{1.2}a^{m+\lambda-2}.\xi^2 k(\lambda-3) + \frac{(m+\lambda-1)(m+\lambda-2)}{1.2.3}a^{m+\lambda-3}.\xi^3 k(\lambda-5) + \ldots$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

3) Ordinando hanc feriem fecundum poteftates τῦ a, obtinetur $\frac{p^m}{m} =$

$$\frac{a^m}{m} + a^{m+1}\xi k1 + a^{m+2}\left(\xi k2 + \frac{(m+3)}{1.2}\xi^2 k1\right) + a^{m+3}\left(\xi k3 + \frac{(m+4)}{1.2}\xi^2 k2 + \frac{(m+5)(m+4)}{1.2.3}\xi^3 k1\right)$$

Ll

$+ a^{m+4}$

$$+ a^{m+4}\left(\xi k_4 + \frac{(m+5)}{1.2}\xi^2 k_3 + \frac{(m+6)(m+5)}{1.2.3}\xi^3 k_2 + \frac{(m+7)(m+6)(m+5)}{1.2.3.4}\xi^4 k_1\right)$$

$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$+ a^{m+\lambda-1}\left(\xi k(\lambda-1) + \frac{(m+\lambda)}{1.2}\xi^2 k(\lambda-2) + \frac{(m+\lambda+1)(m+\lambda)}{1.2.3}\xi^3 k(\lambda-3)\right.$$
$$\left. + \ldots + \frac{(m+2\lambda-3)(m+2\lambda-4)\ldots(m+\lambda)}{1.2\ldots\lambda-1}\xi^{\lambda-1} k\,1\right)$$

$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

Euehendo autem feriem $x - cx^2 + dx^3 - ex^4 + \ldots, = x - \xi x = x(1-\xi) = a$, ad potentiam $-m-(\lambda-1)^{tam}$, prodit

$$a^{-m-(\lambda-1)} = x^{-m-(\lambda-1)}\left(1 + (m+\lambda-1)\xi + \frac{(m+\lambda-1)(m+\lambda)}{1.2}\xi^2 + \ldots\right);$$

cuius feriei fingulis membris ope expreffionum pro $\xi, \xi^2, \xi^3, \ldots$ (2) actu fecundum x euolutis, prodit coëfficiens λ^{tus}

$$= (m+\lambda-1)\xi k(\lambda-1) + \frac{(m+\lambda-1)(m+\lambda)}{1.2}\xi^2 k(\lambda-2) + \ldots,$$

i. e. = producto ex $m+\lambda-1$ in coëfficientem τȣ̃ $a^{m+\lambda-1}$ in ferie pro $\frac{p^m}{m}$. Exinde manifeftum eft, hanc feriem cum fupra (§. XIII. 5.) inuenta confentire, dummodo pro p, x et pro a, z fcribatur.

CAP. II.

DE THEOREMATE POLYNOMIALI COMBINATORIE TRACTATO, EIVSQVE APPLICATIONE AD REVERSIONEM SERIERVM.

ARTICVLVS PRIMVS.

Lemmata ex Doctrina Combinatoria.

Praenotanda.

§. XVII. Ex formula reuerforia §. XIII. inuenta intelligitur, reuerfionem ferierum omnem reduci ad determinationem poteftatum infinitinomii: ficque *problema*, de quo hic maxime agitur, *reuerforium*, arctiffimo vinculo iungi cum *theoremate polynomiali*. Huius igitur theorematis fuccinctam explicationem proponere, eiusdemque deinceps ad problema illud applicationem oftendere, omnino e re effe videtur (y).

Ducem

(y) Sicuti problema de reuerfione ferierum accuratiori inueftigationi theorematis polynomialis originem dediffe videtur, ita plerique auctores, qui illud problema pertractarunt, de hoc etiam theoremate

Ducem hic maxime sequar Celeberr. HINDENBVRGIVM, qui quippe de theoremate polynomiali ad vsum aptissime euoluendo, inprimis meritus, ex eoque primo occasionem nactus est (z), nouum vniuersae Doctrinae Combinatoriae systema condendi, eiusdemque nexum cum Analysi intimum clarissime illustrandi (a). Quatenus equidem inuentis addendo, eaue illustrando magis ac confirmando, vel pro re nata paulisper immutando, vltra id, quod Commentatoris poscit officium, praestare quicquam haud frustra laborauerim, peritiorum sit iudicium. Quibus obseruationes etiam historicas et literarias largiter sparsas haud ingratas fore spero.

1) Priusquam vero hoc theorema ipsum aggrediamur, praemittendum est problema, quod vulgo sub nomine *discerptionis* siue *partitionis numerorum* satis notum est, hoc autem loco *combinatorie* magis quam *arithmetice* consideratur, hisque verbis ab Analysta modo laudato enuntiari solet: *Reperire combinationes numeri propositi siue summae datae* (b). Quorum verborum sensus vt rite intelligatur, haec tenenda sunt.

2) Dum quaeritur de combinationibus *rerum datarum (elementorum)*, quae *literis* ex ordine alphabetico (a, b, c, d, ...) insigniri solent, hae simul notantur *numeris*, plerumque ex ordine naturali 1, 2, 3, 4, ... progredientibus, ita vt cuiuis literae suus respondeat numerus, qui *exponens* vocatur. Saepe etiam pro rebus, ipsos numeros ponere refert (c). Semper autem res siue literae, et numeri siue exponentes hunc in modum coniunguntur:

$$\begin{pmatrix} a, & b, & c, & d, & e, & \ldots \\ 1, & 2, & 3, & 4, & 5, & \ldots \end{pmatrix}$$

ex qua notatione (*indicem* appellat HINDENBVRGIVS) constat, quinam numeri literis ex ordine respondeant. Cuius igitur indicis ope a numeris ad literas, ac vice versa ab his ad illos statim transire licet; quare perinde est, siue elementa literalia, siue numerica combinentur (cc).

3) Iam

remate verba fecerunt: quorum inter recentiores laudasse hoc loco sufficiat *Hieron. Christoph. Wilh. Eschenbachium* (*de serierum reuersione formulis analytico-combinatoriis exhibita specimen*, *Lips. 1789. 4.*)

(z) cf. *Eschenbach* l. c. p. 13.

(a) Scripta huc pertinentia *Hindenburgii*, aliorumue, maxime illius discipulorum, in sequenti tractatione laudandi occasio erit.

(b) *Noui systematis permutationum, combinationum ac variationum primae lineae etc. Lips. 1781. 4.* p. XI. n. 31. cf. *Mauricii de Prasse Vsus logarithmorum infinitinomii in theoria aequationum, Lips. 1796. 4.* p. 3. §. II. n. 7.

(c) *Nou. Syst.* p. X. n. 30. cf. *Toepfer* l. c. p. 47.

(cc) Eum designandi modum, quo *loco literarum ponuntur ipsi numeri*, a *Leibnitio* primum fuisse adhibitum, deinceps vero haud satis frequenter in vsum vocatum, memorat *Hindenburgius* (*Infinitinomii Dignitatum — Historia, Leges ac Formulae etc. Goettingae 1779. 4.* Praef. p. XVIII.; cf.

 Toepfer

4) Iam satis manifestum est, dum rerum datarum modo vsitato combinationes inuestigentur, (combinationes *simpliciter* vocantur ab HINDENBVRGIO) tumque in singulis complexionibus elementorum exponentes numerici addantur, varias diuersasque prodituras esse summas. Quodsi nunc seligantur eae complexiones, in quibus summa istorum exponentium certo numero aequetur, hae complexiones praebent constituuntque combinationes huius summae siue numeri. Obtinentur itaque *combinationes numeri propositi summaeue datae*, elementorum combinandorum iungendo ea, quorum exponentes numerici inuicem additi conficiunt summam datam. Quae igitur combinationes concipi possunt, tanquam decerptae ex combinationibus *simpliciter*, ceu pars ex Toto (d).

4) Sicuti combinationes strictius sic dictae, siue combinationes *simpliciter*, diuiduntur in *Classes*, primam *Vnionum*, secundam *Binionum*, tertiam *Ternionum*, etc.: ita etiam Combinationum summae definitae Classes discernuntur, pro numero elementorum in quauis complexione seu coniunctione singulari occurrentium: quae classes, ex ordine prima, altera, tertia, quarta, quinta etc. a HINDENBVRGIO his characteribus, literis nimirum maioribus latinis A, B, C, D, E, . . . insigniuntur: ${}^{n}A$, ${}^{n}B$, ${}^{n}C$, ${}^{n}D$, ${}^{n}E$, . . . (e) vbi litera n a laeua signo Classis addita denotat *summam* numerorum rebus siue literis combinatis respondentium, quarum *multitudinem* signum Classis indicat (f).

5) Satis porro constat, *combinationes* a *variationibus* in eo differre, quod in *illis* certorum elementorum *vna* tantum coniunctio consideretur (e. g. abc), in *his* contra ad eorun-

Toepfer l. c. p. 47. 143. not. β.). Literas ad numeros, illarum seriem ad horum progressum referre, Analystis magis erat vsitatum. Sic *Moiureus* literis a, b, c, d, . . . tribuit *exponentes* numericos, qui illarum locum indicant (*Philos. Transact.* Vol. XX. p. 190: — *by the Exponent of a Letter I mean the Number which expresses what Place it has in the Alphabet*). Huc etiam spectat mos satis notus, isque in Analysi communiter receptus, coëfficientes seriei communi aliqua litera exprimendi, eosque inuicem discernendi numeris literae additis (*Infinit.* l. c. et p. 63.). Cuiusmodi numeros, qui alias *exponentes* seu *indices* vocantur, *Fischerus* appellat idiomate germanico: *Marken* (*Theorie der Dimensionszeichen* T. I. 1792. pag. 7.); cumque *index Hindenburgii* aliam habeat significationem, nec non denominationi *exponentis* aliqua insit ambiguitas, commodum interdum videtur, istos numeros *notas* literarum vocare.

(d) cf. *E. G. Fischer über den Ursprung der Theorie der Dimensionszeichen und ihr Verhältniss gegen die combinatorische Analytik des Hrn. Prof. Hindenburg*, (*Halle 1794. 4.*) pag. 25.

(e) Classes combinationum *simpliciter* sic exprimuntur: ${}'A$, ${}'B$, ${}'C$, ${}'D$, . . . dum nimirum literae latinae, quae sunt Classium signa, *apicibus indefinitis* notantur (*Hindenburg Nou. Syst.* p. XLII. 13.)

(f) Ad designandas Classes *indefinitas* adhibentur literae maiores latinae *alius Alphabeti*: ita signum ^{n}M exprimit classem combinationum summae n duodecimam, (ex ordine literae M in alphabeto), signum contra ${}^{n}\mathfrak{M}$ classem indefinitam m^{tam} (*Hindenburg Infinitin. Dign.* p. 85. 93; cf. *Prasse* p. 5. Schol. 11; *Rothe*, de *Reuers. ser.* p. 36.)

eorundem elementorum *diuersos situs* respiciatur, siue pro quauis complexione simul omnium eius *permutationum* ratio habeatur (abc, acb, bac, bca, cab, cba). Quo nunc istud discrimen rite seruetur, atque complexiones superfluae facilius euitentur, ea lex pro inuestigandis *combinationibus* praescribitur: complexiones semper accipiendas esse *rite ordinatas*. Vocatur autem complexio *rite ordinata*, cuius elementa numerica (literalium exponentes) a finistra ad dextram continuo crescunt, sic quidem vt elementum, quod praecedit a sinistra, nunquam maius sit sequenti, siue literae eodem ordine relatiuo sibi inuicem succedant, ac in suo alphabeto (g). Talis complexio rite ordinata est veluti *repraesentatrix* ceterarum, quae inde oriuntur, eadem illius elementa varie permutando, siue alio tantum ordine collocando, omnibusue quibus licet modis sedibus suis transponendo. Tunc *combinationes* cum singularum complexionum omnibus *permutationibus* sistunt *variationes*, quarum classes simili ratione, ac classes combinationum, verumtamen discriminis ostendendi caussa, literis maioribus *Italicis* exprimuntur (h).

6) Saepenumero etiam refert, ipsas *Classes* complexionum *rite ordinatas exhibere*. *Classis* nimirum *rite ordinata* vocatur, quando singulae eius complexiones eo ordine sunt dispositae, vel ita sibi inuicem succedunt, vti numeri crescentes progrediuntur. Quamlibet nimirum complexionem considerare licet tanquam numerum, conflatum ex elementis numericis ceu partibus vel potius cifris. Cum vero elementa quaepiam numerum 9 superantia, siue literae alphabeti litera i posteriores in complicatione occurrant, systema vulgare decadicum haud sufficit, sed respiciendum est ad *systemata altiora*, quae habent plures figuras, ceu numerorum simplicium notas: nec tamen necesse est, peculiaria signa adhibere, dummodo caueatur, vt ne elementa numerica ex nostro systemate composita confundantur cum pluribus iisque inuicem seiunctis complexionis elementis. Hinc tandem classis rite ordinata breuiter ceu ea definiri potest, in qua complexio minor semper praecedit, nunquam sequitur maiorem (i).

7) Vocabula et signa hactenus exposita vno exemplo illustrasse sufficiat. Complexiones nimirum *rite ordinatae* summae 6, secundum classes *rite ordinatas*, pro indice

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & . & . & . & . \\ a & b & c & d & e & f & . & . & . & . \end{pmatrix}$$ ita procedunt:

$^6A =$

(g) *Nou. Syst.* p. IX. *Toepfer* l. c. p. 47. *Rasse* l. c. p. 2. 3. De vtilitate ac necessitate, in inuestigandis combinationibus singularum complexionum elementa ordine naturali disponendi, admonuit etiam *Iac. Bernoullius* in *Arte Coniectandi* pag. 22, vbi haec verba extant: "ad hoc cauendum (ne praecedentes quidam modi redeant) semper opus est, vt nulla partium prior minor constituatur vlla sequentium". Infra iniicietur mentio Algebraistae seculo XVI. haud incelebris, *Ioan. Buteonis*, qui et ipse eandem regulam (inuerso tantum elementorum ordine, vti secundum *Hindenburgium*) praescripsit, nec non pro rebus combinandis *numeros tanquam notas* adhibuit.

(h) *Nou. Syst.* p. XIII. p. XLVI. Hinc sponte intelligitur significatio characterum: ^nA, ^nB, ^nC, . . .; ^1A, ^1B, ^1C, . . .; quorum illi *variationum summae* n, hi *variationum simpliciter* classes, primam, alteram, tertiam, . . . denotant.

(i) *Nou. Syst.* p. IX. XXIII. *Toepfer* p. 48. 73. 74. Not. *tt*.

^6A	=	6	=	f;
		15	=	ae
^6B	=	24	=	bd
		33	=	cc
		114	=	aad
^6C	=	123	=	abc
		222	=	bbb
^6D	=	1113	=	aaac
		1122	=	aabb
^6E	=	11112	=	aaaab
^6F	=	111111	=	aaaaaa

8) Nunc ab hoc exemplo particulari ad problema generale (1) progrediendum eſt: *reperire combinationes numeri propoſiti, ſiue ſummae datae*, et quidem *admiſſis repetitionibus*, id eſt, ſub hac conditione, vt in vna complexione liceat eandem literam ſaepius ponere. Cuius problematis *triplicem ſolutionem* exponemus, quarum *prima* et *altera* docebitur, quomodo omnes omnino combinationes ſummae datae ſint inueniendae, i. e. ſingulae complicationes, quarum elementa addita conficiunt iſtam ſummam: differunt autem binae ſolutiones in eo, quod ex priore, complexiones ſecundum claſſes, ex poſteriore, eaedem ordine ſic dicto lexicographico ſint diſpoſitae. *Tertia* ſolutione ſeorſim exhibentur eae complexiones ſummae datae, quae ad certam claſſem combinatoriam pertinent, quin complexionum ex aliis claſſibus ſimul ratio ſit habenda.

Problema.

§. XVIII. Reperire combinationes numeri propoſiti, ſiue ſummae datae, admiſſis repetitionibus.

A) Solutio prima.

Inuentio omnium claſſium, cuiusuis ex proxime praecedente, ſiue Inuolutio Claſſium (k).

1) *Regula pure combinatoria*, cuius ope claſſes ſucceſſiue, ſuo quaeuis ordine, definiuntur, (ſiue combinationes omnes ſummae datae, ſecundum claſſes diſpoſitae, obtinentur, ſumma igitur elementorum in ſingulis complexionibus manente conſtante, eorum, autem multitudine continuo creſcente), huc redit (l):

a) *Pri-*

(k) Denominatio *Inuolutionis claſſium* ſiue *ſecundum claſſes* diſpoſitae, infra (B, 7.) illuſtrabitur. Proceſſus combinatorius ſolutionis primae ab *Hindenburgio* vocatur etiam *deductio ex claſſe in claſſem*, (*Toepfer* l. c. p. 82.).

(l) cf. *Der polynomiſche Lehrſatz — — neu bearbeitet und dargeſtellt von Tetens, Klügel, Kramp, Pfaff und Hindenburg; zum Druck befördert und mit — einem — Abriſſe der combinatoriſchen Methode*

a) *Primam classem* summae n constituit elementum ntum. Nec minus manifestum est, *secundam classem* prodire, praemittendo elementum 1 (siue a) elemento n — 1to, ac permutando successiue illud cum elementis proxime maioribus, hoc cum proxime minoribus, donec ad duo elementa vel omnino vel proxime aequalia perueniatur.

b) Iam ad reperiendas *classes altiores* complexionum *ordines* discernendi sunt: eae nimirum complexiones cuiusuis classis ad *eundem ordinem* referuntur, quae *eodem elemento incipiunt.* Nunc duplici praecepto opus est: *primo* (α) pro ordine 1 vel a cuiusuis classis ex classis praecedentis complexionibus petendo; *deinde* (β) pro deriuandis ex ordine primo ceteris ordinibus eiusdem classis.

α) Ordo 1 siue a cuiusuis classis prodit, praemittendo elementum 1 siue a singulis complexionibus classis praecedentis, quae non habent in fine duo elementa aequalia, tumque permutando elementum vltimum cum proxime minori.

β) Ex ordine quouis r reperitur eiusdem classis ordo sequens r + 1, permutando in singulis complexionibus illius ordinis, quae ab initio simulque in fine habent duo elementa diuersa, elementum primum cum proxime maiori, vltimum cum proxime minori (m).

Regulae illustrandae inseruit exemplum infra adpositum, quo classes combinatoriae summae 9 tam in numeris quam in literis pro indice

$$\begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9 & . & . & . \\ a, & b, & c, & d, & e, & f, & g, & h, & i & . & . & . \end{pmatrix}$$

exhibentur. Etiamsi statim in literis operari liceat (vti HINDENBVRGIVS in recentiori opere solet), iis tamen, qui nondum satis in his calculis exercitati sunt, consultum iri existimaui, operationes in numeris praemittendo, atque ab his ad literas progrediendo: quare etiam in exemplis sequentibus, praecepta generalia illustrantibus, vtrumque operandi modum coniungendum duxi.

2) Loco

thode und ihrer Anwendung auf die Analysis verfehen von C. F. Hindenburg, Leipzig 1796. 8. pag. 183. §. 42. Delineatio doctrinae combinatoriae in hoc libro exhibita, eo etiam praestat, quod *Hindenburgius* egregia simplicitate et ἀκριβείᾳ regulas *pure combinatorias* tradiderit, hasque regulis *mixtis* siue ex parte *arithmeticis* prius adhibitis substituerit. Operationes nimirum pure combinatoriae peraguntur, elementa data adiiciendo tantum, vel demendo, permutando, rite disponendo: quin operationibus arithmeticis, veluti additione ac subtractione, opus sit.

(m) Elementa, proxime maius et minus, appellari etiam possunt, sequens et praecedens. Istis similibusque deinceps adhibendis formulis loquendi arithmeticis, regularum pure combinatoriarum indolem haud affici, nec eas in arithmeticas transmutari, satis manifestum est. Elementa *proxime aequalia* breuitatis gratia ea appellare liceat, quae vnitate tantum differunt, seu in alphabeto sibi inuicem proxima sunt.

2) Loco huius regulae pure combinatoriae, aliam prius tradiderat HINDENBVRGIVS (n), quae, sub *forma magis arithmetica* (o) expressa, huc fere redit:

Classis quaeuis ex praecedente deriuatur, dum in singulis complexionibus huius classis numeri versus dextram extremi (reliquis manentibus inuariatis) discerpuntur in duas partes, hac addita conditione, vt ne partes priores successiue vnitate crescentes minores sint elementis in eadem complexione proxime praecedentibus, vel maiores fiant partibus posterioribus continuo vnitate decrescentibus.

3) Cum ratio legum praescriptarum, quas ipsa quasi natura problematis dictitauit, attentiori statim in oculos incurrat, *demonstrationem* addere vix opus esse videtur, quam tamen, ne quid deesse videatur, breuiter exponam (p).

Primo complexiones omnes rite ordinatas esse exinde apparet, quod ipsa operandi lege impediatur, ne vnquam elementum praecedens maius sit sequenti.

Deinde classes ipsas itidem rite ordinatas prodire, sic efficitur. Ordines sequentes ex praecedentibus secundum seriem elementorum initialium crescentium deriuantur, vnde res omnis redit ad ordines primos; cum vero ordinis cuiusque primi complexiones ex classe proxime praecedente petantur, illi vti numeri crescentes progredientur, dummodo haec classis rite sit ordinata. Sic tandem ad classem secundam recurritur, quam rite ordinatam esse liquet.

Complexiones *denique* in quauis classe *omnes* haberi, sic perspicitur. Quodsi deficeret ordinis cuiuspiam r complexio rite ordinata haec: r, s, . . . u, x; tum deficere simul deberet ordinis praecedentis complexio: r — 1, s, . . . u, x + 1 (ex 1, b, β); vnde regrediendo ex ordine 1 aliqua complexio deficeret, haec nimirum 1, s, . . . u, x + r — 1. Quare etiam in classe praecedente deficere deberet complexio s, . . . u, x + r — 1 (ex 1, b, α). Hinc sponte sequitur, ascendendo ad classes anteriores, classem secundam fore incompletam, quod est absurdum.

Simili ratione regulam (2) demonstrare licet (q).

4) Obseruatione dignum est, quod complexiones *summae cuiusquam* n praedicto modo (1, 2) secundum classes dispositae exhibeant seu *inuoluant* simul complexiones *summarum*

(n) *Infinitin. Dign.* p. 73. sq.

(o) regulas secundum *formam* posse esse *arithmeticas*, et tamen *re ipsa combinatorias*, liquet etiam ex iis, quae monet *Hindenburgius* (*Archiv IV. Heft* p. 395. §. 13.)

(p) In sequentibus etiam *regulas rite et vniuersaliter demonstrare* omni cura allaboraui, quam lectores intelligentes haud superfluam, nec me in eo vel post *Hindenburgium* acta tantum egisse iudicabunt, (cf. quae *Fischerus* de regulis a *Toepfero* propositis monet: *Ursprung der Theorie der Dimensionszeichen und ihr Verhältniss gegen die combinatorische Analytik des Herrn Prof. Hindenburg*, *Halle* 1794. 8. pag. 26. §. 40).

(q) cf. *Hindenburg Infinit. Dign.* pag. 76.

marum omnium minorum n — 1, n — 2, n — 3 ..., n — r, ... 2, 1, ſimiliter ordinatas. Separetur nimirum terminus extremus complexionis primae in claſſe r + 1ta, qui eſt n — r, per lineam verticalem, eaque linea producatur per claſſes omnes ſequentes: tum elementa ad dextram ſeparata ea, quae a laeua praecedit elementum 1, conficient complexiones omnes ſummae n — r (r).

Exemplum.

5) *Index* $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \ldots \\ a, & b, & c, & d, & e, & f, & g, & h, & i \ldots \end{pmatrix}$

^9A	=	9	=	i
^9B	=	18	=	ah
		27		bg
		36		cf
		45		de
^9C	=	117	=	aag
		126		abf
		135		ace
		144		add
		225		bbe
		234		bcd
		333		ccc
^9D	=	111|6	=	aaa|f
		112|5		aab|e
		113|4		aac|d
		122|4		abb|d
		123|3		abc|c
		222|3		bbb|c
E^9	=	111|15	=	aaa|ae
		111|24		aaa|bd
		111|33		aaa|cc
		112|23		aab|bc
		122|22		abb|bb
F^9	=	111|114	=	aaa|aad
		111|123		aaa|abc
		111|222		aaa|bbb
G^9	=	111|1113	=	aaa|aaac
		111|1122		aaa|aabb
H^9	=	111|11112	=	aaa|aaaab
I^9	=	111|111111	=	aaa|aaaaaa

B) Solu-

(r) *Hindenburg Infinit. Dign.* pag. 79. cf. *Archiv I. Heft* pag. 26.

B) Solutio altera.

Inuentio complexionum omnium, ordine lexicographico dispositarum, siue Inuolutio summae lexicographica.

1) Complexiones summarum 1, 2, 3, 4, ... vsque ad summam propositam successiue ita exhibere licet, vt complexiones cuiusuis summae comprehendant simul siue inuoluant complexiones summarum omnium minorum. *Regula* ad hunc finem accommodata eaque pure combinatoria, haec est:

a) Pro summa 1 ponitur, instar vnius complexionis, ipsum elementum 1, siue a.

b) Complexiones summae cuiusuis n ex complexionibus summae praecedentis n — 1 deriuantur, dum:

α) Singulis his complexionibus praemittitur elementum 1; deinde

β) in complexionibus eiusdem summae n — 1, quae quidem duo elementa priora diuersa habent, primum elementum cum proxime maiori permutatur; ac complexiones ita oriundae suo ordine complexionibus ex (α) deriuatis deorsim adiiciuntur (s).

2) Regula (1) sic *demonstratur.* Primo quidem complexiones singulae rite ordinatae prodeunt: quoniam per ipsam regulam (1, b) cauetur, ne vnquam elementum maius praecedat; minusue sequatur. Iam porro, si complexio aliqua rite ordinata summae n, veluti r, s, ... x, deficeret, tum deficere simul deberet summae n — 1 complexio r — 1, s, ... x, vel pro r = 1, complexio s, ... x, quippe harum complexionum alterutra illam, vi regulae (1. b), produxisset. Sic vlterius ad summas continue minores regrediendo, pro summa tandem 1 complexio deficeret, id quod est absurdum. Hinc complexiones omnes rite ordinatas omnium classium pro qualibet summa reperiri, siue solutionem completam esse, manifestum est.

3) *Ex-*

(s) *Polyn. Lehrs.* p. 184. §. 43.

3) *Exemplum* prius (A. 5), ſecundum regulam expoſitam (1) reſolutum, ita ſe habet:

1 1 1 1 1 1 1 1 1	a a a a a a a a a
1 1 1 1 1 1 1 2	a a a a a a a b
1 1 1 1 1 1 3	a a a a a a c
1 1 1 1 1 2 2	a a a a a b b
1 1 1 1 1 4	a a a a a d
1 1 1 1 2 3	a a a a b c
1 1 1 1 5	a a a a e
1 1 1 2 2 2	a a a b b b
1 1 1 2 4	a a a b d
1 1 1 3 3	a a a c c
1 1 1 6	a a a f
1 1 2 2 3	a a b b c
1 1 2 5	a a b e
1 1 3 4	a a c d
1 1 7	a a g
1 2 2 2 2	a b b b b
1 2 2 4	a b b d
1 2 3 3	a b c c
1 2 6	a b f
1 3 5	a c e
1 4 4	a d d
1 8	a h
2 2 2 3	b b b c
2 2 5	b b e
2 3 4	b c d
2 7	b g
3 3 3	c c c
3 6	c f
4 5	d e
9	i

Significatio linearum horizontalium et verticalium mox explicabitur.

4) Quod ad *ordinem* attinet, ex quo complexiones cuiusuis ſummae, ſecundum regulam (1) inuentae, inter ſe progrediuntur: huius ordinis indoles breuiter ac apte omnino exprimitur, dum is vocatur *lexicographicus*. Quodſi nimirum fingamus, complexiones, tanquam aggregata literarum, vocabulorum vim habere, tum haec *quaſi verba* (t) in lexico iſtiusmodi *linguae fictae* ſimili ordine ſibi inuicem ſuccedent, quo iſtae complexiones ad regulam (1) diſpoſitae progrediuntur: i. e. complexioni priori vel poſteriori reſpondebit etiam verbum in lexico praecedens vel ſubſequens. Quem conſenſum ſic demonſtrare licet. Supponamus, ordinem talem alphabeticum pro complexionibus ſummae

n — 1

(t) cf. *Archiv* II. p. 166.

n — 1 obtinere, tum praemittendo iis elementum 1 siue a, pro complexionibus summae n inde deriuatis (ex 1, b, α) iste ordo adhuc seruabitur; nec idem turbabitur, permutando in complexionibus summae n — 1 elementum primum cum proxime maiori, pro complexionibus ceteris summae n (ex 1, b, β), quas ipsas quippe ab elemento a liberas prioribus cum a incipientibus loco posteriores esse debere, manifestum est. Cum igitur conclusio a summa n — 1 ad summam n valeat, facile apparet, pro omnibus summis complexiones secundum ordinem alphabeticum seu lexicographicum progredi.

5) Earundem complexionum ordinem alia insuper ratione arithmetica considerare licet, dum abstrahatur a *lexicis* et *verbis*, atque respiciatur ad *numeros* atque *systemata numerorum*, verum non tantum systema nostrum decadicum, sed etiam, vti supra § XVII. 6, systemata altiora, plures figuras numerosue simplices habentia. Concipiatur nimirum, singularum complexionum elementa numerica, tanquam cifras seu figuras alicuius systematis, referre fractiones ex lege huius systematis, quales in nostro systemate vocantur *decimales*, notanturque praemittendo figuris zero cum commate: tum ex ordine literarum transeundo ad ordinem numerorum, sponte apparet, *fractiones minores praecedere, maiores sequi*, seu illas respondere complexionibus loco prioribus, has posterioribus. Cum igitur in dispositione *secundum Classes*, quam Solutio prima (A) ostendit, complexiones vti *numeri integri crescentes* procedant, (non tantum quamuis classem, quippe rite ordinatam, seorsim considerando, verum etiam ad classium inter se progressum respiciendo): in dispositione altera *lexicographica* complexiones vti *numeri fracti ex lege alicuius systematis* (u), *itidem crescentes*, progrediuntur.

Hinc itaque consensus et nexus memorabilis inter dispositionem secundum classes, et lexicographicam patet. Porro in illa dispositione (A), complexiones cuiusuis classis inter se ordinem etiam lexicographicum seruant: quaeuis enim classis rite ordinata simul est lexicographice disposita. Ex altera parte, in dispositione lexicographica colligendo complexiones eiusdem classis (siue elementorum multitudinis), deorsum eundo, classes apparent rite ordinatae, quod quidem ipse ordo lexicographicus secum fert. Sicque ab ordine alphabetico statim atque sponte transitus patet ad ordinem classium (v).

6) Modum (1) hactenus expositum ac illustratum, inueniendi atque exhibendi complexiones summae datae, appellat HINDENBVRGIVS *Inuolutionem lexicographicam*: *Inuolutionem* quidem ideo, quoniam in complexionibus istis summae datae simul complexiones summarum omnium praecedentium ita *inuolutae* sunt, vt hae per angulos siue lineas verticales et horizontales (in exemplo nr. 2. adpositas) ex illis separari seu resecari queant: cum contra complexiones summarum maiorum vel sequentium definire liceat, inuolutionem vel figuram iam descriptam extendendo tantum in latum ac profundum (w).

Quan-

(u) *Systems - Brüche* ex determinatione *germanica Hindenburgii* (*Archiv* II. p. 166. not. *).
(v) *Polyn. Lehrs.* p. 198. §. 60.
(w) *Archiv* I. 13.

Quanquam haec solutio per inuolutionem *dependens* esse videatur, quippe ad summas praecedentes *recurrendum* est: eadem tamen *independenti* (x) omnino aequiparatur, quoniam complexiones quaesitae pro quauis summa ita obtinentur, vt nihil superflui sit admixtum, i. e. nihil scribatur, quod mediatum duntaxat ac praeparatorium vsum haberet, nec ad rem ipsam quaesitam proxime atque essentialiter pertineret. In quo consistit virtus singularis ac propria Inuolutionum.

7) Modus in solutione *prima* (A) expositus, inueniendi complexiones summae cuiusquam n, ab HINDENBVRGIO itidem adpellatur *inuolutio*, et quidem *inuolutio classium*, siue *secundum classes* disposita. Ratio huius denominationis ex obseruatione supra (A, 4.) commemorata apparet: quippe reuera complexiones summae n *inuoluunt* simul complexiones summarum omnium minorum. Cum vero complexionibus summarum huiusmodi minorum per lineas verticales separatis admixtae sint *superfluae* siue *inutiles* (eae nimirum omnes, quae a sinistra non praecedit elementum 1): cum porro ad complexiones summarum maiorum non sic progredi liceat, vt ex lege constante nouos tantum terminos *adiicere*, vel inuolutionem *extendere* opus sit; quae inquam cum ita sint, dispositioni secundum classes forma inuolutoria adscribi nequit sensu eo strictiori, quo dispositio lexicographica vocatur Inuolutio (6). Quare illa dispositio habenda saltem videtur pro inuolutione *imperfecta* (y).

8) Quo

(x) Formulae *dependentes* siue *recurrentes* ab *independentibus* eo differunt, quod illae quantitatem quaesitam, quae ceu terminus seriei alicuius consideratur, per huius seriei terminos antecedentes exprimant: independentes contra eandem immediate exhibeant, quarum igitur formularum auxilio quemuis seriei terminum seorsim extra ordinem assignare licet, cum in formulis alterius generis recursu ad terminos priores opus sit, et singuli termini successiue tantum, suo quiuis ordine, prodeant, sic quidem, vt dum quaeritur certus terminus, definiendi prius sint alii, quos ipsos nosse forte non, saltem nunc non, interest. Hoc igitur sensu et respectu formulae dependentes seu recurrentes non immediate, sed per ambages atque operationes superfluas ad finem perducunt: illae autem formulae, quae nil superflui admixtum habent, vel independentes sunt, vel his aequipollent.

(y) Inuolutiones secundum classes esse *imperfectas*, id monuit etiam censor libri de *theoremate polynomiali* in Ephemeridibus literariis Ienensibus (*Allgemeine Literatur-Zeitung, 1796.* Nr. 381.), vir profecto Analyseos combinatoriae insigniter peritus. Verum de argumento, quod ille profert, equidem paullo aliter sentio. Etenim *anguli* (*Winkelhaken*) ad constituendam ac ostendendam formam inuolutoriam haud necessario requiri videntur. In iis quidem dispositionibus, quas *Hindenburgius* inuolutionibus perfectissimis adnumerat (*Polyn. Lehrs.* p. 202. 204. cf. infra §. XX. XXI.), inuolutiones inferiores separantur, ducendo tantum lineas horizontales. Inuolutionis imperfectae exemplum aliud deinceps occurret. Ceterum cum vix modum aptum concipere liceat, complexiones omnes summae cuiusquam exhibendi, qui non ostendat simul vel inuoluat, ex parte saltem, complexiones summarum minorum: haud incongruum videtur, vocabulo, praeter sensum strictiorem, latiorem etiam concedere, sic quidem vt *Inuolutio* summae n *sensu latiori* denotet *Aggregatum omnium complexionum rite ordinatarum huius summae*, quocunque demum modo eae complexiones sint dispositae. *Inuolutionis* contra *sensu strictiori* duplex maxime virtus est: 1) quod exinde

8) Quo rite inuicem discernantur inuolutiones secundum classes, ac inuolutiones lexicographice dispositae, illas litera I, has charactere J denotat HINDENBVRGIVS. Est igitur

$$ {}^{n}I = {}^{n}A + {}^{n}B + {}^{n}C + \ldots + {}^{n}N $$

$$ \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ a, & b, & c, & d, & \ldots \end{pmatrix} $$

vbi literae A, B, C, ... N significant classes 1mam, 2dam, ... ntam; porro

$$ {}^{2n}J = {}^{2n}\mathfrak{A} + {}^{2n}\mathfrak{B} + {}^{2n}\mathfrak{C} \ldots + {}^{2n}\mathfrak{N} + {}^{2n}\overset{n}{\mathfrak{N}}, $$

$$ {}^{2n+1}J = {}^{2n+1}\mathfrak{A} + {}^{2n+1}\mathfrak{B} + {}^{2n+1}\mathfrak{C} \ldots + {}^{2n+1}\mathfrak{N} + {}^{2n+1}\overset{n+1}{\mathfrak{N}}, $$

vbi literis $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, ... exprimuntur diuersi *ordines* inuolutionis lexicographicae, dum nimirum complexiones incipientes elementis 1 vel a; 2 vel b; 3 vel c; ... referuntur ad ordinem primum a; alterum b; tertium c; ... (z). Litera $\mathfrak{N}$ denotat ordinem ntum, literae $\overset{n}{\mathfrak{N}}$ et $\overset{n+1}{\mathfrak{N}}$ vi exponentium distantiae n et n+1, ordines (n+n)tum = 2ntum, et (n+n+1)tum = (2n+1)tum.

In formula pro ${}^{n}I$ literae A, B, C, ... a prima vsque ad ntam omnes adsunt; in formulis autem binis pro ${}^{2n}J$ et ${}^{2n+1}J$ occurrit lacuna inter literas penultimam et vltimam, quae haud difficulter explicatur, quippe manifestum est, in inuolutione lexicographica summae 2n vel 2n+1, ex ordine nto vnam tantum adesse complexionem duorum elementorum, hancque sequi complexionem vnam monadicam ex ordine 2nto vel 2n+1to (a).

C) So-

exinde inuolutiones summarum minorum lineis ita separare liceat, vt figurae resectae nil superflui admixtum sit, nec etiam quicquam deficiat; deinde 2) quod ad summam proxime maiorem *progressus* sic pateat, vt partes omnes iam paratae immediate sint vtiles, iisque ex lege constanti nouas tantum adiicere, siue figuram supplemento extendere opus sit. Quoad primum requisitum, sc. *regressum* ad summas minores, inuolutio classium laborat *excessu*, inuolutio autem lexicographica inuersa Boscovichiana (§. XXIV.) *defectu*.

(z) cf. *Archiv* IV. p. 397.

(a) De insigni compendio, quod praebent formae inuolutoriae, distinctius loquitur *Hindenburgius* primum in collectione inscripta: *Leipziger Magazin zur Naturkunde, Mathematik und Oekonomie, herausgegeben von C. B. Funk, N. G. Leske und C. F. Hindenburg, Jahrg. 1781.* p. 461. *Jahrg. 1782.* p. 440. cf. *Leipziger Magazin für reine und angewandte Mathematik, herausgeg. v. J. Bernoulli u. C. F. Hindenburg, Jahrg. 1786.* p. 323. not. x.

Inuolutio imperfecta classium (7) breuiter iam descripta extat in eius Tractatu *de Infinit. Dignit.* etc. p. 79.; atque dispositio figurata haud absimilis occurrit in eiusdem libro: *Beschreibung einer ganz neuen Art, nach einem bekannten Gesetze fortgehende Zahlen durch Abzählen oder Abmessen bequem und sicher zu finden, Leipz. 1776.* p. 97. 99. sq. *Idem* deinceps amplius pertractauit argumen-

C) Solutio tèrtia.

Classis cuiusuis complexionum datae summae, extra ordinem, inuentio.

1) Quodsi non omnes complexiones datae summae, sed tantum complexiones certae cuiuspiam classis desiderantur, tum sequenti modo procedendum est (b):

a) Elementum *maximum* (c), quod classis pro data summa admittit, primum ponitur; huic praemittitur elementum 1: ex hac binione ordinis 1 biniones reliquorum ordinum deriuantur, permutando successiue primum elementum alicuius binionis cum proxime maiori, alterum cum proxime minori, vsque dum duo elementa vel proxime vel omnino aequalia prodeant.

b) Ex hisce binionibus terniones, ex ternionibus quaterniones, porro classes altiores summarum crescentium, et sic tandem complexiones classis quaesitae pro summa data obtinentur, obseruando hanc *regulam:*

α) Singulis complexionibus classis modo inuentae praemittitur elementum 1 siue a, sicque obtinetur nouae classis ordo 1.

β) Ex

gumentum de inuolutionibus in *Archivi mathem. fasc. I*, (*Ueber combinatorische Involutionen und Evolutionen, und ihren Einfluss auf die combinatorische Analytik*, pag. 13-46.), nec non in fasciculis sequentibus, vbi in primis applicatio inuolutionum ad fractiones continuas, nec non ad formulas *Moivrei* et *Boscovichii* pro polynomii potestatibus vberrime exposita est.

Analogon formae inuolutoriae, licet imperfecte expressae, reperitur apud *Leibnitium* (in *Arte Combinatoria*, *Francof. 1690. 4.* p. 58.), qui in exhibendis permutationum singulis speciebus. (vt propriis ipsius verbis vtar), *quasi limitibus* (expressis per lineas verticales et horizontales) *distinguit variationes exponentis antecedentis ab iis quas superaddit sequens.*

Aliud exemplum, in quo *Leibnitius* figura inuolutionibus simili, diuerso autem consilio, vsus est, commemoratur ab *Hindenburgio* (*Archiv* II. p. 245. not. **, cf. *Leibnit. opp.* T. II. p. 392. 352.).

Ceterum ex *dispositionibus figuratis* Analysin vberiores adhuc fructus, ac hactenus quidem ea exinde percepit, capere posse, vix dubium est. Praeter inuolutiones specimen satis notum alius generis offert *Parallelogrammum Neutonianum.* Quod ipsas inuolutiones attinet, cum hucusque prolatae sint *planae*, quae in longum et latum extenduntur, mente concipi possunt etiam inuolutiones *solidae*, siue trium dimensionum: sic vt partes ad certum finem coniungendae sitae sint omnes in plani alicuius vel horizontalis vel verticalis vel diagonalis interfectione, siue eae etiam solidum tali plano terminatum compleant. Meminisse hic iuuat functionum plurium variabilium, atque tabularum illis computandis inseruientium. Porro etiam singulae partes alicuius inuolutionis rursus tanquam inuolutiones considerari possunt.

(b) *Polyn. Lehrs.* p. 188. §. 51.

(c) Pro classi k^{ta} summae n est elementum maximum $= n - (k - 1) = n - k + 1$; hoc igitur *arithmetice* definiendum est, eoque respectu solutio non omnino *pure combinatoria*, sed *mixta* esse videtur. Quomodo pro *definita* seu limitata serie elementorum 1, 2, 3, ... n, vbi $m < n - k + 1$, classis k^{tae} summae n complexiones eruendae sint, ostendit *Toepferus* (p. 83-86.). cf. *Iac. Bernoulli Ars coniectandi* p. 21; *Nou. Comment. Petrop.* T. XIV; *L. Euler de partitione numerorum in partes tam numero quam specie datas*, p. 168-187; *Hindenburg Infinit. Dign.* p. 86. 87.

β) Ex ordine 1 prodit ordo 2 siue b, ex hoc ordo 3 siue c, et sic porro, dum in iis complexionibus ordinis praecedentis, quae elementa duo priora simulque posteriora diuersa habent, elementum primum cum proxime minori permutatur.

2) *Exempli* gratia Classis quinta summae [illegible], ^{15}E, secundum hanc regulam sic reperitur:

$$^{15}E\begin{pmatrix}1 & 2 & 3 & 4 & \ldots & 9 & 10 & 11 & \ldots\\ a, & b, & c, & d, & \ldots & i, & k, & l & \ldots\end{pmatrix}$$

= 1	1	1	1	11	=	a	a	a	a	l
1	1	1	2	10		a	a	a	b	k
1	1	1	3	9		a	a	a	c	i
1	1	1	4	8		a	a	a	d	h
1	1	1	5	7		a	a	a	e	g
1	1	1	6	6		a	a	a	f	f
1	1	2	2	9		a	a	b	b	i
1	1	2	3	8		a	a	b	c	h
1	1	2	4	7		a	a	b	d	g
1	1	2	5	6		a	a	b	e	f
1	1	3	3	7		a	a	c	c	g
1	1	3	4	6		a	a	c	d	f
1	1	3	5	5		a	a	c	e	e
1	1	4	4	5		a	a	d	d	e
1	2	2	2	8		a	b	b	b	h
1	2	2	3	7		a	b	b	c	g
1	2	2	4	6		a	b	b	d	f
1	2	2	5	5		a	b	b	e	e
1	2	3	3	6		a	b	c	c	f
1	2	3	4	5		a	b	c	d	e
1	2	4	4	4		a	b	d	d	d
1	3	3	3	5		a	c	c	c	e
1	3	3	4	4		a	c	c	d	d
2	2	2	2	7		b	b	b	b	g
2	2	2	3	6		b	b	b	c	f
2	2	2	4	5		b	b	b	d	e
2	2	3	3	5		b	b	c	c	e
2	2	3	4	4		b	b	c	d	d
2	3	3	3	4		b	c	c	c	d
3	3	3	3	3		c	c	c	c	c

3) Deductio hic tradita exhibet simul *inuolutionem:* quaeuis nimirum classis inuoluit classes praecedentes pro summis continue decrescentibus; quae forma inuolutoria in exemplo (2) per angulos expressa est.

4) *Com-*

4) *Complexiones rite ordinatae* prodeunt, quoniam ipſa regula (1) prohibet, quominus vnquam elementum maius praecedat. Porro ipſae *claſſes rite ordinatae* ſunt: ſi enim claſſis inuoluta proxime praecedens rite eſt ordinata, tum idem etiam locum habebit quoad complexiones ordinis 1 claſſis ſequentis, indeque etiam quoad ceteras complexiones, quarum vnusquisque ordo ex praecedente ordine deriuatur.

5) *Alia regula magis arithmetica* ſiue mixta, quamuis claſſem extra ordinem inueniendi, haec eſt (e):

a) Initium fit a complexione ſimpliciſſima, quae ceterarum tanquam numerorum eſt minima, quaeque tot conſtat vnitatibus, demta vna, quot claſſis habet partes, quibus deinceps vnitatibus adiicitur ſummae praeſcriptae complementum.

b) Ex hac complexione nunc ſecundam, ex ſecunda tertiam, porro ex quauis proxime ſequentem ſic definire licet:

α) Quoties elementa duo poſteriora alicuius complexionis non ſint vel plane vel proxime aequalia, tunc ad obtinendam ſequentem elementum penultimum vnitate augetur, poſtremum minuitur.

β) Contra ſi accidat, ad elementa illius complexionis praecedentia progredi oportet, donec ad elementum perueniatur, quod ab vltimo magis quam vnitate differat, tumque illud vnitate augetur, aucto aequalia ad dextram adiiciuntur (elementis ad ſiniſtram manentibus iisdem), poſtremo autem loco ponitur ſummae complementum. Si iſte proceſſus non amplius locum habeat, tum claſſis eſt completa (f). Regulae huic illuſtrandae inſeruit praecedens exemplum (a): illa nimirum complexiones easdem eodem ordine producit, ac regula prior (1), i. e. complexiones et claſſes rite ordinatas.

6) Hanc regulam arithmeticam ſub forma paullo aliter expreſſa exhibuit HINDENBVRGIVS (g), quae huc fere redit:

Ex

(e) *Toepfer* p. 80. 81.

(f) Sit complexio aliqua praecedens haec: $\ldots s-r,\ s-\rho^{I},\ s-\rho^{II},\ s-\rho^{III},\ \ldots s-\rho^{N},\ s1$ vbi $\rho^{I}, \rho^{II}, \rho^{III}, \ldots \rho^{N}$ ſunt $=0$ vel $=1$, r autem >1; tum complexio proxime ſequens erit: $\ldots s-r+1,\ s-r+1,\ s-r+1,\ s-r+1,\ \ldots s-r+1,\ s+x$. Quo ſumma elementorum eadem maneat, eſſe debet $x=-\rho^{I}-\rho^{II}-\rho^{III}\ldots-\rho^{N}+n(r-1)-1$, hinc pro iisdem n et r valor τῶ x minimus eſt $=-n+n(r-1)-1=n(r-2)-1=$ $=-r+1+(n+1)(r-2)$, i. e. ſemper maior valore $-r+1$, vel huic aequalis pro $r=2$. Quare complementum ſummae, $s+x$, ſemper maius eſt elemento praecedenti, $s-r+1$, vel huic aequale pro $r=2$. Haec quidem ad regulam illuſtrandam atque comprobandam afferre, haud ſuperfluum mihi videbatur.

(g) *Infinit. Dign.* p. 80. 81.

Ex prima complexione deriuantur proximae, variando, quoad licet, elementa bina poſtrema (5, b, α); tum elementum a fine tertium (1) vnitate augetur, aucto aequale (2) adiicitur, ac vltimo loco complementum ſummae. Ex hac complexione rurſus proximae deriuantur, variando elementa duo poſtrema; porro elementum tertium rurſus vnitate augetur, et prior proceſſus reiteratur; idemque continuatur, donec quartum elementum (1) vnitate augere opus ſit, cui rurſus aequalia (2) adiiciuntur, ac in fine complementum. Ex hac complexione rurſus nouae deducuntur, augendo ſucceſſiue ſecundum et tertium elementum; tandem ipſum quartum iterum augendum eſt. Sic ad praecedentia elementa progreſſus fit, ex lege vniformi hac, vt elemento cuilibet, poſtquam vnitate auctum fuerit, adiiciantur aequalia, (ad dextram, ad ſiniſtram nil mutando), ac in fine complementum ſummae, quod elemento praecedenti nunquam minus eſſe debet: vnde quoties elementum quodpiam vltimo vel omnino vel proxime aequale eſt, illud non amplius augendum, ſed ad elementum praecedens progrediendum.

Corollarium 1.

Reſolutio claſſis ſingularis altioris in ſummam claſſium plurium inferiorum.

§. XIX. 1) Claſſis quaelibet ſingularis ex §. XVIII. C. exhibita, ad aliam formam eamque magis compendioſam reuocari poteſt, adhibendo pro elemento primo a *exponentes repetitionis* (h), i. e. per numeros huic literae exponentium inſtar adſcriptos indicando, quoties ea in aliqua complexione ſit repetita. Sic pro exemplo ſuperiori (§. XVIII. C. 2.) habetur:

$$
{}^{15}E \;=\; \begin{matrix} a^4\,l \\ a^3\,(bk,\ ci,\ dh,\ eg,\ ff) \\ a^2\,(bbi,\ bch,\ bdg,\ bef,\ ccg,\ cdf,\ cce,\ dde) \\ a^1\,(bbbh,\ bbcg,\ bbdf,\ bccf,\ bcde,\ bddd,\ ccce,\ ccdd) \\ a^0\ bbbbg \end{matrix}
\qquad \begin{pmatrix} 1, & 2, & 3, & \ldots \\ a, & b, & c, & \ldots \end{pmatrix}
$$

dum ſpatii contrahendi cauſſa complexiones, cum quibus ſingulis elementum a aliquoties repetitum coniungi debet, in lineis horizontalibus ſuo ordine collocentur. Hae complexiones in eadem linea horizontali aequalem numerum elementorum, eandemque ſummam habent: ita quidem, vt deorſum eundo tam numerus iſte quam ſumma haec ſucceſſiue vnitate creſcant, quippe ille eſt numeri elementorum claſſis ſiue *exponentis claſſis* ad multitudinem τῶν a complementum, haec autem complementum ſummae integrae Claſſis ad exponentem repetitionis τȣ a. Quare in lineis horizontalibus, prima, ſecunda, tertia, quarta, quinta, occurrunt vnio, biniones, terniones, quaterniones, quinio, ſummarum

(h) *exponentes repetitionis* diſcernuntur ab exponentibus *poteſtatum*: in combinationibus nimirum plura elementa = a tantum iuxta ſe inuicem poſita, haud neceſſario per multiplicationem coniuncta intelliguntur (*Polyn. Lehrſ.* p. 189. §. 53.). Signum a^0 innuit, in complexione aliqua deficere elementum a.

rum 11, 12, 13, 14, 15: ·um hae complexiones ex elementis datis ita formentur, vt excludatur primum elementum a, quippe quod iam separatum ac factoris instar communis praemissum est.

Ex modo dictis sponte sequitur haec *formula generalis pro resolutione classis* n^{tae} *summae* $n+r$ (i):

$$\underset{\begin{pmatrix}1&2&3&4&\dots\\a,&b,&c,&d,&\dots\end{pmatrix}}{{}^{r+n}N} = a^{n-1}\,\underset{\begin{pmatrix}2&3&4&\dots\\b,&c,&d,&\dots\end{pmatrix}}{{}^{r+1}A} + a^{n-2}\,{}^{r+2}B + a^{n-3}\,{}^{r+3}C + \dots + a^{n-m}\,{}^{r+m}M + \dots$$

vbi complexiones ad dextram partem aequationis ab elemento a liberae accipiendae sunt, quare suppositus est index $\begin{pmatrix}2&3&4&\dots\\b,&c,&d,&\dots\end{pmatrix}$, quo innuitur, seriem elementorum primo tantum elemento truncatam esse, i. e. singula elementa eosdem exponentes siue comites numericos seruare, ac in classe primitiua, primum tantum elementum a combinationibus excludi.

2) Ex hac formula facile deriuatur alia resolutio simplicior vtiliorque classis altioris in aggregatum plurium inferiorum. Quodsi nimirum ad dextram partem aequationis (1) numeri elementis b, c, d, . . . respondentes singuli vnitate minuantur, siue index $\begin{pmatrix}2&3&4&\dots\\b,&c,&d,&\dots\end{pmatrix}$ mutetur in indicem $\begin{pmatrix}1&2&3&\dots\\b,&c,&d,&\dots\end{pmatrix}$; tum summae pro vnione, binionibus, ternionibus, . . . complexionibus cuiusuis m^{tae} classis, minuuntur vnitatibus, vna, duabus, tribus, . . . m, siue ${}^{r+m}M$ abit in ${}^{r}M$, quippe quoduis elementorum m minuendo vnitate, summa eorum perdit m vnitates (k). Hinc manifestum est, loco summarum crescentium in complexionibus formulae prioris (1) prodire summas inuicem aequales, cunctas $= r$; sicque haec obtinetur *formula resolutoria:*

$$\underset{\begin{pmatrix}1&2&3&4&\dots\\a,&b,&c,&d,&\dots\end{pmatrix}}{{}^{r+n}N} = a^{n-1}\,\underset{\begin{pmatrix}1&2&3&\dots\\b,&c,&d,&\dots\end{pmatrix}}{{}^{r}A} + a^{n-2}\,{}^{r}B + a^{n-3}\,{}^{r}C + \dots + a^{n-m}\,{}^{r}M + \dots$$

vbi aggregatum classium continuandum est, donec fiat $m = n$ vel $= r$, i. e. vsque dum per-

(i) *Polyn. Lehrs.* p. 219. §. 96.

(k) Haec transmutatio suppeditat hanc aequationem:

$$\underset{\begin{pmatrix}2&3&4&\dots\\b,&c,&d,&\dots\end{pmatrix}}{{}^{r+m}M} = \underset{\begin{pmatrix}1&2&3&\dots\\b,&c,&d,&\dots\end{pmatrix}}{{}^{r}M}. \quad \text{Similiter est:} \quad \underset{\begin{pmatrix}3&4&5&\dots\\c,&d,&e,&\dots\end{pmatrix}}{{}^{r+2m}M} = \underset{\begin{pmatrix}1&2&3&\dots\\c,&d,&e,&\dots\end{pmatrix}}{{}^{r}M};$$

et sic porro, statuendo exponentes summae a dextra parte aequationis successiue $= r+3m$, $r+4m$, $r+5m$, . . .

peruentum fuerit ad $a^0\ {}^{r}N$ vel ad $a^{n-r}\ {}^{r}R$, prouti fuerit $r > n$ vel $< n$. Sic exempli gratia est:

$$\underset{\begin{bmatrix}1 & 2 & 3 & 4 & \dots \\ a, & b, & c, & d, & \dots\end{bmatrix}}{{}^{15}E} = \underset{\begin{bmatrix}1 & 2 & 3 & 4 & \dots \\ b, & c, & d, & e, & \dots\end{bmatrix}}{a^4\ {}^{10}A + a^3\ {}^{10}B + a^2\ {}^{10}C + a^1\ {}^{10}D + a^0\ {}^{10}E};$$

$${}^{9}E = a^4\ {}^{4}A + a^3\ {}^{4}B + a^2\ {}^{4}C + a^1\ {}^{4}D.$$

3) Haud superfluum videtur, ex formula resolutoria modo inuenta (2) alias insuper easque satis concinne expressas deriuare. Est nimirum $A^r + {}^{r}B + {}^{r}C + \dots + {}^{r}R = {}^{r}I$, denotante ${}^{r}I$ summae r inuolutionem secundum classes dispositam (§. XVIII. B, 8.). Hinc ex formula (2), pro $r < n$ vel $= n$, sponte sequitur haec:

$$a)\ \underset{\begin{bmatrix}1 & 2 & 3 & \dots \\ a, & b, & c, & \dots\end{bmatrix}}{{}^{r+n}N} = \underset{\begin{bmatrix}1 & 2 & 3 & \dots \\ b, & c, & d, & \dots\end{bmatrix}}{a^{n-*}\ {}^{r}I}$$

dum asterisco * varii valores numerici tribuantur pro singulis diuersis classibus, quibus constat inuolutio, ita quidem vt ille semper sit aequalis exponenti classis, siue $* = 1, 2, 3, \dots r$, pro classibus, prima, secunda, tertia, ... rta (1). Quod si fuerit $r > n$, tum eandem formulam sub aliqua restrictione adhibere licet, dummodo nimirum notetur, exponentem repetitionis τ8 a non posse fieri negatiuum, siue asterisci valorem non maiorem quam n, hincque per ipsam rei naturam excludi Inuolutionis ${}^{r}I$ complexiones classium nta altiorum.

Cum sit ${}^{r}I = {}^{r}J$, (quippe vtraque inuolutio, tam quae secundum classes quam quae lexicographice est disposita, constat aggregato omnium complexionum rite ordinatarum summae n), in locum formulae praecedentis haec etiam substitui potest:

$$b)\ \underset{\begin{bmatrix}1 & 2 & 3 & \dots \\ a, & b, & c, & \dots\end{bmatrix}}{{}^{r+n}N} = \underset{\begin{bmatrix}1 & 2 & 3 & \dots \\ b, & c, & d, & \dots\end{bmatrix}}{a^{n-*}\ {}^{r}J}$$

vbi iam asterisci valor numericus semper aequalis sumendus est numero literarum vel elementorum in quauis complexione inuolutionis lexicographicae occurrentium. Hinc sponte apparet, secundum hanc formulam complexiones singulas classis ${}^{r+n}N$ alio inter se ordine pro-

(1) Manifestum est, a^{n-*} hoc loco haud accipiendum esse ceu vnum factorem, in ${}^{r}I$ ducendum; is potius exponens *repraesentat plures diuersosque factores*, *pluribus partibus* aggregati per ${}^{r}I$ expressi respondentes. Hoc sensu factores *asteriscis* instructi passim ab *Hindenburgio* commode adhibentur. (cf. *Polyn. Lehrs.* p. 265.)

progredi, quam ex formula priore (a), quae Claſſem rite ordinatam praebet: nec pro illa (b) exponentes τῶ a ſucceſſiue vnitatibus decreſcere, vti pro hac (a).

Denotando per $^{n}[C]$ aggregatum omnium complexionum rite ordinatarum ſummae n, quocunque ordine eae inter ſe procedant (m), formulas (a) et (b) ſub hac communi comprehendere licet:

$$c)\quad {}^{r+n}\underset{\begin{Bmatrix}1 & 2 & 3 & \ldots\\ a, & b, & c, & \ldots\end{Bmatrix}}{N} = \underset{\begin{Bmatrix}1 & 2 & 3 & \ldots\\ b, & c, & d, & \ldots\end{Bmatrix}}{a^{n-*}\,{}^{r}[C]}$$

vbi aſteriſci valor numericus = multitudini elementorum in quauis complexione ſumendus eſt.

4) Accuratius conſiderando inuolutionem, quam quaeuis claſſis (n^{ta}) ſecundum (§. XVIII. C.) exhibita ſiſtit, apparebit, talem claſſem in duas partes reſolui poſſe: quarum prior obtinetur, complexionibus ſingulis claſſis praecedentis ($n - 1^{tae}$) et ſummae proxime minoris ($n + r - 1$), praemittendo elementum 1 ſeu a; complexiones vero ab a liberas, quibus altera pars claſſis iſtius conſtat, deriuare licet ex complexionibus claſſis eiusdem (n^{tae}) pro ſumma proxime minori ($n + r - 1$), dum harum elementa prima, quae quidem a ſecundis diuerſa ſunt, cum proxime maioribus permutantur. Eadem haec reſolutio ſequitur ex lege ſupra (§. XVIII. C.) obſeruata, qua ab inuolutione ſummae cuiuspiam praecedentis ad inuolutionem ſummae ſequentis progreſſus fit: quae lex neceſſario etiam pro ſingulis harum binarum inuolutionum claſſibus ſeiunctim conſideratis obtinere debet.

Idem, *quin ad prius demonſtrata recurrendum ſit*, ſic facile oſtenditur. Supponitur nimirum, binas claſſes, ex quibus claſſis quaeſita componitur, in quasue haec reſoluitur, eſſe completas. Iam ſi pro hac claſſe, rite facta compoſitione, deficeret aliqua complexio r, s, ... x, tum pro $r = 1$ deficere deberet claſſium illarum prioris complexio s, ... x, pro $r > 1$ autem claſſium illarum alterius complexio $r - 1$, s, ... x, quod eſt contra hypotheſin. Manifeſtum porro eſt, claſſem compoſitam cum ſuis complexionibus rite ordinatam prodire, ſi claſſes componentes ita ſint ordinatae.

Quaeuis igitur claſſis m^{ta} ſummae ϱ ſiue $^{\varrho}M$ componitur ex claſſibus $^{\varrho-1}\overset{-1}{M}$ et $^{\varrho-1}M$, (denotante $\overset{-1}{M}$ claſſem $m - 1^{tam}$, vi (n) *exponentis diſtantiae* $- 1$), et quidem ex illius complexionibus ſingulis, dum iis praemittitur elementum 1, ex huius autem claſſis complexionibus iis tantum, quarum elementum primum a ſecundo diuerſum eſt, dum illud cum proxime maiori permutetur. Quae obſeruatio in ſequentibus vtilis erit.

Corol-

(m) de Signo $^{n}[C]$ cf. *Archiv* IV. p. 417; *Polyn. Lehrſ.* p. 265.

(n) de *exponentibus diſtantiae* eorumque vſu commodo ad deſignandos ſeriei alicuius terminos, per eorum diſtantiam relatiuam ab alio termino, cf. *Nov. Syſt. Perm.* p. XXVII ſq.

Corollarium 2.

Inuolutio fummae *indeterminatae* n, feruato *ordine claſſium*.

§. XX. 1) Adhibeantur pro inuolutione lexicographica (cuius legem et formam fupra §. XVIII. B. defcripfimus) exponentes repetitionis elementi primi a, ficque hoc elementum a fingulis complexionibus feparetur: tum fponte prodit haec aequatio pro inuolutione lexicographica fummae cuiusquam indeterminatae n (o):

$$\underset{\begin{bmatrix}1, & 2, & 3, & 4, & \ldots \\ a, & b, & c, & d, & \ldots\end{bmatrix}}{{}^{n}J} = a^{n-1}\,{}^{1}J + \underset{\begin{bmatrix}2, & 3, & 4, & \ldots \\ b, & c, & d, & \ldots\end{bmatrix}}{a^{n-2}\,{}^{2}J} + a^{n-3}\,{}^{3}J + a^{n-4}\,{}^{4}J + \ldots + a^{n-\varrho}\,{}^{\varrho}J + \ldots + a^{1}\,{}^{n-1}J + a^{0}\,{}^{n}J$$

vbi pro finiftra parte aequationis index prior, pofterior pro dextra valet. Veritas huius aequationis manifefta eft, *quin ad prius* (§. XVIII. B.) *oſtenſa recurrere opus ſit.* Conftat nimirum, in inuolutione lexicographica complexiones progredi fecundum exponentes elementi a fucceffiue vnitatibus decrefcentes, idque elementum $n-\varrho^{ies}$ repetitum coniungi debere cum fingulis complexionibus reliquorum elementorum fiue ab a liberis, quae fummam $=\varrho$, numeri exponentis $n-\varrho$ ad fummam inuolutionis $=n$ complementum, conficiunt, quaeque et ipfae lexicographice funt difpofitae, quarum igitur aggregatum per ${}^{\varrho}J$ pro indice $\begin{bmatrix}2, & 3, & 4, & \ldots \\ b, & c, & d, & \ldots\end{bmatrix}$ exprimendum eft.

2) Cum fit ${}^{\varrho}J = {}^{\varrho}I$, quippe vtraque inuolutio complectitur omnes et fingulas complexiones rite ordinatas fummae ϱ, ponere licebit:

$${}^{\varrho}J = {}^{\varrho}A + {}^{\varrho}B + {}^{\varrho}C + {}^{\varrho}D + \ldots$$

quod quidem aggregatum claffium pro indice $\begin{bmatrix}2, & 3, & 4, & \ldots \\ b, & c, & d, & \ldots\end{bmatrix}$ continuandum eft vsque ad claffem λ^{tam}, exiftente $\lambda = \frac{\varrho}{2}$ vel $= \frac{\varrho-1}{2}$, prouti ϱ fuerit numerus par vel impar: quoniam elementorum 2, 3, 4, . . . plura quam λ coniunctim praebent fummam maiorem quam ϱ. Hinc in formula (1) repetitiones τȣ a cum exponentibus decrefcentibus in ferie deorfum fcribendo, iuxtaque in lineis horizontalibus claffes cum illis coniungendas collocando, haec obtinetur aequatio:

${}^{n}J =$

(o) *Polyn. Lehrſ.* p. 219. §. 94.

$$
\begin{array}{ll}
{}^{n}J = {}^{n}I \begin{Bmatrix} 1 & 2 & 3 & 4 & \ldots \\ a, & b, & c, & d, & \ldots \end{Bmatrix} = & a^{n-1}a \\
& + a^{n-2}\,{}^{2}A \\
& + a^{n-3}\,{}^{3}A \\
& + a^{n-4}(\,{}^{4}A + {}^{4}B) \\
& + a^{n-5}(\,{}^{5}A + {}^{5}B) \\
& + a^{n-6}(\,{}^{6}A + {}^{6}B + {}^{6}C) \\
& + a^{n-7}(\,{}^{7}A + {}^{7}B + {}^{7}C) \\
& + a^{n-8}(\,{}^{8}A + {}^{8}B + {}^{8}C + {}^{8}D) \\
& + a^{n-9}(\,{}^{9}A + {}^{9}B + {}^{9}C + {}^{9}D) \\
& + a^{n-10}(\,{}^{10}A + {}^{10}B + {}^{10}C + {}^{10}D + {}^{10}E) \\
& + a^{n-11}(\,{}^{11}A + {}^{11}B + {}^{11}C + {}^{11}D + {}^{11}E) \\
& + \quad . \quad . \quad . \quad . \quad . \\
& + a^{n-r}(\,{}^{r}A + {}^{r}B + {}^{r}C + \ldots + {}^{r}M + \ldots) \\
& + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
& + a^{0}(\,{}^{n}A + {}^{n}B + {}^{n}C + {}^{n}D + \;.\;.\;.\;.\;.\;.\;) \\
& \quad \begin{Bmatrix} 2 & 3 & 4 & . & . & . \\ b, & c, & d, & . & . & . \end{Bmatrix}
\end{array}
$$

3) Quo clarius intelligatur, fecundum quam legem claffes in lineis horizontalibus ex fe inuicem deriuari ac pro lubitu continuari queant, conuenit indicem $\begin{Bmatrix} 2 & 3 & 4 & \ldots \\ b, & c, & d, & \ldots \end{Bmatrix}$ eodem modo ac §pho praecedente, in hunc mutare: $\begin{Bmatrix} 1 & 2 & 3 & \ldots \\ b, & c, & d, & \ldots \end{Bmatrix}$; quo facto quaeuis claffis ${}^{r}M$ abit in ${}^{r-m}M$ (§. XIX. 2.). Expreffio pro ${}^{n}I$ (2) hunc in modum transformata difponatur fecundum figuram adfcriptam:

${}^{n}I =$

$^{n}\mathrm{I} = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ a, & b, & c, & d, & \ldots \end{pmatrix}$

a^{n-1}	a				
a^{n-2}	^{1}A				
a^{n-3}	^{2}A				
a^{n-4}	^{3}A	^{2}B			
a^{n-5}	^{4}A	^{3}B			
a^{n-6}	^{5}A	^{4}B	^{3}C		
a^{n-7}	^{6}A	^{5}B	^{4}C		
a^{n-8}	^{7}A	^{6}B	^{5}C	^{4}D	
a^{n-9}	^{8}A	^{7}B	^{6}C	^{5}D	
a^{n-10}	^{9}A	^{8}B	^{7}C	^{6}D	^{5}E
a^{n-11}	[illegible]	^{9}B	^{8}C	^{7}D	^{6}E
etc.			etc.		
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ b, & c, & d, & e, & f, & g, & \ldots \end{pmatrix}$					

ita quidem, vt classes cum iisdem repetitionibus τȢ à singulatim coniungendae inter lineas horizontales *zonis*, classes autem homonymae inter lineas verticales *columnis* contineantur ac discernantur, sicque cuiuis classi singulari locus suus siue *cellula* assignetur. Iam in memoriam reuocanda est compositio antea (§. XIX. 4.) demonstrata cuiusuis classis certae summae ex duabus classibus summae proxime minoris. Hinc nimirum quaeuis cellula figurae nostrae oritur, praemittendo elementum 1, siue vi indicis literam b, singulis complexionibus in cellula columnae praecedentis et zonae duobus locis altioris, ac deinceps in cellula columnae non praecedentis sed eiusdem, et zonae vno tantum loco altioris, permutando complexionum elementa prima a secundis diuersa cum proxime maioribus. Hanc deductionem illustrat sequens figura, quae sistit inuolutionis *fragmentum generale* (p).

a^{n-r+2}

(p) verbum analogum denominationi *termini generalis* serierum.

a^{n-r+2}	${}^{r-3}A$	. . . .	${}^{r-m-1}\overset{-1}{M}$	
a^{n-r+1}	${}^{r-2}A$	. . . .		${}^{r-m-1}M$
a^{n-r}	${}^{r-1}A$	. . . .		${}^{r-m}M$

Hic nimirum ex lege modo expoſita claſſis ${}^{r-m}M$, in columna m^{ta} et zona $r-1^{ta}$, deriuatur ex claſſe ${}^{r-m-1}\overset{-1}{M}$, in columna $m-1^{ta}$, zona $r-3^{ta}$, ſimulque ex claſſe ${}^{r-m-1}M$, in columna m^{ta}, zona $r-2^{ta}$.

4) Quodſi claſſes ſingulae cellulis comprehenſae ex lege praeſcripta actu euoluantur, eaeque igitur cum ſuis complexionibus rite ordinatae prodeant, tum haec obtinebitur Inuolutio ſummae indeterminatae n (q):

(q) *Polyn. Lehrſ.* p. 204. §. 62.

 I^n

$$ {}^{n}1 \left(\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \ldots \\ a, & b, & c, & d, & e, & f, & g, & h, & i, & k, & l & \ldots \end{matrix} \right) $$

a^{n-1}	a				
a^{n-2}	b				
a^{n-3}	c				
a^{n-4}	d	b^2			
a^{n-5}	e	bc			
a^{n-6}	f	bd c^2	b^3		
a^{n-7}	g	be cd	b^2c		
a^{n-8}	h	bf ce d^2	b^2d bc^2	b^4	
a^{n-9}	i	bg cf de	b^2e bcd c^3	b^3c	
a^{n-10}	k	bh cg df e^2	b^2f bce bd^2 ^2d	b^3d b^2c^2	b^5
a^{n-11}	l	bi ch dg ef	b^2g bcf bde c^2e cd^2	b^3e b^2cd bc^3	b^4c
etc.			etc.		

Cuius itaque inuolutionis constructio hac lege pure combinatoria eaque simplicissima perficitur:

In serie verticali deorsum ponantur repetitiones τȣ a cum exponentibus decrescentibus; iuxta illas in prima columna elementa singula, tum cellulae reliquarum columnarum efformantur, dum

α) com-

(α) complexionibus singulis cellulae in columna praecedenti ac zona duobus locis altiore praemittatur elementum b, sine respectu indicis pro I$^{\text{u}}$ elementum 2;

(β) in complexionibus eius cellulae, quae modo descriptae (α) versus sinistram subiacet, elementa prima a secundis diuersa permutentur cum proxime maioribus;

γ) complexiones ex (α) et (β) oriundae, illae primum, deinceps hae, suo quouis ordine, in noua cellula collocentur.

Sicuti pro elemento primo a, ita etiam pro reliquis elementis b, c, d, . . . exponentes repetitionis adhibere licet: id quod inuolutionem magis compendiosam reddit.

5) Ex figura descripta (4) *separare licet inuolutiones omnes inferiores*, pro summis minoribus determinatis, e. g. pro $n = 10$, ducendo tantum lineam horizontalem infra a^0, e. g. a^{n-10} pro $n = 10$.

Complexiones supra hanc lineam scriptae conficiunt simul inuolutionem summae determinatae. Pari facilitate *progredi licet ad inuolutiones altiores*, pro summis maioribus, figuram descriptam extendendo, eiue adiiciendo nouos terminos ex eadem lege.

6) Quanquam complexiones, quae simul conficiunt summae inuolutionem hactenus expositam, haud immediate secundum classes dispositae appareant, eas tamen facillime hoc ordine disponere, sicque ex figura (4) inuolutionem secundum classes rite ordinatas statim deducere licet.

Complexiones nimirum ad eandem classem pertinentes collocatae sunt in *eadem linea diagonali*, quae pro classe prima per vltimum elementum in prima columna (iuxta a^0), pro secunda per penultimum (iuxta a^1), pro tertia per proximum, et sic porro, transit, quaeque semper deorsum producitur vsque ad lineam horizontalem infra a^0. Singulae classes deorsum in diagonalibus eundo rite ordinatae sunt, pro classe proxime altiori ad diagonalem a dextra adiacentem procedendum seu ascendendum est. Sic igitur quamuis etiam classem singularem ex inuolutione excerpere licet.

Vix opus est, vt moneam, inuolutionem eiusque classes hic pro indice $\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ a, & b, & c, & d, & \ldots \end{pmatrix}$ intelligi, et singulis complexionibus cellularum elementum a cum debito exponente, quem series verticalis prima ostendit, praemissum concipi debere. Assumto nunc indice altero: $\begin{pmatrix} 1, & 2, & 3, & 4, & \ldots \\ b, & c, & d, & e, & \ldots \end{pmatrix}$ quaeuis cellula classem singularem rite ordinatam sistit, et in singulis diagonalibus debite productis reperiuntur inuolutiones completae summarum crescentium 1, 2, 3, . . . secundum classes deorsum procedentes dispositae. Quod quidem ex figura (3) manifestum est.

Ceterum ex eadem figura apparet, fingulas feries tam verticales, quam horizontales et diametrales inuolutionis (4), ope eorum, quae §. XVIII. tradita funt, extra ordinem fiue independenter a praecedentibus exhiberi poffe.

Corollarium 3.

Inuolutio fummae *indeterminatae* n ordine *lexicographico*.

§. XXI. 1) Cum ex modo oftenfis in inuolutione §. XX. 4. defcripta ordo claffium fatis fit manifeftus, fiue indicem $\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ a, & b, & c, & d, & \ldots \end{pmatrix}$, fiue alterum $\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ b, & c, & d, & e, & \ldots \end{pmatrix}$ refpicias: difficilius contra videtur, iftam inuolutionem in lexicographicam transformare. Id quidem manifeftum eft, ob decrefcentes exponentes τȢ a, zonas integras inter fe ordine alphabetico progredi: quare oftendendum reftat, qua lege complexiones fingulae in eadem zona fiue inter lineas horizontales comprehenfae difponendae fint ita, vt eae etiam ipfae inter fe ordinem alphabeticum feruent.

2) Ex §. XX. 3. conftat, quamuis zonam determinari per zonas binas praecedentes, ita quidem, vt in illa occurrant complexiones omnes zonae duobus locis altioris, praemiffo iis elemento 2 ex indice fiue b, fimulque complexiones zonae proxime altioris eae, quae duo elementa priora diuerfa habent, elemento earum primo cum proxime maiori permutato. Exinde facile *regula* condi poteft, vi cuius complexiones cuiusuis zonae ex praecedentibus ftatim ita deriuantur, vt ordinem lexicographicum teneant.

Elemento nimirum 1, feu a, poftquam illud exponente rite inftructum eft, adiiciuntur complexiones iuxta fe inuicem in linea horizontali pofitae, hac lege:

α) Complexionibus fingulis lineae horizontalis duobus locis altioris praemittitur elementum 2, feu b;

β) Complexiones ita ortas fequuntur deinceps complexiones lineae horizontalis proxime altioris eae, quarum elementa duo priora inter fe diuerfa funt, quarumque elementum primum cum proxime maiori permutandum eft.

Hinc fequens oritur fummae indeterminatae n inuolutio lexicographica (r):

$^{n}J =$

(r) *Polyn. Lehrf.* p. 202. §. 66.

$$
\overset{n}{\mathfrak{J}}\binom{1\ 2\ 3\ 4\ \ldots}{a, b, c, d, \ldots} = \begin{array}{l}
a^{n-1}\, a \\
a^{n-2}\, b \\
a^{n-3}\, c \\
a^{n-4}\, [b^2, d] \\
a^{n-5}\, [bc, e] \\
a^{n-6}\, [b^3, bd, c^2, f] \\
a^{n-7}\, [b^2c, be, cd, g] \\
a^{n-8}\, [b^4, b^2d, bc^2, bf, ce, d^2, h] \\
a^{n-9}\, [b^3c, b^2e, bcd, bg, c^3, cf, de, i] \\
a^{n-10}\, [b^5, b^3d, b^2c^2, b^2f, bce, bd^2, bh, c^2d, cg, df, e^2, k] \\
a^{n-11}\, [b^4c, b^3e, b^2cd, b^2g, bc^3, bcf, bde, bi, c^2e, cd^2, ch, dg, ef, l] \\
\text{etc.} \qquad\qquad \text{etc.}
\end{array}
$$

3) Sic *omnes* complexiones obtineri, quae ad eandem zonam pertinent, ex dictis manifestum est. Eas autem per regulam (2) necessario secundum ordinem alphabeticum disponi, hunc in modum demonstratur. Sumamus, complexiones duarum linearum horizontalium praecedentium isto ordine progredi, tum complexiones ex 2, α prodeuntes inter se, nec minus complexiones ex 2, β similiter erunt ordinatae; at vero hae, quae omnes ab elemento b liberae sunt, illas quae contra cum ipso hoc elemento incipiunt, sequi debent: id quod ipsum regula (2) praescribit. Sic a duabus lineis horizontalibus ad proxime sequentem concludere licet: indeque ordinem alphabeticum in omnibus obtinere euidens est (s).

Scholion 1.

Alia deductio inuolutionum (§. XX. XXI.), immediate ex inuolutione lexicographica (§. XVIII. B.).

§. XXII. Figura §. XX. 4. statim sistit omnes complexiones summarum 1, 2, ... vsque ad 11, quas sub forma simpliciore exhibere vel in spatium arctius contrahere vix licet.

(s) Pro zonis prima, secunda, tertia, ... vsque ad septimam inuolutionis §. XX. 4. statim transire licet in ordinem alphabeticum, dum complexiones cuiusuis zonae colliguntur, a dextra versus sinistram progrediendo, ac in quauis cellula deorsum eundo. Hanc vero legem nequaquam generalem esse, iam pro complexionibus, quibus praemissum est a^{n-9}, ex intuitione inuolutionis hoc §pho descriptae manifestum fit. Inuerse autem ab hac inuolutione semper ad priorem §. XX. 4. transitus patet, ex lege constante ac facili, dum nimirum complexiones eiusdem classis siue aeque multorum elementorum colligantur ac suo ordine rite disponantur, vti figura §. XX. 4. docet.

licet. Huic commodo accedit alterum hoc, quod inuolutio a fummis minoribus ad maiores facile extendi queat; quare HINDENBVRGIVS merito iftam inuolutionem perfectiffimam ac abfolutam praedicat. Equidem §pho praecedente alio ratiociniorum ordine vfus fum, quem ad cuncta fatis diftincte et generaliter demonftranda aptum iudicaui: etenim rem aeque ingeniofe excogitatam ac vtilem ex alio vifus puncto intueri haud fuperfluum videbatur. Ceterum fequenti infuper ratione inuolutionem §. XXI. immediate ex inuolutione lexicographica §. XVIII. B. deducere, atque ex illa deinceps ad inuolutionem §. XX. transire licet.

Cum nimirum inuolutio lexicographica §. XVIII. B. extenditur a quauis fumma ad proximam, complexiones iam inuentae, abftrahendo ab elemento primo a, haud mutantur, quippe praemittitur tantum nouum a: at vero praeterea nouae adiiciuntur complexiones, dum prioris inuolutionis complexionum prima elementa a fecundis diuerfa, augentur vnitate. Hinc noua accedit zona, ad quam formandam tantum ad zonas duas proxime praecedentes eft refpiciendum, quippe prioris fummae zonae penultimam praecedentes incipiunt cum a^2, a^3, ... vel I I, I I I, ... i. e. cum elementis aequalibus, zona vero penultima vnum habet elementum a, quod igitur cum proxime fequente b eft permutandum: in zona vltima, quae erat ab a libera, complexionum elementa prima fecundis haud aequalia cum proxime maioribus permutantur. Hac ratione fequens oritur inuolutio:

I I I I I I I	1	fiue	a^{n-1}	a
I I I I I I	2		a^{n-2}	b
I I I I I	3		a^{n-3}	c
I I I I	2 2		a^{n-4}	b b
	4			d
I I I	2 3		a^{n-5}	b c
	5			e
I I	2 2 2		a^{n-6}	b b b
	2 4			b d
	3 3			c c
	6			f
I	2 2 3		a^{n-7}	b b c
	2 5			b e
	3 4			c d
	7			g
	2 2 2 2		a^{n-8}	b b b b
	2 2 4			b b d
	2 3 3			b c c
	2 6			b f
	3 5			c e
	4 4			d d
	8			h

Cuius

Cuius lex, quae ex modo dictis fronte sequitur, cum lege inuolutionis §. XXI. omnino consentit.

Quae porro inuolutio in hanc transformari potest: dum complexiones zonarum in cellulas rediguntur, siue ita ordinantur, vt complexiones eiusdem classis in eadem cellula reperiantur.

IIIIIIII	1		
IIIIIII	2		
IIIIII	3		
IIIII	4	22	
IIII	5	23	
III	6	33 24	222
II	7	34 25	223
I	8	44 35 26	233 224
	9	45 36 27	333 234 225

a^{n-1}	b c d		
a^{n-2}	e	bb	
a^{n-3}	f	bc	
a^{n-4}	g	cc bd	bbb
a^{n-5}	h	cd be	bbc
a^{n-6}	i	dd ce bf	bce bbe
	k	de cf bg	ccc bcd bbf

Haec inuolutio cum inuolutione §. XX. exhibita quoad formam et legem conspirat.

Scholion 2.

Notitiae historicae de partitione numerorum summaeue datae combinationibus.

§. XXIII. De Problemate hactenus exposito iam pridem inter Analystas constat. Satis notum est, in primis L. EVLERVM quaestionem de *partitione numerorum* accurate pertractasse (t). Is vero, quum maxime in *numero* discerptionum inuestigando occupatus esset, de earum *formis* seu de complexionibus datae summae actu exhibendis minus fuit sollicitus, nec certam pro hoc negotio atque generalem regulam tradidit. Attamen considerando modum, quo e. g. classem tertiam pro summa 12 (^{12}C), classes tertiam et quar-

(t) *Introd. ad Analys. Infinit.* T. I. Cap. XVI.; *Nou. Comment. Acad. Sc. Petrop.* T. III. p. 125 etc. *Eulerus* commemorat, problema sibi fuisse propositum a *Naudaeo:* eiusdem vero mentionem aliquam obiter iam fecerat *Leibnitius* (*Ars combinatoria* etc. Francofurti 1690. 4. pag. 55. Opp. T. II. p. 382. cf. *G. G. Leibnitii* et *Ioh. Bernoullii* Commercium philos. et mathem. T. I. ab a. 1694. ad 1699.) Obseruandum praeterea est, Leibnitium saepius insignem vsum Artis combinatoriae summis laudibus extulisse, atque praeclara quaeuis ab vlteriori eius perfectione expectasse.

quartam pro summa 10 (^{10}C, ^{10}D) inuestigauit (u), sicque exhibuit, vt tam complexiones quam classes ipsae rite sint ordinatae: probabile videtur, eum regulam arithmeticam supra commemoratam (§. XVIII. C. 5.) in mente habuisse, ac normae instar adhibuisse. Eandem hanc regulam pro classi singulari datae summae in simili disquisitione de partitione numerorum, etiamsi non verbis disertis, re tamen ipsa ac signis generalibus expressit PAOLI (v).

Complexiones summarum 6 et 7 secundum omnes classes exhibuit EVLERVS eo ordine, quo illae prodeunt ex regula infra (§. XXV. 4.) exponenda (w). MONTMORTIVS quaerens, quot iactus dati tesserarum numeri producere queant numerum datum, discerptiones numerorum 2, 3, 4, ... vsque ad 12 in duas et tres partes, siue classes ^nB et ^nC ab $n = 2$ ad $n = 12$ exhibuit, certamque pro hac operatione obseruauit regulam infra explicandam (x) (§. XXVI. 6.).

HAVSENIVS problema "de inueniendis combinationibus numerorum 1, 2, 3, 4, 5, 6, ... quae summam efficiant datam" ad *reuersionem serierum* adplicauit, vtque eius praxin aliquo exemplo ostenderet, combinationes summae 7 exhibuit, eo vero procedendi modo, vt vix inde regulam constantem ac vniuersalem, quam auctor secutus fuerit, deducere liceat: quippe ne ipsae quidem complexiones omnes rite sunt ordinatae (y). CASTILLONEVS (z) eiusdem problematis vsu in *theoremate polynomiali* ostenso in quatuor exemplis complexiones (v. c. pro summa 6) secundum classes rite ordinatas disponit, sic quidem,

(u) Nou. Comm. Tom. III. pp. 126. 141. 142. Classis tertia pro summa 12 sic exhibetur:

$$12 = 1 + 1 + 10;\quad 12 = 1 + 2 + 9;\quad 12 = 1 + 3 + 8;$$
$$12 = 1 + 4 + 7;\quad 12 = 1 + 5 + 6;\quad 12 = 2 + 2 + 8;$$
$$12 = 2 + 3 + 7;\quad 12 = 2 + 4 + 6;\quad 12 = 2 + 5 + 5;$$
$$12 = 3 + 3 + 6;\quad 12 = 3 + 4 + 5;\quad 12 = 4 + 4 + 4.$$

(v) *P. Paoli* Opuscula analytica. *Liburni* MDCCLXXX 4. p. 49 etc.

(w) Introd. in Analys. inf. T. I. pag. 258. 270.

(x) Essay d'Analyse sur les Jeux de Hazard, Sec. edit. Paris MDCCXIII. 4. pag. 47. 48. cf. pag. 203., vbi tabula extat, ad decem tesseras extensa. cf. *Iac. Bernoullii* Ars coniectandi, Basileae 1713. 4. pag. 21.

(y) *Hausenius* inter alias complexiones partim directe partim inuerse rite ordinatas, has etiam inuenit nullo modo rite ordinatas 2311, 241, 232. Fateor, me haud intelligere, qua ratione ex complexione 2311 deduxerit complexionem 232, nec potius ex 22111, 223. Verum etiamsi casum simpliciorem processu minus ordinato resoluerit, nec regulam certam ac firmam tenuisse videatur, quis tamen est qui dubitet, *Hausenium* quem *Kaestnerus* praeceptorem laudat, casibus etiam difficilioribus proprio Marte resoluendis parem fuisse. Quo itaque exemplo apprime illustrantur ea, quae *Hindenburgius* ad mentem suam clarius explicandam contra aliquam *Fischeri* obiectionem profert (*Archiv der Mathem. II. Heft*, pag. 253.)

(z) *If. Newtoni* Arithmetica vniuersalis — cum commentario *Ioh. Castillonei*, p. 33. 34.

quidem, vt regulam arithmeticam supra (§. XVIII. A. 2.) commemoratam ante oculos habuisse videatur, quam tamen verbis nec ille expressit, nec modo generali concepit.

Ex his breuiter commemoratis abunde iam colligere licet, problematis praecedentis vsum Analystas bene intellexisse, atque etiam data occasione exempla particularia resoluisse, regulas vero combinatorias simplices ac vniuersales plerumque eos neglexisse (a). Cuiusmodi tamen regulas nequaquam pro superfluis habendas esse, in aperto est (b). Earundem potius necessitatem ac vtilitatem dudum agnouerunt Duumuiri praestantissimi, MOIVREVS et BOSCOVICHIVS. Quorum meditationes huc facientes paucis saltem commemorare omnino e re esse videtur, cum vt multiplicitas ac varietas, quam offert Ars Combinatoria, inde elucescat, tum vt ex comparatione clarius perspiciatur, quantum in hoc genere praestiterit HINDENBVRGIVS.

Scholion 3.
Moivrei solutio problematis §. XVIII.

§. XXIV. 1) MOIVREVS sumta occasione a theoremate polynomiali, classem indeterminatam m^{tam} summae $m+r$, denotante r numerum determinatum, siue ex HINDENBVRGII signo ^{m+r}M ita inuestigare docet, vt ab $r=0$ incipiendo, vbi ^{m}M est $=$ elemento 1 siue a m^{ies} repetito $= a^{m}$, ad $r = 1, 2, 3, \ldots$ successiue progrediendum sit, hac quidem lege combinatoria: In omnibus complexionibus summae proxime minoris $m+r-1$, elementorum 1 siue a vltimum versus dextram permutetur cum elemento 2 seu b; in singulis porro complexionibus summarum minorum $m+r-2$; $m+r-3$; $m+r-4$; . . . m; ex ordine itidem vltimum elementum 1 seu a permutetur cum elementis 3; 4; 5; . . . siue c; d; e; . . . nisi post illud elementum a sequatur elementum minus substituendo, seu litera ordine alphabetico prior.

Hac quippe ratione MOIVREI sermo algebraicus in combinatorium vertitur: is nimirum permutationem elementi a cum b, c, d, . . . tanquam multiplicationem per $\frac{b}{a}, \frac{c}{a}, \frac{d}{a}$, . . . exprimit, iubetque complexiones summae $m+r-1$ singulas ducere in $\frac{b}{a}$, complexiones

(a) Sic *Iac. Bernoullius* (l. c.) problema iam memoratum de tesseris exemplo illustraturus haec praeloquitur: "quae quidem omnia melius exemplis quam regulis addisci possunt." Etiam *Castillioneus* l. c. p. 34. asserit, sufficere primas arithmeticae regulas.

(b) Valorem huiusmodi regularum haud diffessus est *Fischerus* (*Ueber den Ursprung der Theorie der Dimensionszeichen und ihr Verhältniss gegen die combinatorische Analytik des Herrn Prof. Hindenburg*, *Halle 1794.* pag. 52.); rectius autem de eo statuit, praeter *Hindenburgium* (*Archiv* l. c. pag. 253. 194; *Polynom. Lehrs.* pag. 211.), *Klügelius* (*Pol. Lehrs.* pag. 89.). Quod quidem in iuris scientia praecipitur, legibus cauendum esse, ne in arbitrium iudicis solutum sit ac liberum, idem etiam in quaestionibus analyticis obseruandum videtur.

plexiones summarum $m+r-2$; $m+r-3$; $m+r-4$; . . . eas tantum, in quibus non occurrunt elementa b; b, c; b, c, d; . . . ducere in $\frac{c}{a}$; $\frac{d}{a}$; $\frac{e}{a}$; . . . (c). Regula MOIVREI his binis exemplis illustratur:

${}^{m}M$	a^{m}	${}^{5}E$	11111
${}^{m+1}M$	$a^{m-1}b$	${}^{6}E$	11112
${}^{m+2}M$	$a^{m-2}bb$	${}^{7}E$	11122
	$a^{m-1}c$		11113
${}^{m+3}M$	$a^{m-3}bbb$	${}^{8}E$	11222
	$a^{m-2}bc$		11123
	$a^{m-1}d$		11114
${}^{m+4}M$	$a^{m-4}bbbb$	${}^{9}E$	12222
	$a^{m-3}bbc$		11223
	$a^{m-2}bd$		11124
	$a^{m-2}cc$		11133
	$a^{m-1}e$		11115
${}^{m+5}M$	$a^{m-5}bbbbb$	${}^{10}E$	22222
	$a^{m-4}bbbc$		12223
	$a^{m-3}bbd$		11224
	$a^{m-2}bcc$		11233
	$a^{m-2}be$		11125
	$a^{m-2}cd$		11134
	$a^{m-1}f$		11116
etc.	etc.	etc.	etc.

2) Hoc modo *complexiones* singulas *rite ordinatas* (d) prodire manifestum est, quippe ipsa regula (1) cavetur, ne elementum maius praecedat. *Omnes* porro ita complexiones obtineri,

(c) Producta (seu complexiones), in quibus dignitatem τῦ a (elementum a aliquoties repetitum) proxime sequitur b, vel c, vel d, ... *Moivraeus* nominat primae, secundae, tertiae, ... *Classis*.

(d) *Classes* non sunt rite ordinatae, uti iam ex ${}^{m+4}M$, ${}^{m+5}M$ apparet: nec illae secundum exponentes τῦ a procedunt, quippe qui iam pro ${}^{m+6}M$ non amplius continue crescunt.

obtineri, sic probatur. Ponamus, classes summarum praecedentium ${}^{m+r-1}M$, ${}^{m+r-2}M$, ${}^{m+r-3}M$, ... ${}^{m}M$ esse completas. Iam referat $a^{m-l}ux\ldots z$ quamuis complexionem rite ordinatam, quae in classe ${}^{m+r}M$ occurrere debet. Denotet ϑ exponentem numericum literae u, tum pro certo l, ϑ maiorem valorem recipere nequit, ac si ponantur elementa u, x, ... z cuncta inter se aequalia: vnde ob numerum horum elementorum $= l$, fit $\vartheta l = r + l$, siue $\vartheta = 1 + \frac{r}{l}$, quare numerus ϑ non potest fieri maior quam $1 + r$. Consideretur igitur classis praecedens ${}^{m+r-\vartheta+1}M$, ea, cum sit ex hypothesi completa, comprehendet etiam complexionem rite ordinatam hanc: $a^{m-l+1}x\ldots z$, quippe cuius elementa efficiunt summam $m+r$ vnitate auctam et exponente ϑ minutam, i. e. summam $m+r-\vartheta+1$. Iam vero ad producendam classem ${}^{m+r}M$ ex regula (1) in singulis complexionibus classis ${}^{m+r-\vartheta+1}M$ elementum a permutandum est cum u, nisi minus quam u sequatur; quare complexio $a^{m-l+1}x\ldots z$ necessario suppeditabit pro classe ${}^{m+r}M$ complexionem $a^{m-l}ux\ldots z$; hincque euidens est, in hac classe nullam complexionem deficere posse, si quidem classes summarum praecedentium fuerint completae; vnde sponte sequitur, classes cunctas esse completas. Haec addenda esse duxi, quoniam MOIVREVS regulam suam demonstratione haud muniuit.

3) Ex regula (1) apparet, quamuis classem ex classibus homonymis summarum omnium minorum (vsque ad m) determinari. Nihilominus tamen MOIVREVS bene vidit, idque diserte enuntiauit, quamuis classem etiam *independenter* reperiri posse (e). Alia insuper obseruatio, hoc etiam respectu notatu dignissima, eiusdem auctoris sagacitatem haud effugit: quod nimirum exponentes literarum b, c, d, ... omnes vnitate minuendo, ita vt exponens literae b euadat 1; literae c, 2; literae d, 3; et sic deinceps, tunc complexiones elementorum b, c, d, ..., seiuncto elemento a, in quauis classe ${}^{r+m}M$ efficiant summam eandem r (f). Quae obseruatio symbolice expressa suppeditat formulam supra commemoratam:

$$ {}^{m+r}M \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ a, & b, & c, & d, & \ldots \end{pmatrix} = a^{m-*}\,{}^{r}GP \begin{pmatrix} 1 & 2 & 3 & \ldots \\ b, & c, & d, & \ldots \end{pmatrix}. $$

4) Ac-

(e) *Miscellanea Analytica* de Seriebus et Quadraturis. Londini 1730. 4. pag. 88. cf. *Archiv der Mathem. IV. Heft.* pag. 400.

(f) *Misc. Anal.* p. 90. *Archiv* l. c.

4) Accuratius considerando ordinem, secundum quem apud MOIVREVM complexiones elementorum b, c, d, ... pro quauis classe dispositae sunt, mox apparebit, illas ordinem alphabeticum sequi, ac inuolutionem certae summae ostendere, quae cum inuolutione lexicographica supra exposita (§. XVIII.) prorsus consentit, dummodo index illic assumtus $\binom{1\ 2\ 3\ 4\ \ldots}{a,\ b,\ c,\ d,\ \ldots}$ permutetur cum indice $\binom{1\ 2\ 3\ \ldots}{b,\ c,\ d,\ \ldots}$ siue elementa a, b, c, d, ... cum b, c, d, ... Optimo igitur iure HINDENBVRGIVS MOIVREVM *inuentorem inuolutionum* praedicat, in quas vero is nescio quasi se ipso incidit quarumque eximia commoda haud satis perspexit (g). Ipsa etiam lex, secundum quam apud MOIVREVM complexiones elementorum b, c, d, ... deducuntur, a regula Hindenburgiana (§. XVIII.) differt: quare ratio est reddenda, cur ex illa etiam lege complexiones necessario ad ordinem lexicographicum procedere debeant. Supponamus, hunc ordinem locum habere pro complexionibus summarum r — 1, r — 2, ... 1. Tunc complexiones summae r ex regula MOIVREI obtinentur, dum complexionibus omnibus summae r — 1 praemittatur elementum b, deinde complexionibus summae r — 2, quarum elementum primum non est minus quam c, praemittatur elementum c, similiterque porro complexionibus summarum r — 3; r — 4; r — 5; ... quae incipiunt elementis non minoribus quam d; e; f; .... praeponantur elementa haec ipsa d; e; f; .... Iam ex hypothesi complexiones pro quauis summa praecedente ordine inter se alphabetico progrediuntur, atque etiam complexiones pro diuersis summis successiue decrescentibus praedicto modo quoad elementa initialia immutatae simili ordine procedunt, quippe quae elementis successiue crescentibus b, c, d, ... incipiunt. Exinde sponte sequitur, ordinem alphabeticum etiam ad summam r, hincque ad omnes summas extendi. Ceterum MOIVREVS, qui formam inuolutoriam silentio praetermisit, ordinis etiam lexicographici mentionem nusquam fecit. Quare iure miratur HINDENBVRGIVS, quod vir tantae sagacitatis ac in arte combinatoria peritiae haud vlterius progressus fuerit (h).

Scholion 3.

De BOSCOVICHII solutione duplici, inprimis Inuolutione summae inuerse lexicographica.

§. XXV. Quod MOIVREVS intellexit quidem, nec tamen distinctius euoluit, id alia ratione perfecit BOSCOVICHIVS (i). Is enim cum videret, quod ille iam indicauerat,

(g) *Archiv* l. c. pag. 391.
(h) *Archiv* IV. 401.
(i) *Boscovichius* methodum suam, inueniendi potestates infinitinomii, cui ipse incredibilem facilitatem tribuit, primo publicauit in Ephemeridibus Literariis, quae Romae annis 1747 et 1748 prodierunt (*Giornale de' Letterati di Roma*, per l'anno 1747 e nel 1748), vnde *Hindenburgius* notatu digniora excerpsit (*Archiv* IV, 402 sq.). Breuius deinceps eandem quaestionem ille retractauit in differ-

uerat, etiam claſſis ſingularis indeterminatae inuentionem redire ad reperiundas complexiones ſummae cuiusuis determinatae (§. XXIII. 3.), regulam tradidit, vi cuius complexiones certae ſummae independenter a ſummis praecedentibus inuenire licet: quod ipſum MOIVREVS etiam in mente habuerat.

Regula ſimplex omnino, nec tamen pure combinatoria haec eſt: Prima complexio tot conſtat vnitatibus, quot ſumma; pro ſecunda ponatur vltimo loco binarius, huicque iungantur vnitates reliquae; tum veniunt duo binarii, deinceps tres, nec non plures, completurque ſumma per vnitates. Exhauſtis binariis ſequitur vltimo loco ternarius, cum reliquis vnitatibus, idem cum vno, duobus, tribus, pluribusue binariis; deinceps duo ternarii cum vnitatibus, iidem cum vno, duobus, tribus, . . . binariis, porro tres ternarii, pluresque; quos excipiunt quaternarii, vnus primo, deinde etiam plures. Sic ad numeros finales ſucceſſiue maiores progrediendum eſt, ita vt ſi quispiam numerus prima vice ſcribatur, huic iungantur primo merae vnitates, deinde vnus binarius, porro plures; quos excipiunt ternarii, quaternarii . . . et ſucceſſiue numeri creſcentes, nec vero maiores vltimo. Reſiduum ſummae ſemper vnitatibus completur. Regula ſequenti exemplo illuſtratur:

```
⁸CP   11111111
       1111112
        111122
         11222
          2222
        111113
         11123
          1223
          1133
           233
         11114
          1124
           224
           134
            44
          1115
           125
            35
           116
            26
            17
             8
```

2) Accu-

diſſertatione ſingulari, quae eſt inſerta operi inſcripto: *Delle progreſſioni e ſerie Libri due del P. Franceſco Luino — coll' aggiunta di due Memorie del P. R. G. Boscovich.* In Milano MDCCLXVII. 4. pag. 257-265: Methodo di alzare un infinitinomio a qualunque potenza indefinita. — Prior expoſitio, quam equidem non legi, *Hindenburgio* iudice alteri praeferenda, locupletiorque eſt variis iisque eximiis de arte combinatoria obſeruationibus — — Ceterum *Boscovichii* methodus nondum nota erat *Hindenburgio*, cum primum de Infinitinomio ſcriberet (*Archiv* IV, 420.)

a) Accuratius considerando complexiones, quas BOSCOVICHII regula praebet, primo apparet, singulas illas *rite esse ordinatas*, deinde eas inter se progredi secundum numeros *finales* crescentes, cum in inuolutione *Moivreo-Hindenburgiana* (§. XVIII.) complexionum elementa *initialia* successiue crescant. Sicuti iam hae complexiones ordine lexicographico progrediuntur *directe*, ita illae *inuerse*, i. e. BOSCOUICHII complexiones seruant ordinem alphabeticum, dum eae retro legantur, sic quidem vt ciphrae vltimae in singulis complexionibus fiant primae, penultimae secundae, et sic porro. Hoc quidem modo BOSCOVICHIVS ipse in dissertatione recentiori complexiones disponit, v. c.

$$
{}^5CP = \begin{array}{l} 11111 \\ 2111 \\ 221 \\ 311 \\ 32 \\ 41 \\ 5 \end{array}
$$

Praeterea insigne discrimen cernitur inter dispositionem Boscovichianam et inuolutionem lexicographicam directam, cum ad vtriusque relationem erga inuolutionem classium attendatur. Quod quidem discrimen vel HINDENBVRGII, in detegendis relationibus combinatoris alioquin profecto felicissimi, perspicaciam effugit. Is nimirum asserit (k), ex dispositione Boscovichiana prodire complexiones rite ordinatas classium rite ordinatarum, dum colligantur complexiones ad eandem classem pertinentes, ita vt a regione inferiori ad superiorem ascendatur; vnde ab illa dispositione facile transire liceret ad inuolutionem classium. Illud vero assertum iam pro complexionibus summae 9 haud amplius valet: namque secundum BOSCOVICHIVM complexio 144 praecedit ordine complexionem 225, quare ascendendo praecedit contra 225, sequitur 144, hincque classis tertia non rite prodit ordinata. Docet porro HINDENBVRGIVS (l), ex inuolutione classium vice versa deriuari posse dispositionem BOSCOVICHII, dum colligantur, rursus ascendendo, eae diuersarum classium complexiones, quae desinunt elemento eodem, primo 1, deinde 2, 3, et sic porro. Nec vero hoc etiam praeceptum rite se habere, duobus exemplis ostendisse sufficiat. Pro summa 9 adhibendo inuolutionem classium reperitur in classe quarta (${}^9D$) complexio 2223, et in classe sequente (${}^9E$) complexio 11133. Iam ascendendo ex regula HINDENBVRGII complexio 11133 praecedere deberet complexionem 2223, in dispositione contra BOSCOVICHII inuerse lexicographica praecedit 2223, sequitur 11133. Pro summae 14 classe quinta rite ordinata complexio 13334 praecedit complexionem eiusdem classis 22244, nec minus secundum BOSCOVICHII dispositionem illa complexio prior occurrit. Ex his exemplis satis manifestum est, dispositionem inuerse lexicographicam Boscovichianam neutiquam ad inuolutionem classium simili modo referri, ac inuolutionem directe lexicographicam seu Moivreo-Hindenburgianam.

Com-

(k) *Archiv* IV, 408. 430.
(l) *Polynom. Lehrs.* p. 195. §. 60.

Comparationi hactenus expositae inter vtramque dispositionem lexicographicam, tam directam quam inuersam, ne quid deesse videatur, commemorandum insuper est, regulam Boscovichianam (1) tum praestare, ac insigne compendium afferre posse, cum in serie elementorum aliqua deficiant, siue elementa per *saltus* procedant (m). Sic e. g. ex elementis 1, 3, 5, pro summa 8 hae statim complexiones prodeunt:

11111111
111113
1115
35

3) Praeter regulam (1) *alteram* insuper solutionem exhibuit BOSCOVICHIVS (n), qua complexiones cuiusuis summae (n) deducuntur ex complexionibus summae proxime praecedentis (n — 1). Quae quidem regula ita se habet:

a) Singulis complexionibus summae praecedentis praemittitur elementum 1.

b) In iis complexionibus summae praedictae, quae non habent duo elementa priora aequalia, primum elementum permutetur cum proxime maiori.

c) Complexiones ex (a) et (b) oriundae eo inter se ordine collocentur, vt pro quauis complexione summae n — 1 operationes (a) et (b) successiue instituantur (quatenus vtraque locum habet), tumque ad complexionem sequentem illius summae progressu facto idem processus reiteretur.

Exempli gratia ex complexionibus summae 8 (1) complexiones summae 9 ita prodeunt:

111111111

(m) *Archiv* IV, 410; *Polyn. Lehrs.* p. 261; *Giornale de' Letterati di Roma*, 1748. p. 86. 87.

(n) *Polyn. Lehrs.* p. 184.; *Archiv* IV, 404. In recentiori dissertatione hanc regulam praetermisit *Boscovichius*; similem regulam ad elementa *literalia* respicientem, eamque ad potestates infinitinomii immediate applicandam tradidit *Giornal.* 1748. p. 18.; 1747. p. 402. (cf. *Archiv* IV, 406.).

111111111
11111112
1111122
111222
12222
1111113
111123
11223
2223
11133
1233
333
111114
11124
1224
1134
234
144
11115
1125
225
135
45
1116
126
36
117
27
18
9

Cum igitur ex MOIVREI regula complexiones cuiusuis fummae deducendae fint ex complexionibus fummarum *omnium* minorum, hac contra regula BOSCOVICHIVS oftendit, quomodo recurfu ad *vnam* tantum fummam proxime minorem opus fit. Quae quidem regula quoad effentialia a, b), abftrahendo ab ordine (c), prorfus confentit cum ea, quam HINDENBVRGIVS pro inuolutione Moivreana ftabiliuit (§. XVIII.). Sufpicari etiam licet, BOSCOVICHIO legis iftius obferuandae occafionem forte dediffe inuolutionem MOIVREI, quippe cuius efformatio hanc legem clarius oftendit, quam difpofitio altera (1); vt ne tamen complexiones alio ordine prodirent, quam ex regula (1), addendum erat praeceptum (c). Hoc igitur praecepto confenfum inter binas fuas folutiones obtinuit BOSCOVICHIVS; at vero commoda *inuolutionis*, difpofitioni Moivreanae propria, fic neglexit. Etenim fi a fumma ad fummam proximam progrediendum fit, tum vtriusque fummae complexiones *feorfim fcribere* oportet: quare cum folutio ex lege (§. XVIII.) a fuperfluis libera fit ac *independenti* aequipolleat, folutio contra ex regula BOSCOVICHII incommoda *dependentiae* habet (o). Hinc apparet, auctorem infigne compendium, quod affert

(o) *Archiv* IV, 409.

affert forma inuolutoria, haud perspexisse. Idem nec ad ordinem lexicographicum attendit.

4) Eodem ordine, ac secundum BOSCOVICHII regulas (1, 3), prodeunt complexiones summae n, si iungatur elemento 1 complexio summae n — 1 ex meris vnitatibus, porro iungantur elementis successiue crescentibus 2; 3; 4; . . . complexiones summarum n — 2; n — 3; n — 4; . . . ex elementis 1 et 2; 1, 2 et 3; 1, 2, 3 et 4; . . . conflatae: quippe cauendum est, ne in hisce complexionibus occurrat elementum, minus elemento, cui eae abiectae sunt.

Huic praecepto, quod BOSCOVICHII verbis innuitur saltem (p), similem omnino regulam obseruasse videtur EVLERVS; eo tantum discrimine, quod hic eam complexionem primo loco ponat, quae illi est vltima, ac vice versa. Sic complexiones summarum 6 et 7 EVLERVS (q) sequentem in modum exhibuit:

6	7
5+1	6+1
4+2	5+1+1
4+1+1	4+3
3+3	4+2+1
3+2+1	4+1+1+1
3+1+1+1	3+3+1
2+2+2	3+2+2
2+2+1+1	3+2+1+1
2+1+1+1+1	3+1+1+1+1
1+1+1+1+1+1	2+2+2+1
	2+2+1+1+1
	2+1+1+1+1+1
	1+1+1+1+1+1+1

5) Accuratius porro, quam MOIVREVS distinxit BOSCOVICHIVS *duo* problemata, vnum idque hactenus expositum de inueniendis complexionibus omnibus certae summae, alterum de inuenienda classi quapiam singulari pro data summa. Pro classi mta summae t, siue ${}^{t}M$, hanc tradit regulam: Prima complexio oritur, praemittendo numero t — m + 1 vnitates m — 1; tum ponantur vltimo loco numeri successiue decrescentes t — m;

(p) Clarius demum illud expressit *Hindenburgius* (*Archiv* IV. 407.), indeque ostendit (p 409.), quatenus *Boscovichii* dispositio praebeat inuolutionem: dum nimirum secernantur linea verticali elementa complexionum maxima, et lineis horizontalibus eae complexiones, quae idem elementum maximum commune habent. Haec vero inuolutio *imperfecta est*, quippe exinde nec ad summas *omnes* minores regredi, nec ad summam proxime maiorem inuolutione progredi licet. Ceterum inuolutionem Boscovichianam eiusque ordines eodem modo designat *Hindenburgius*, ac inuolutionem directe lexicographicam, huiusque ordines.

(q) Introduct. in Analys. infinit. T. I. p. 258. 270.

t — m; t — m — 1; t — m + 2; etc. iisque iungantur complexiones classis m — 1tae complementorum summae t, siue numerorum m; m + 1; m + 2; etc. quae quidem complexiones conflandae sunt ex numeris vltimo non maioribus. Exempli gratia classis tertia summae 9 sic prodit:

${}^{9}C = 117$	Sic igitur classis quaelibet ad classem proxime praecedentem reducitur,
126	verum in hac classe non vnius tantum summae, vti ex regulis Hinden-
135	burgianis, sed complexionum eiusdem classis ad diuersas summas per-
225	tinentium ratio est habenda. Satis praeterea manifestum est, regu-
144	lam istam Boscovichianam imperfectam esse, ipsique quoad vsum com-
234	modum atque expeditum omnino praeferendas regulas supra descriptas.
333	Ceterum notandus est consensus mutuusque nexus inter regulam BOS-

COVICHII pro classe singulari, eiusdemque dispositionem complexionum pro certa summa. Ex ista nimirum regula haud equidem prodeunt classes rite ordinatae (r), sed complexiones cuiusuis classis eodem ordine sibi inuicem succedunt: quem seruant complexiones in dispositione lexicographica ascendendo collectae. Illa porro regula hanc suppeditat formulam:

$$ {}^{m+r}M = a^{m-r}J $$

$$ \begin{pmatrix} 1 & 2 & 3 & 4 \ldots \\ a, & b, & c, & d, \ldots \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \ldots \\ b, & c, & d, \ldots \end{pmatrix}. $$

Classis nimirum quaeuis mta summae m + r ad normam praescriptam efformata immediate simul ostendit complexiones omnes summae, secundum ipsam inuolutionem lexicographicam Boscovichianam dispositas, dum alter index accipiatur.

6) Regulam aeque simpliciter expressam, pro adornanda classi singulari ita, vt ea simul immediate ostendat inuolutionem lexicographicam directam Moivreo-Hindenburgianam, talem inquam regulam equidem haud noui. At vero regulam haud absimilem, quae praebet classes rite ordinatas, obseruasse iam videtur MONTMORTIVS (s), hanc nimirum:

α) Pro

(r) Vsque ad summam t = 8 classes etiam ex *Boscovichii* regula prodeunt rite ordinatae, at non ita pro summis maioribus: v. c. pro t = 9 in exemplo. Quod si vero complexionum elementa retro legantur, tum illae vt numeri continuo decrescentes procedunt, hocque igitur sensu classes etiam rite ordinatae apparent.

(s) *Essay d'Analyse sur les jeux de hazard*, pag. 49: quanquam hoc loco regula haud generaliter enuntiata sit, sequentia tamen verba eam satis clare innuunt: "Pour former ces points (complexiones classis tertiae pro summis 3, 4, 5, 6 . . .) on joint l'as (1) du troisiéme dé à tous les points de la Table précedente pour deux dés (complexionibus classis praecedentis), et ensuite le deux du troisiéme dé avec tous les points de la Table précedente ou l'as ne se trouve point, et ensuite le trois du troisiéme dé, avec tous les points de la Table précedente ou il ne se trouve ni l'as, ni le deux, et ainsi du reste." Haec regula iam in *Moivrei* praecepto latet: quod nimirum de inuolutione summae n (tanquam aggregato classium) ex inuolutionibus summarum n — 1, n — 2, . . . deducenda valet, id etiam de singulis classibus obtinet, dum illius inuolutionis complexiones ad classem mtam pertinentes, harum vero complexiones classis m — 1tae accipiantur.

α) Pro deducenda classi ${}^{t}M$ iungantur a dextra elemento 1 singulae complexiones classis proxime praecedentis pro summa proxime minore, seu classis ${}^{t-1}\overset{-1}{M}$;

β) porro elementis successiue crescentibus 2; 3; 4; . . . adiiciantur complexiones classium ${}^{t-2}\overset{-1}{M}$; ${}^{t-3}\overset{-1}{M}$; ${}^{t-4}\overset{-1}{M}$; . . . quae incipiunt elementis non minoribus, ac istis 2; 3; 4; . . . Sic quaeuis classis ad complexiones classis proxime praecedentis, pro summis tamen diuersis ac successiue decrescentibus, reducitur. Quoad primam partem (α) haec regula consentit cum Hindenburgiana (§. XVIII. C. 1. b. α), quae ceteroquin praestantior ac vsui omnino aptior est.

ARTICVLVS SECVNDVS.

De theoremate polynomiali.

PROBLEMA.

§. XXVI. Seriei $y = ax^{\mu} + bx^{\mu+\delta} + cx^{\mu+2\delta} + dx^{\mu+3\delta} + \ldots$ potestatem mtam reperire, sic quidem, vt huius potestatis coëfficientem quemuis extra ordinem independenter a praecedentibus assignare liceat.

Solutio.

A) Pro exponente m integro affirmatiuo.

1) Sumatur simplicitatis gratia $\mu = \delta = 1$, tum erit y^m = producto m serierum sibi inuicem aequalium,

$$= (ax + bx^2 + \ldots + rx^{\varrho} + \ldots)(ax + bx^2 + \ldots + sx^{\sigma} + \ldots)\ldots$$
$$(ax + bx^2 + \ldots + tx^{\tau} + \ldots)$$

Concipiamus, rite esse factam multiplicationem, primae seriei in alteram, producti duarum in tertiam, trium in quartam, et sic porro: tum colligendo eos terminos, in quibus eadem potestas τῦ x occurrit, potestas y^m hanc formam induet:

$$y^m = Px^m + P^{I}x^{m+1} + P^{II}x^{m+2} + P^{III}x^{m+3} + \ldots + P^{N}x^{m+n} + \ldots$$

Iam terminorum, ex multiplicatione ista repetita prodeuntium, quilibet complectitur m factores, quorum primus (veluti rx^{ϱ}) ex serie prima, secundus (sx^{σ}) ex altera, . . . mtus (tx^{τ}) ex serie mta sumtus est: inter quos etiam factores aequales seu repetiti admittuntur, quippe ob series ipsas inuicem aequales perinde est, num factores isti ad suam quiuis seriem, vel cuncti ad vnam seriem y referantur. Sit igitur eorum terminorum,

 qui

qui collectiue sumti constituunt membrum per $P^N x^{m+n}$ expressum, aliquis $= rx^\varrho \cdot sx^\sigma \ldots tx^\tau$; tunc esse debet, addendo exponentes $\tau\tilde{\varrho}$ x, $\varrho+\sigma+\ldots+\tau = m+n$. Assumto nunc indice, $\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ a, & b, & c, & d, & \ldots \end{pmatrix}$ siue *notas* (t) literarum aequales ponendo exponentibus potestatum $\tau\tilde{\varrho}$ x, quibus illae iunctae sunt: facile apparet, coëfficientem P^N componi ex illis *tantum* productis literalibus, quae ex m elementis seriei a, b, c, ... ita efformari possunt, vt summa notarum sit $= m+n$. Talium autem productorum *nullum omnino* deesse poterit, vtcunque elementa eligantur, et quocunque demum eo ordine collocentur, dummodo conditioni praedictae satisfiat: si enim deesset v. c. productum r.s ...t, tum termini rx^ϱ, sx^σ, ... tx^τ, ex serie prima secunda, ... m^ta non in se inuicem essent multiplicati, quod statui nequit, quippe in multiplicatione plurium serierum quiuis terminus alicuius seriei in cunctos reliquarum serierum terminos ducendus est, vnde productum serierum sit aggregatum omnium productorum, quae vtcunque formare licet, dummodo ex quauis serie vnus terminus pro factore accipiatur.

2) Haec porro producta literalia, quae coniunctim efficiunt P^N, haud omnia reuera inter se esse *diuersa*, liquet: ea enim pro identicis habenda sunt, quae non ipsis elementis seu factoribus, sed horum tantum situ vel ordine differunt. Talia igitur producta aequalia coniungere licet, dum vnum instar omnium accipiatur, idque multiplicetur per numerum, qui indicat, quoties eadem elementa diuerso inter se ordine poni, i. e. *permutari* inuicem queant: tot enim vicibus illud productum repetitum occurret. Exprimatur hocce productum, cetera aequalia repraesentans, per $a^\alpha b^\beta c^\gamma d^\delta \ldots$, vbi exponentes α, β, γ, ... (qui euanescunt, deficiente aliquo elemento) admissas elementorum repetitiones innuunt: tum aliunde satis constat (u) esse *numerum permutationum*, siue multitudinem diuersorum situum pro iisdem elementis locum habentium, $= \frac{1.2.3\ldots(\alpha+\beta+\gamma\ldots)}{1.2..\alpha\,.\,1.2\ldots\beta\,.\,1.2\ldots\gamma\ldots}$; qui itaque numerus seu coëfficiens producto literali iungendus est, indeque *coëfficiens polynomialis* appellatur. Quoniam multitudo elementorum iunctorum in quouis producto constanter est $= m$, loco $\alpha+\beta+\gamma\ldots$ ponere etiam licet m.

3) Ex quibus hactenus expositis haec iam sponte consequitur *regula*, determinandi coëfficientem generalem $n+1^{tum}$ (P^N) potestatis n^{tae} polynomii propositi y: a)

(t) Hoc vocabulo quod respondet *Fischeri* denominationi germanicae, *Marken*, vtor hoc loco ad vitandam ambiguitatem, cum *exponentes* alio sensu appellantur. (cf. §. XVII. not. cc.)

(u) Expressionem generalem numeri permutationum solide ac perspicue, pro more suo demonstrauit Cel. *Lorenzius* (*Elemente der Mathematik, Erster Theil, die reine Mathematik, II. Ausg.* p. 455.)

a) Quaerantur methodo supra tradita (§. XVIII. C.) complexiones rite ordinatae classis mtae, summae $m+n$, pro indice $\begin{pmatrix} 1 & 2 & 3 & 4 & \dots \\ a, & b, & c, & d, & \dots \end{pmatrix}$; tum

b) cuiuis complexioni iungatur coëfficiens polynomialis seu numerus permutationes elementorum combinatorum indicans;

c) quo facto aggregatum istarum complexionum literalium (a) per debitos numeros (b) multiplicatarum praebebit coëfficientem $n+1$tum potestatis quaesitae $(ax+bx^2+cx^3+\dots)^m$.

4) Ex hac regula *formula* simplex deriuari potest. Sicuti nimirum complexiones rite ordinatae classis mtae pro summa $m+n$ coniunctim designantur per ${}^{m+n}M$, ita praemittendo *literam minorem germanicam* classi M homonymam $\mathfrak{m}$, signo $\mathfrak{m}^{m+n}M$ indicatur, singulas istas complexiones ductas insuper esse in suum quemuis numerum permutationum. Inde haec nascitur expressio: $P^N = y^m k(n+1) = \mathfrak{m}^{m+n}M$.

5) A serie hactenus considerata ad generaliorem $y = ax^{\mu} + bx^{\mu+\delta} + cx^{\mu+2\delta} + \dots$ facile transire licet. Est nimirum $y = x^{\mu-\delta}(ax^{\delta} + bx^{2\delta} + cx^{3\delta} + \dots) = x^{\mu-\delta}(a\chi + b\chi^2 + c\chi^3 + \dots)$, posito $x^{\delta} = \chi$: hinc fit, vti antea, $y^m k(n+1) = \mathfrak{m}^{m+1}M$; atque $y^{m\tau}(n+1) = \mathfrak{m}^{m+n}M \,.\, x^{\mu m + \delta n}$.

$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots \\ a, & b, & c, & d, & \dots \end{pmatrix}$

6) *Exemplum.*

Quaeritur dignitatis 4tae seriei $y = ax + bx^2 + cx^3 + dx^4 + ex^5 + \dots$ terminus quintus. Pro $m = 4$, $n = 4$ est ${}^{m+n}M = {}^8D =$ 1115
siue transeundo a numeris ad literas, ac praefigendo coëffi- 1124
cientes polynomiales, $\mathfrak{D}\,{}^8D = \frac{1.2.3.4}{1.2.3}a^3e +$ 1133
1223
2222
$\frac{1.2.3.4}{1.2}a^2bd + \frac{1.2.3.4}{1.2.1.2}a^2c^2 + \frac{1.2.3.4}{1.2}ab^2c + \frac{1.2.3.4}{1\dots4}b^4$.

Hinc prodit $y^4k7 = 4a^3e + 12a^2bd + 6a^2c^2 + 12ab^2c + b^4$. Ceterum complexiones immediate pro elementis literalibus exhiberi, iisque statim coëfficientes numerici iungi possunt.

B) Pro exponente quouis indeterminato m.

7) Seriem $y = a + bx + cx^2 + dx^3 + ex^4 + \dots$ considerando tanquam binomium, cuius pars prima $= a$, altera $= bx + cx^2 + dx^3 + \dots = p$, fit

fit $y^m = a^m + {}^m\mathfrak{A}a^{m-1}p + {}^m\mathfrak{B}a^{m-2}p^2 + {}^m\mathfrak{C}a^{m-3}p^3 + \ldots + {}^m\mathfrak{N}a^{m-n}p^n$ $+ \ldots$ Euoluendo iam poteſtates τῦ p ſecundum ea, quae de exponentibus integris modo ſunt demonſtrata (4), obtinetur $p = \mathfrak{a}^1 A x + \mathfrak{a}^2 A x^2 + \mathfrak{a}^3 A x^3 + \ldots + \mathfrak{a}^n A x^n + \ldots$

$$p^2 = \mathfrak{b}^2 B x^2 + \mathfrak{b}^3 B x^3 + \mathfrak{b}^4 B x^4 + \ldots + \mathfrak{b}^n B x^n + \ldots$$
$$p^3 = \mathfrak{c}^3 C x^3 + \mathfrak{c}^4 C x^4 + \mathfrak{c}^5 C x^5 + \ldots + \mathfrak{c}^n C x^n + \ldots$$
$$p^n = \mathfrak{n}^n N x^n + \mathfrak{n}^{n+1} N x^{n+1} + \ldots$$

Inde ſubſtituendo ac colligendo terminos in easdem poteſtates τῦ x ductos, prodit:

$$\begin{array}{l} y^m = a^m + {}^m\mathfrak{A}a^{m-1}\mathfrak{a}^1 A x \\ \quad + {}^m\mathfrak{A}a^{m-1}\mathfrak{a}^2 A x^2 + {}^m\mathfrak{A}a^{m-1}\mathfrak{a}^3 A x^3 + \ldots + {}^m\mathfrak{A}a^{m-1}\mathfrak{a}^n \mathfrak{A} x^n + \ldots \\ \quad + {}^m\mathfrak{B}a^{m-2}\mathfrak{b}^2 B \quad + {}^m\mathfrak{B}a^{m-2}\mathfrak{b}^3 B \quad + \ldots + {}^m\mathfrak{B}a^{m-2}\mathfrak{b}^n B \quad + \ldots \\ \qquad\qquad\qquad\qquad + {}^m\mathfrak{C}a^{m-3}\mathfrak{c}^3 C \quad + \ldots + {}^m\mathfrak{C}a^{m-3}\mathfrak{c}^n C \quad + \ldots \\ \qquad\qquad\qquad\qquad\qquad\qquad + \ldots + \ldots \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + {}^m\mathfrak{N}a^{m-n}\mathfrak{n}^n N \quad + \ldots \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + \ldots \qquad\qquad + \ldots \end{array}$$

8) Hinc ſponte ſequitur: $y^m k(n+1) = {}^m\mathfrak{A}a^{m-1}\mathfrak{a}^n A + {}^m\mathfrak{B}a^{m-2}\mathfrak{b}^n B + {}^m\mathfrak{C}a^{m-3}\mathfrak{c}^n C + \ldots + {}^m\mathfrak{N}a^{m-n}\mathfrak{n}^n N$, cuius formulae, coëfficientem generalem exprimentis, conſtructio ac interpretatio ex ſignis Hindenburgianis ſatis manifeſta eſt. Cum ea inuoluat complexiones ſummae n ſecundum claſſes 1mam, 2dam, . . . ntam diſpoſitas (${}^nA$, ${}^nB$, . . . ${}^nN$): adhiberi poterit 1) modus ſupra traditus (§. XVIII. A.), claſſes pro data ſumma ex ordine, quamuis ex proxime praecedente, deriuandi. Tum 2) literae $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, . . . $\mathfrak{n}$, innuunt, ſingulis complexionibus coëfficientes polynomiales iungendos eſſe; praetereaque 3) complexiones omnes cuiusuis claſſis ſingularis ϱtae ducendae ſunt in coëfficientem binomialem claſſi homonymum ſiue ϱtum dignitatis mtae, ſimulque 4) eaedem multiplicandae in $a^{m-\varrho}$.

9) Pro forma generaliore ſeriei $y = a x^\mu + b x^{\mu+\delta} + c x^{\mu+2\delta} + d x^{\mu+3\delta} + \ldots$ coëfficiens generalis poteſtatis mtae eadem omnino formula exprimitur, qui ductus in $x^{\mu m + \delta n}$ praebet terminum generalem ſiue $y^m \tau(n+1)$ (5).

Corol-

Corollarium 1.

§. XXVII. Cum formula praecedentis §phi (8) etiam pro exponente m integro affirmatiuo valeat, aequando eam formulae priori (4) obtinetur (v):

$$m^{n+m}M = {}^{m}\mathfrak{A}a^{m-1}\,{}_{a}{}^{n}A + {}^{m}\mathfrak{B}a^{m-2}\,{}_{b}{}^{n}B + {}^{m}\mathfrak{C}a^{m-3}\,{}_{c}{}^{n}C + \dots$$
$$\begin{pmatrix}1 & 2 & 3 & 4 & \dots\\ a, & b, & c, & d, & \dots\end{pmatrix} \begin{pmatrix}1 & 2 & 3 & 4 & \dots\\ b, & c, & d, & e, & \dots\end{pmatrix} + {}^{m}\mathfrak{R}a^{m-r}\,{}_{r}{}^{n}R + \dots + {}^{m}\mathfrak{N}a^{m-n}\,{}_{n}{}^{n}N.$$

Quae expressio similis est formulae supra (§. XIX. 2.) traditae pro resolutione classis altioris in summam plurium inferiorum. Discrimen in eo tantum cernitur, quod hic complexionibus singulis classis mtae iuncti sint coëfficientes polynomiales. Est nimirum pro complexione quauis $a^{\alpha}b^{\beta}c^{\gamma}d^{\delta}\dots$ (§. XXV. 2.), vbi $\alpha+\beta+\gamma+\delta+\dots = m$, coëfficiens polynomialis $= \frac{1.2\dots\alpha\,.\,(\alpha+1)\dots(\alpha+\beta+\gamma+\delta\dots)}{1.2\dots\alpha\,.\,1.2\dots\beta\,.\,1.2\dots\gamma\,.\,1.2\dots\delta\dots} =$

$\frac{(\alpha+1)\dots m}{1\dots\beta\,.\,1.2\dots\gamma\,.\,1.2\dots\delta\dots} =$ (posito $\alpha = m - p$, siue $\beta+\gamma+\delta+\dots = p$)

$\frac{m(m-1)\dots(m-p+1)}{1\dots p}\cdot\frac{1\dots p}{1\dots\beta\,.\,1\dots\gamma\,.\,1\dots\delta\dots} = {}^{m}\mathfrak{P}p$, i. e. = producto duorum factorum, quorum vnus est coëfficiens binomialis ptus dignitatis mtae, alter = coëfficiente polynomiali complexionis $b^{\beta}c^{\gamma}d^{\delta}\dots$ ab elemento a liberae. Hinc intelligitur, a parte dextra aequationis pro $m^{n+m}M$ signis ${}^{m}\mathfrak{A}a$, ${}^{m}\mathfrak{B}b$, ${}^{m}\mathfrak{C}c$, ... exprimi ipsos coëfficientes polynomiales, quos a parte altera postulat classis mta $m^{n+m}M$. Qua ratione semper complexionis coëfficiens polynomialis resolui potest in polynomialem complexionis quae est residua, dum ab illa elementum quodpiam separatur, simulque in coëfficientem binomialem homonymum. Ceterum $n > m$, manifestum est, seriem a parte dextra non vsque ad classem ntam N, verum tantum ad classem mtam continuandam esse, illamque abrumpere termino ${}^{m}\mathfrak{M}m^{n}M$; id quod cum supra (§. XIX.) obseruatis consentit, atque etiam ex ipsorum coëfficientium binomialium indole colligitur.

Corollarium 2.

§. XXVIII. 1) Accuratius considerando formulam (§. XXVI, 8.), qua pro exponente indeterminato m coëfficiens generalis $n+1$tus exhibetur, quaeque pro exponente integro affirmatiuo aeque valet ac pro negatiuo et fracto, perspicitur, formulae illa comprehendi complexiones omnes rite ordinatas classium omnium summae n (${}^{n}A$, ${}^{n}B$, ${}^{n}C$, ... ${}^{n}N$). Exprimatur itaque aggregatum complexionum cunctarum summae n signo ${}^{n}(C)$, tum formulam istam in hanc contrahere licet:

$y^{m}k(n+1)$

(v) cf. Nou. Syst. Permut. p. LV, 9; *Polyn. Lehrs.* p. 236.

$$\frac{y^{m}k(n+1)}{(w)\; y\,(a,\, b,\, c,\, d,\, \ldots)} = \frac{\left({}^{m}\mathfrak{A}\overset{*-1}{a}\right)\overset{m-*}{a}\;{}^{n}(C)}{\begin{pmatrix}1 & 2 & 3 & \ldots\\ b, & c, & d, & \ldots\end{pmatrix}}$$ vbi quantitas

$\left({}^{m}\mathfrak{A}\overset{*-1}{a}\right)\overset{m-*}{a}$ factoris instar communis praemissa pro singulis complexionibus sub ${}^{n}(C)$ comprehensis peculiariter definienda est. *Asterisci* nimirum $*$ valorem numericum sumere oportet $=$ multitudini elementorum in quauis complexione aggregati ${}^{n}(C)$ inuicem iunctorum, siue $=$ exponenti Classis ad quam pertinet complexio. Idem Asteriscus vnitate minutus literis $\mathfrak{A}$ et a communiter superimpositus pro vtraque vim habet exponentis distantiae, sic quidem vt fiat ${}^{m}\mathfrak{A}\overset{r-1}{a} = {}^{m}\mathfrak{R}\,r$, siue pro $* = 2;\ 3;\ 4;\ \ldots\ n$, ${}^{m}\mathfrak{A}\overset{*-1}{a}$ $= {}^{m}\mathfrak{B}\,\mathfrak{b};\ {}^{m}\mathfrak{C}\,\mathfrak{c};\ {}^{m}\mathfrak{D}\,\mathfrak{d};\ \ldots\ {}^{m}\mathfrak{N}\,\mathfrak{n}$; quippe $\mathfrak{B}, \mathfrak{b}$; $\mathfrak{C}, \mathfrak{c}$; $\mathfrak{D}, \mathfrak{d}$; ... $\mathfrak{N}, \mathfrak{n}$ distant ab $\mathfrak{A}, \mathfrak{a}$, vnitatibus vna, duabus, tribus, ... $n-1$, respectu nimirum habito loci, quem coëfficientes binomiales suo quisque ordine occupant, nec non exponentis classium, ad quas coëfficientes polynomiales spectant. Illi quidem coëfficientes aeque ac valores asterisci pendent tantummodo a classe siue multitudine elementorum, polynomiales contra pendent simul a numeris $\alpha, \beta, \gamma, \delta, \ldots$, i. e. ab exponentibus repetitionis elementorum aequalium in singulis complexionibus (x).

2) Ex formula modo tradita colligere licet *regulam generalem*, determinandi coëfficientem $n+1^{tum}$ seriei $y = ax^{\mu} + bx^{\mu+\delta} + cx^{\mu+2d} + \ldots$ ad potestatem quamuis m^{tam} eleuatae. α) Quaerantur nimirum pro indice $\begin{pmatrix}1 & 2 & 3 & 4 & \ldots\\ b, & c, & d, & e, & \ldots\end{pmatrix}$ complexiones rite ordinatae summae n secundum omnes classes; β) singulis complexionibus literalibus iungantur coëfficientes debiti polynomiales; γ) praeterea complexionibus ad eandem classem p^{tam} pertinentibus siue elementorum numero p praefigatur ${}^{m}\mathfrak{P}\,a^{m-p}$, denotante ${}^{m}\mathfrak{P}$ coëfficientem binomialem p^{tum} dignitatis m^{tae}. Aggregatum productorum literalium sic deductorum aequabitur coefficienti quaesito.

3) Qua igitur ratione determinatio coëfficientis generalis semper redit ad inuentionem complexionum summae certae: qui processus etiam pro exponente integro affirmatiuo sub-

(w) Charactere y (a, b, c, d, ...), quem *Rothius* sub nomine *Scalae* introduxit (Formulae de serierum reuersione demonstratio vniuersalis, *Lipsiae 1793. 4.* pag. 1.), indicatur, seriei y secundum potestates variabilis x progredientis coëfficientes esse ex ordine a, b, c, d, ...

(x) *Archiv* IV, 416, 417.

substitui potest in locum inuentionis classis singularis (§. XXV. 4.). Iam vero ex supra demonstratis constat, varios existere modos, complexiones istas summae datae inueniendi. Hinc etiam partes coëfficientem componentes diuerso ordine eruere ac dispónere licebit. Triplex hic inprimis varietas obseruanda est:

α) Eligendo modum *Hindenburgianum*, *classes* ex classibus deducendi (§. XVIII. A.), complexiones per Inuolutionem secundum classes dispositam exhibentur; tum $^{n}[C]$ abit in ^{n}I, atque habetur $y^{m} k(n+1) = \left(\begin{matrix} *-1 \\ {}^{m}\mathfrak{A}\, a \end{matrix}\right) a^{m-*}\, {}^{n}I$, siue etiam $= \overset{*-1}{{}^{m}\mathfrak{A}}\;\; a^{m-*}\, \mathfrak{j}\, {}^{n}I$; vbi litera minori $\mathfrak{j}$, ex analogia literarum $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ significatur, complexionibus singulis quibus constat Inuolutio ^{n}I, iungendos esse coëfficientes polynomiales: ita quidem vt sit $\mathfrak{j}\,{}^{n}I = \mathfrak{a}\,{}^{n}A + \mathfrak{b}\,{}^{n}B + \mathfrak{c}\,{}^{n}C + \ldots + \mathfrak{n}\,{}^{n}N$.

β) Porro complexiones summae n exhiberi etiam possunt per *Inuolutionem directe lexicographicam Moivreo-Hindenburgianam* (§. XVIII. B.); quam Inuolutionem exprimendo litera J (§. XVIII. B. 8.), fit $y^{m} k(n+1) = \left(\begin{matrix} *-1 \\ {}^{m}\mathfrak{A}\, a \end{matrix}\right) a^{m-*}\, {}^{n}J = \overset{*-1}{{}^{m}\mathfrak{A}}\;\; a^{m-*}\, \mathfrak{j}\, {}^{n}J$, vbi litera $\mathfrak{j}$ itidem habet significatum modo declaratum (α).

γ) Tertio denique ad reperiundas complexiones summae n in vsum vocare potest *Inuolutio inuerse lexicographica Boscovichiana* (§. XXV.), quae inuolutio cum etiam litera J insigniatur, formula praecedens (β) pro coëfficiente recurrit.

Corollarium 3.

§. XXIX. Formulae §pho praecedenti expositae, et regulae illis respondentes pro inueniendo coëfficiente quouis $n+1^{to}$, vniuersales sunt, atque etiam pro exponente m integro affirmatiuo adhiberi possunt. Pro tali exponente alia insuper habetur formula (§. XXVI. 4.), $y^{m} k(n+1) = m^{n+m}M \left(\begin{matrix} 1 & 2 & 3 & 4 & \ldots \\ a, & b, & c, & d, & \ldots \end{matrix}\right)$; perinde autem est, vtra harum formularum adhibeatur, i. e. num classis m^{ta} summae $n+m$ quaeratur, num vero in locum huius processus substituatur inuentio complexionum omnium summae n. Idem consensus etiam ex formulis supra (§. XIX. 3.) expositis apparet, ex quibus est

$$^{n+m}M \left(\begin{matrix} 1 & 2 & 3 & 4 & \ldots \\ a, & b, & c, & d, & \ldots \end{matrix}\right) = a^{m-*}\, {}^{n}(C) \left(\begin{matrix} 1 & 2 & 3 & 4 & \ldots \\ b, & c, & d, & e, & \ldots \end{matrix}\right) = a^{m-*}\, {}^{n}I = a^{m-*}\, {}^{n}J,$$

dum complexionibus fingulis classis m^{tae} iungantur coëfficientes polynomiales, hique pro complexionibus Inuolutionum a parte dextra aequationis refoluantur modo §. XXVI. often-fo, quilibet nimirum in coëfficientem binomialem et polynomialem. Cum vero formulae pro ${}^{n+m}M$ reftringantur ad $n \overline{\gtrless} m$, nec amplius valeant, nifi addita limitatione (§. XIX. 3.): expreffiones contra pro $y^m k(n+1)$ per inuolutiones ifta limitatione haud egent, fed potius fenfu analytico fiue refpectu *theoriae* pro vniuerfalibus haberi poffunt, quippe coëfficientibus binomialibus poft m^{tum} euanefcentibus fponte excluduntur complexiones ad claffem m^{tam} maiorem pertinentes.

2) Verumtamen exiftente m numero integro *minore* quam n, refpectu *praxeos* obferuanda eft differentia, prouti adhibeantur Inuolutiones fecundum claffes vel lexicographicae. Ex illis nimirum prodeunt complexiones indeque coëfficientis quaefiti $y^m k(n+1)$ fingulae partes eodem omnino ordine, ac fecundum formulam $m^{n+m}M$ fiue ope claffis fingularis m^{tae}; cumque in iftiusmodi inuolutionibus a claffibus inferioribus ad altiores progreffus fiat, attendere tantum oportet, vt proceffus *poft inuentam claffem* m^{tam} finiatur: qua obferuata cautione vtraque formula, tum ea quae ${}^{n+m}M$ quam altera quae ${}^{n}I$ inuoluit, aequa propemodum facilitate gaudere videtur. Quodfi contra inuolutiones binae lexicographicae adhibeantur, tum primo complexiones coëfficientem quaefitum praebentes alio ordine difpofitae funt, quam fecundum formulam alteram, quae claffem rite ordinatam ${}^{n+m}M$ praebet: deinde quum iftae inuolutiones a claffi fumma n^{ta} incipiant, quaerendo ad ductum formularum §. XXVIII. (3. β, γ) complexiones omnes fummae n, pro valore $n > m$ reperiuntur complexiones fuperfluae, quae ad determinationem coëfficientis quaefiti haud pertinent, quaeque ipfae rurfus exeunt in coëfficientium binomialium euanefcentium. Hinc manifeftum eft, formulis binis §. XXVIII. 3, β, γ pro exponente m integro affirmatiuo quaefitum non femper via breuiffima obtineri, fed potius illas nonnunquam (pro $n > m$) iufto prolixiores euadere, ac fuperflua admixta habere.

Quibus hactenus dictis illuftrari magis ftrictiusque definiri mihi videntur ea, quae fimili confilio breuius pronuntiauit HINDENBVRGIVS (y). Iure nimirum is defiderauit apud MOIVREVM et BOSCOVICHIVM hoc, quod duos cafus, fpecialem exponentis integri affirmatiui, et generalem exponentis indeterminati, haud fua quemque methodo, verum vtrumque methodo communi tractauerint: indeque idem aptiffime omnino duplicem exhibuit formulam, quarum vna cafum priorem fine ambagibus refoluit, altera cafum pofteriorem.

Corol-

(y) *Archiv* IV, 419.

Corollarium 4.

§. XXX. Quanquam ex hactenus traditis pro exponentibus tantum integris positiuis coëfficientis expressio ad determinationem classis singularis reduci videatur, pro exponente autem indeterminato ad inuentionem complexionum certae summae omnium secundum diuersas classes: exhibere tamen licet formulam, quae coëfficiens quilibet pro exponente etiam indeterminato per classem singularem exprimitur.

1) Est nimirum

$$y^m k(n+1) = {}^m\mathfrak{A}\, a^{m-1}\, {}^n_a A + {}^m\mathfrak{B}\, a^{m-2}\, {}^n_b B + {}^m\mathfrak{C}\, a^{m-3}\, {}^n_c C + \ldots$$
$$y(a,b,c,d,\ldots) \quad \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ b, & c, & d, & e, & \ldots \end{pmatrix} \qquad + {}^m\mathfrak{N}\, a^{m-n}\, {}^n_n N.$$

Posito nunc in formula §. XXVII. $m = n$, habetur

$$n^2\, {}^n N = {}^n\mathfrak{A}\, a^{n-1}\, {}^n_a A + {}^n\mathfrak{B}\, a^{n-2}\, {}^n_b B + {}^n\mathfrak{C}\, a^{n-3}\, {}^n_c C + \ldots$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ a, & b, & c, & d, & \ldots \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & \ldots \\ b, & c, & d, & \ldots \end{pmatrix} \qquad + {}^n\mathfrak{N}\, a^0\, {}^n_n N.$$

Quas binas formulas comparando apparet, prioris singulas partes prodire ex alterius partibus respondentibus, dum hae ducantur, suo quaeuis ordine respectiue in $\frac{{}^m\mathfrak{A}}{{}^n\mathfrak{A}} a^{m-n}$; $\frac{{}^m\mathfrak{B}}{{}^n\mathfrak{B}} a^{m-n}$; $\frac{{}^m\mathfrak{C}}{{}^n\mathfrak{C}} a^{m-n}$; . . . $\frac{{}^m\mathfrak{N}}{{}^n\mathfrak{N}} a^{m-n}$. Exinde haec obtinetur expressio: $y^m k(n+1) = a^{m-n} \cdot \frac{{}^m\overset{*}{\mathfrak{A}}}{{}^n\overset{*}{\mathfrak{A}}} n^2\, {}^n N$, pro scala et indice $y\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ a, & b, & c, & d, & \ldots \end{pmatrix}$, vbi factor $\frac{{}^m\overset{*}{\mathfrak{A}}}{{}^n\overset{*}{\mathfrak{A}}}$ sub forma factoris communis exhibitus (ad analogiam signi n plures coëfficientes polynomiales simul exhibentis, et signorum § XXVIII. simili modo adhibitorum) diuersos repraesentat factores, qui singuli seorsim definiendi sunt ex asterisco. Asterisci autem valor pro singulis complexionibus classis $2\,{}^nN$ coëfficientibus suis polynomialibus instructis, pendet ab exponente repetitionis elementi a, qui quidem si fuerit $= n-1$, $n-2$, $n-3$, . . . 0, erit asteriscus $* = 0, 1, 2, 3, \ldots n-1$; siue $\overset{*}{\mathfrak{A}} = \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \ldots \mathfrak{N}$, vel etiam asterisci valor numericus prodit, a multitudine elementorum excluso a, in quauis complexione subtrahendo vnitatem.

2) Formulam modo inuentam sequenti ratione transformare licet:

$$\text{Est } \frac{{}^m\mathfrak{A}}{{}^n\mathfrak{A}} = \frac{m}{n} = \frac{m(m-1)\ldots(m-n+1)}{1.2\ldots n} \cdot \frac{1.2\ldots n-1}{(m-1)\ldots(m-n+1)}$$

$$= \frac{{}^m\mathfrak{N}}{m-1\overline{\mathfrak{N}}^{1}};\quad \frac{{}^m\mathfrak{B}}{{}^n\mathfrak{B}} = \frac{m(m-1)}{n(n-1)} = \frac{m(m-1)\ldots(m-n+1)}{1.2\ldots n}\cdot\frac{1.2\ldots n-2}{(m-2)\ldots(m-n+1)}$$

$= \dfrac{{}^m\mathfrak{N}}{m-2\overline{\mathfrak{N}}^{2}}$; simili modo prodit $\dfrac{{}^m\mathfrak{C}}{{}^n\mathfrak{C}} = \dfrac{{}^m\mathfrak{N}}{m-3\overline{\mathfrak{N}}^{3}}$; $\dfrac{{}^m\mathfrak{D}}{{}^n\mathfrak{D}} = \dfrac{{}^m\mathfrak{N}}{m-4\overline{\mathfrak{N}}^{4}}$; et sic porro; vbi numeratore manente semper eodem lex progressus pro denominatore euidens est. Hinc

fit ${}^m yk(n+1) = a^{m-n}\cdot{}^m\mathfrak{N}\cdot\dfrac{{}^{2n}_{\;n}N}{m-*\overline{\mathfrak{N}}^{*}}$ $y\left(\begin{smallmatrix}1 & 2 & 3 & 4 & \ldots\\ a, & b, & c, & d, & \ldots\end{smallmatrix}\right)$, vbi rursus diuisor formam denominatoris Communis mentiens cum asterisco varios recipit valores pro diuersis complexionibus classem ${}^{2n}N$ constituentibus. Asteriscus semper est = asterisco prioris formulae (1) vnitate aucto, siue = multitudini elementorum, excluso a: hinc pro complexionibus, quae incipiunt cum a^{n-1}, a^{n-2}, a^{n-3}, ... a^0, est $* = 1, 2, 3, \ldots n$. (z)

Corollarium 5.

§. XXXI. 1) Posito §. XXVI. $x = 1$, series $y = ax + bx^2 + cx^3 + dx^4 + \ldots$ abit in hanc: $y = a + b + c + d + \ldots$ quae igitur hoc respectu pro specialiori haberi potest (a). Iam existente *primo* exponente m numero integro affirmatiuo,

$$\text{fit } p^m\tau(n+1) = m^{n+m}M\left(\begin{smallmatrix}1 & 2 & 3 & 4 & \ldots\\ a, & b, & c, & d, & \ldots\end{smallmatrix}\right);$$

hincque $p^m = m^mM + m^{m+1}M + m^{m+2}M + m^{m+3}M + \ldots = m'M$, vbi charactere 'M, virgula signo classis M a laeua adposita, secundum HINDENBVRGIVM notantur elementorum a, b, c, d, ... *combinationes* sic dictae simpliciter (ad summam definitam haud respicientes) classis m^{tae}, et litera m more hactenus seruato innuitur, singulis complexionibus rite ordinatis iungendum esse numerum, qui multitudinem permutationum eorundem elementorum indicat. Liquet nimirum, aggregatum complexionum cuiusuis classis pro summis definitis *omnibus* praebere ipsas complexiones *simpliciter*, in quibus nullius summae *certae* ratio habetur.

Eadem

(z) Ad has binas formulas generales (1, 2) nondum exhibitos, qua mihi quidem ad legem characteristicae combinatoriae iuste et satis concinne expressae videntur, perductus sum ex iis, quae *Hindenburgius* simili omnino ratione de formula reuersoria ad *vnam classem* reducenda obseruauit (*Problema solutum maxime vniuersale ad serierum reuersionem — absoluendam Paralipomenon*, pag. XXI. Coroll. III.)

(a) Alio respectu *Iac. Bernoullius* (Opp. II. 993.) et *Hindenburgius* (*Infinitin.* pag. 4. §. XV, 4. pag. 43. §. XXVIII, 7. pag. 147.) formulam $a + b + c + d + \ldots$ generaliorem esse statuunt.

Eadem formula $p^m = m'M$, quin ad feriem generalem (§. XXVI.) recurrendum fit, exinde etiam fequitur, quod poteftas m^{ta}, ex multiplicatione iterata feriei $a + b + c + d + \ldots$, in fe ipfam orta, comprehendere debeat partium feu elementorum a, b, c, d, ... complicationes omnes fecundum numerum m, admiffis quoque repetitionibus feu elementis aequalibus, et quibusuis eorundem elementorum diuerfis fitibus.

2) Pro exponente quouis indeterminato m habetur, vi theorematis binomialis, pofito $q = b + c + d + e \ldots$ et affumto indice $\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ b, & c, & d, & e, & \ldots \end{pmatrix}$

$$p^m = a^m + {}^m\mathfrak{A} a^{m-1} p + {}^m\mathfrak{B} a^{m-2} p^2 + {}^m\mathfrak{C} a^{m-3} p^3 + \ldots$$
$$= a^m + {}^m\mathfrak{A} a^{m-1} a'A + {}^m\mathfrak{B} a^{m-2} b'B + {}^m\mathfrak{C} a^{m-3} c'C + \ldots$$

vbi 'A, 'B, 'C, ... exprimunt Vniones, Biniones, Terniones ... fiue combinationes *fimpliciter* claffis 1^{mae}, 2^{dae}, 3^{tiae} ... elementorum b, c, d, ... exclufo a.

3) Hinc apparet, determinationem poteftatum polynomii p redire ad inueniendas *combinationes fimpliciter* cuiusuis claffis pro datis elementis (b). Quae quaeftio cum ad fcopum noftrum minus pertineat, fufficit, fequentem regulam breuiter commemoraffe (c).

Claffem

(b) *Nou. Syft. Permut.* p. XIX, 8-11. cf. *Polyn. Lehrf.* p. 230.

(c) *Polyn. Lehrf.* pag. 174; *Nou. Syft. Permut.* p. XIX, 10. Huiusmodi etiam regulam Algebraiftae, qui de numero combinationum quaefiuerunt, in exhibendis fingulis fpeciebus ante oculos habuiffe videntur. Memorandis hoc loco inprimis eft *Ioann. Buteo*, in cuius *Logiftica* (Lugduni MDLIX, 8.) occurrit p. 305. problema tum temporis forte nouum, hoc: "Ludens aleator tefferis quatuor, quaero quibus et quot modis inter fe diuerfis iacere poffit?" Auctor exhibet in peculiari tabula (p. 308. 309.) numerorum 1, 2, 3, 4, 5, 6, Vniones, Biniones, Terniones, Quaterniones, omnes *rite ordinatas*, quippe ipfe praecipit: (p. 306.) effe *"perpetuo feruandum, vt aequalis nota numeri vel maior praecedentem fe ipfa fequatur, et nusquam minor."* Claffes vocat *tabulas*, ordines claffium *partes* tabularum; quas ex elemento initiali difcerpit, fingulas complexiones *verficulos*. Iam de tabula quinta ex quarta deriuanda fic loquitur: "Partem primam tabulae quintae quarta iam defcripta facit, fi ad fingulos verfuum 126 praenotatur monas, ordinem quintum defcribens. Quem prolongabit dyas tot verficulis, quot habet ipfa quarta ex fecunda parte in finem, qui funt feptuaginta. Similiter et trias verficulis 35, tetras 15, pentas 5, exas vnico." Idem auctor occafione fumta ab alio problemate curiofo (Quaeft 92, pag 312.) exhibet tabulam amplam, (p. 312-27.) pro *variationibus* omnibus, claffium quatuor priorum, *literarum* fex, quas "vt fit traditio commodior" *numeris ad literas refpondentibus* 1, 2, 3, 4, 5, 6 denotat; cuius tabulae condendae hanc praefcribit regulam, claffem fequentem deriuandam effe ex praecedente, praeponendo complexionibus illius omnibus primum 1, deinde 2, 3, 4, 5, 6. Obferuat, numerum variationum in claffe fecunda effe 6 . 6, in tertia 6 . 6 . 6, in quarta 6 . 6 . 6 . 6, et fic porrò. Ceterum ipfe in exorfu folutionis ait: "ad conftructionem tabulae artificio non vulgari opus effe; idque tetigiffe hactenus neminem, nec effe vt inuentionem fortuitam quis expectet." Quae hactenus excerpta commemorare haud fuperfluum duxi, cum liber *Buteonis* (in Bibliotheca Vniuerfitatis noftrae afferuatus) fit rarior, atque auctoris peritia in arte combinatoria minus nota effe videatur. Obiter eum nominat *Leibnitius* in *Arte Combinatoria* pag. 5. De auctore cf. Ill. *Kaeft-*

neri

Classem primam constituunt ipsa elementa. Deinde, ex quauis classe deriuatur sequens, praemittendo elementum 1, tum elementa 2, 3, 4, et sic porro, complexionibus classis praecedentis singulis, exclusis semper iis complexionibus, quae incipiunt ab elemento minori, seu secundum ordinem alphabeticum priori eo ipso, quod esset praemittendum: qua nimirum cautione efficitur, vt complexiones omnes rite ordinatae prodeant. Aliam praeterea regulam tradidit HINDENBVRGIVS (d), idemque docuit, quomodo classes sub *forma inuolutoria* exhibere liceat: ad quam formam hic etiam, vti semper, egregie commodam, peruenitur per regulam ipsam praedictam, disponendo classes ex se inuicem successiue deriuatas eo ordine, quem exemplum adpositum satis manifestum reddit (e):

A

neri Geschichte der Mathematik seit der Wiederherstellung der Wissenschaften bis an das Ende des 18ten Jahrhunderts, Band 1. S. 468.

(d) *Polyn. Lehrs.* p. 175.

(e) *Klügelius* inuentionem potestatum polynomii $a + b + c + d \ldots$ ita proponit, vt ea ad eundem processum combinatorium reducatur, cuius auxilio polynomium $ax + bx^2 + cx^3 + dx^4 + \ldots$ eleuatur, ad discerptionem nimirum numerorum. Exhibeatur in formula (2), producto $b^\beta c^\gamma d^\delta \ldots$ species aliqua seu complexio ex combinationibus *simpliciter* classis r^{tae} elementorum b, c, d, . . . tum sponte liquet, esse debere $\beta + \gamma + \delta + \ldots$ pro singulis huiusmodi speciebus $= r$. Hinc quaerendum est, quot et quibus variis modis numerus r discerpi queat in numeros integros, qui deinceps numeri literis polynomii b, c, d, . . . tanquam exponentes iungantur. Iam eas complexiones istarum literarum, quae exponentes eosdem habent (licet literae ipsae diuersae sint), inter se *similes* appellat *Klügelius*: cuiusmodi igitur complexionibus idem etiam respondet coefficiens polynomialis. Aggregatum complexionum inter se similium cunctarum designetur per $\int b^\beta c^\gamma d^\delta \ldots$, vbi vna complexio instar omnium exprimitur, ex qua ceterae deriuantur, literas tantum mutando, exponentes seruando. Eiusdem obseruationis mentionem etiam fecit *Hindenburgius* (*Infinit.* §. XIII, 7. pag 36. §. XXII, 10. pag. 89-91.), eas complexiones, quae *Klügelio* audiunt *similes*, *eiusdem generis* nuncupans (p. 13. §. VI.) Quum vero neuter vberius explicauerit, quo pacto ex vna complexione reliquas *repraesentante*, hae ipsae (ex *genere* variae *species*) sint deriuandae, pauca de eo addere haud superfluum videtur. Si quidem exponentes $\beta, \gamma, \delta, \ldots$ (siue numeri ex aliqua discerptione τοῦ r orti, quorum multitudo sit k) omnes inter se differant, tum ex literis polynomii b, c, d, e, . . formandae sunt omnes *variationes* (combinationes cum permutationibus) classis k^{tae}, *exclusis repetitionibus*; quo facto literis singularum complexionum iungendo exponentes $\beta, \gamma, \delta, \ldots$, obtinentur omnes species eiusdem generis. Quod si vero inter numeros $\beta, \gamma, \delta, \ldots$ reperiantur aequales, tunc isto modo prodirent complexiones literarum pro identicis habendae indeque superfluae (e. g. bc^2d^2, bd^2c^2). In discerptione summae exponentium (siue numeri r) ratio simul haberi potest *permutationum*, ita quidem, vt summae definitae r exhibeantur *variationes* (*Polyn. Lehrs.* pag. 176. sq.), tunc pro quauis complexione numerica classis k^{tae} sufficit, reperire combinationes literales (rite ordinatas) eiusdem classis ex elementis b, c, d, . . . exclusis repetitionibus. Ex quibus hactenus dictis effici videtur, obseruationis istius de diuersis generibus et variis inde pendentibus speciebus vsum *practicum* difficultatibus haud exiguis premi, hincque forte praestare, in euolutione actuali potestatum polynomii $a + b + c + d \ldots$, abstrahendo a discerptione numerorum, combinationes simpliciter adhibere, quas formula (2) exprimit. Alio autem respectu illa obseruatio memorabilis omnino, nec non vtilis esse videtur, diuersa genera reperire,

'A	a,	b,	c
B'	aa,	ab,	ac
		bb,	bc
			cc
C'	aaa,	aab,	ac
		abb,	abc
			acc
		bbb,	bbc
			bcc
			ccc

a	a	a
a	a	b
a	a	c
a	b	b
a	b	c
a	c	c
b	b	b
b	b	c
b	c	c
c	c	c

Scholion 1.

De MOIVREI theoremate polynomiali.

§. XXXII. De processu combinatorio, quo vsi sunt MOIVREVS et BOSCOVICHIVS ad inueniendum coëfficientem $n+1$tum pro exponente indeterminato m, iudicare licebit iam ex formulis (β) (γ) §. XXVIII. 3. in memoriam reuocando ea, quae supra (§. XXIV. XXV.) exposita sunt. Quare sufficit, de solutionibus horum Analystarum pauca tantum adiicere.

Quanquam nexus doctrinae combinatoriae cum theoremate polynomiali iam ante MOIVREVM haud esset ignotus (f), propria tamen sunt eaque insignia huius Analystae in

nerire, ex quibus statim etiam diuersi coëfficientes polynomiales innotescunt, qui pro omnibus speciebus vtut multis et variis soli locum habere possunt (*Infinitin.* pag. 90.): quippe singulis speciebus eiusdem generis idem competit coëfficiens polynomialis. De modo, quo *Castilloneus* terminos ipsos siue producta literalia potestatum polynomii $a+b+c+d\ldots$ quaesiuit, cf. *Infinit.* §. XIV, 37.

(f) Iam anteaquam *Neutonus* theorema binomiale sub forma vniuersali exhibuisset, Algebraistae in definiendis potestatibus binomii integris, combinationibus ac permutationibus vsi fuisse videntur. Nec minus nota fuit regula, determinandi *numerum permutationum* rerum quotcunque admissis etiam repetitionibus. Hinc *Moivreus* istiusmodi regulam tanquam vulgo cognitam (*a rhule commonly given*) commemorat. Quod porro *coëfficientem polynomialem* attinet ex numero permutationum deriuandum, de illo ante *Moivreum* meditatus est *Leibnitius*. Is nimirum per litteras ($\frac{6}{16}$ Maii 1695. in *Leibnitii* et *Bernoullii* Commerc. T. I. p. 47.) nuntiauit *Bernoullio*, excogitasse se olim regulam, definiendi numerum coëfficientem cuiusuis termini (veluti $a^{\alpha} b^{\beta} c^{\gamma} \ldots$) in polynomii $(a+b+c+d\ldots)$ potestate quacunque occurrentis. Rescripsit *Bernoullius* ($\frac{8}{18}$ Iun. 1695. l. c. p. 54.) sibi quoque rem tentanti illico in mentem venisse regulam desideratam, quam *Leibnitius* deinceps (p. 66.) a sua in effectu non abludentem, in forma tantum diuersam, agnouit. Quod tamen inuentum neuter tum temporis publicauit. *Moivreus* autem sub idem fere tempus, theorematis binomialis Neutoniani extendendi caussa, formam polynomii generaliorem $(ax+bx^2+cx^3+dx^4+\ldots)$ aggressus est, in cuius solutione etiam coëfficientis polynomialis expressionem in perficiendo illo theoremate merita. Is enim discedens a forma polynomii $y = a + b + c + d + \ldots$ per *combinationes* sic dictas *simpliciter* tractata, primus, quantum equidem sciam, vsum *combinationum summae definitae* in extollendo ad potestatem quamuis polynomio formae alterius eiusque generalioris $y = ax + bx^2 + cx^3 + dx^4 + \ldots$ ostendit. Feliciter nimirum animaduertit, literas a, b, c, ... notando numeris ex ordine 1, 2, 3, ..., coëfficientem $n + 1^{tum}$ potestatis cuiusuis integrae m^{tae} seriei y prodire ex complexionibus classis m^{tae} summae $n + m$, iungendo singulis his complexionibus numeros multitudinem permutationum eorundem elementorum indicantes. Quanti fuerit momenti ea obseruatio, satis notum est. MOIVREVS autem haud omnem ex illa fructum cepit. Supra iam vidimus, qualem ille regulam pro inueniendis istis complexionibus praescripserit: talem scilicet, quae coëfficientes potestatum etiam pro exponentibus non integris affirmatiuis praeberet, cuiusmodi exponentes primo intuitu a regula prorsus exclusi esse videbantur. Vi huius regulae complexiones (siue producta literalia) coëfficientis $n + 1^{ti}$ reperiuntur, dum complexiones coëfficientis n^{ti} omnes ducantur in $\frac{b}{a}$; porro complexiones coëfficientium praecedentium $n - 1^{ti}$; $n - 2^{ti}$; $n - 3^{ti}$; ... multiplicentur in factores ex ordine sequentes: $\frac{c}{a}$; $\frac{d}{a}$; $\frac{e}{a}$; ... exclusis cuiusuis coëfficientis iis complexionibus, in quibus occurrit litera factoris respondentis numeratorem ordine alphabetico praecedens. Coëfficientem polynomialem pro complexione seu producto singulari $a^{m-p} b^{\beta} c^{\gamma} d^{\delta} \ldots$ (vbi $p = \beta + \gamma + \delta + \ldots$) exhibuit MOIVREVS formula $\frac{m(m-1)(m-2) \ldots (m-p+1)}{1.2\ldots\beta \,.\, 1.2\ldots\gamma \,.\, 1.2\ldots\delta \ldots}$, quae ita est comparata, vt exponentis m tanquam numeri integri non amplius ratio sit habenda, cum sufficiat numeros β, γ, δ, ... eorumque summam $= p$ esse, vti semper sunt, numeros integros. Quanquam ex regula modo expressa in definiendo coëfficiente aliquo recurrendum sit ad coëfficientes omnes praecedentes, ex iis tamen, quae supra (§. XXIV.) adnotata fuere, constat, MOIVREVM haud fugisse, quemuis coëfficientem independenter etiam a praecedentibus assignari posse.

Praedit, eamque ad exponentes etiam non integros affirmatiuos extendit. Quam solutionem duobus annis post (1697) publicauit auctor in *Transactionibus Philosophicis* (Vol. XIX. Nr. 230. pag. 619. sq). Haec equidem addenda duxi iis, quae de gloria repertae *infinitinomiorum methodi Leibnitio* potius quam *Moivreo* tribuenda, monuit *Hindenburgius* (*Infinitin.* p. 29). Exinde simul intelligitur, quo iure *Ioh. Bernoullius* et ipse idem sibi inuentum adscribat, quod vna cum theoremate de reuersione serierum *Hermanno* per litteras communicasse se refert (Opp. Vol. IV. Nr. CLXVIII. Obseruationes in Moivreum, p. 125; Nr CLXX. Remarques sur le Calc. Integr. de M. Stone, p. 173.). Ceterum in *Leibnitii* et *Bernoullii* epistolis vnice de coëfficiente numerico sermo est, non item de exhibendis singulis speciebus ($a^{\alpha} b^{\beta} c^{\gamma} \ldots$), seu productis ipsis literalibus potestatum polynomii: quae species quomodo per *combinationes simpliciter* euoluendae sint, accuratius ostendit *Iacobus Bernoullius*; quas vero etiam ex *Moivrei* formula posito $x = 1$ deriuare licet.

Praeterea idem obseruauit, cum minuantur elementorum b, c, d, . . . notae numericae, quaeuis vna vnitate, tum complexiones ex his elementis (seiuncto elemento a) eas omnes, quae coëfficientem $n+1$tum constituunt, praebere summam n; vnde in locum classis $m$tae pro summa $m+n$, cuius vsus immediatus per rei naturam ad exponentes tantum integros m restrictus est, substituere licet inuentionem complexionum omnium summae definitae n, qui processus combinatorius semper locum habet.

2) Quae post has etiam obseruationes in MOIVREI solutione desiderari adhuc poterant, felicissime ea suppleuit HINDENBVRGIVS. Qui nimirum acutissime animaduertit, separando elementum a, complexiones istas summae n in MOIVREI formula ordine lexicographico esse dispositas, easque ostendere inuolutionem, cuius construendae regulam facillimam praescripsit. *Vncias* porro, quas sic vocat, *Moivreanas*, i. e. numeros productis litteralibus iungendos, resoluit in coëfficientes binomiales et polynomiales, sicque sponte deductus est ad formulam alteram supra commemoratam (§. XXXVIII. 3. β)

$y^m k(n+1) = \left(\,^m\mathfrak{A}\alpha^{*-1}\right) a^{m-*}\, {}^{n}\mathfrak{J}$, hancque ob expeditam omnino inuolutionis lexicographicae constructionem vincere iudicauit facilitate eam (l. c. α), quam primo ipse ex Inuolutione secundum classes deduxerat (g).

Scholion 2.

De BOSCOVICHII formula pro polynomii potestatibus.

§. XXXIII. Quae in solutione Moivreana imperfecta videbantur, ea perficere alio modo aggressus est BOSCOVICHIVS. Is nimirum a serie $y = a + bx + cx^2 + dx^3 + \dots$ exorsus, statim ab initio literas b; c; d; . . . exclusa prima a, notat numeris 1; 2; 3; . . . Tum obseruat, potestatem seriei y exponentis cuiusuis integri m complecti omnes mioies (?), quae quidem ex singulis membris seriei, etiam repetitis, conflari possunt; porro potestatis eiusdem secundum exponentes τῶ x ordinatae $y^m = M + M^{I}x + M^{II}x^2 + M^{III}x^3 + \dots + M^{N}x^{n} + \dots$ membrum quoduis $M^{N}x^{n}$ componi ex productis omnibus m factorum,

$$a^{\alpha} \cdot (bx)^{\beta} \cdot (cx^2)^{\gamma} \cdot (dx^3)^{\delta} \dots = a^{\alpha} \cdot b^{\beta} \cdot c^{\gamma} \cdot d^{\delta} \dots x^{\beta + 2\gamma + 3\delta + \dots}$$

pro quibus fuerit $\beta + 2\gamma + 3\delta + \dots = n$, hincque coëfficientem generalem M^{N} comprehendere elementorum b, c, d, . . . complexiones quasuis summam n efficientes, hisque deinceps complexionibus singulis praeter a^{m-p} (existente $p = \beta + \gamma + \delta + \dots$ = multitudini elementorum) iungendum esse debitum numerum permutationum $= \frac{m(m-1)\dots(m-p+1)}{1 \dots \beta \cdot 1 \dots \gamma \cdot 1 \dots \delta}$. Quod igitur MOIVREVS obiter tantum obseruasse videbatur,

id

(g) *Archiv* IV, 392.

id ipſum pro baſi ſolutionis ſuae poſuit BOSCOVICHIVS; indeque etiam regulam tradidit, complexiones omnes certae ſummae n independenter a ſummis minoribus inueniendi, quam ipſam regulam ſupra ſatis ample explicauimus. Hanc itaque BOSCOVICHII ſolutionem breuiter exprimere licet formula tertia ſupra (§. XXVIII. 3. γ) tradita: $y^{m} k(n+1) = ({}^{m}\mathfrak{A}\, \overset{*-1}{a}) a^{m-*}\, {}^{n}J$, vbi litera ${}^{n}J$ denotatur Inuolutio ſummae n inuerſe lexicographica, et coëfficientes polynomiales ſecundum HINDENBVRGIVM reſoluti ſunt in binomiales et polynomiales, ob $\frac{m(m-1)\ldots(m-p+1)}{1\ldots\beta \,.\, 1\ldots\gamma \,.\, 1\ldots\delta\ldots} =$ $\frac{m(m-1)\ldots(m-p+1)}{1\ldots p} \,.\, \frac{1.2\ldots p}{1\ldots\beta \,.\, 1\ldots\gamma \,.\, 1\ldots\delta\ldots} = {}^{m}\mathfrak{P}p$. Ceterum BOSCOVICHIVS conſiderationem *claſſis ſingularis* haud prorſus neglexit, regulam vero, illam inueniendi, dedit vſui minus aptam (§. XXV.); nec etiam caſus binos problematis (§. XXVI. A. B), qui reſpectu exponentis ſiue integri - affirmatiui ſiue indeterminati locum habent, ſatis inuicem diſtinxit: (h) quanquam ipſum haud effugit, formulam ſuam generalem applicatam ad m numerum integrum affirmatiuum maiorem quam n, partes ſuperfluas habere, harumque omiſſione illam compendioſiorem reddi (i).

Scholion 3.

IACOBI BERNOVLLII euolutio poteſtatum polynomii $p = a + b + c + d + \ldots$

§. XXXIV. 1) IACOBVS BERNOVLLIVS accepto nuntio de MOIVREI ſolutione duplicem modum tradidit, polynomium $a + b + c + d + \ldots$ (quam formam Moivreanā $ax + bx^{2} + cx^{3} + dx^{4} + \ldots$ generaliorem eſſe putabat), ad poteſtatem *indefinitam* attollendi (k), quorum alter ex combinationibus et permutationibus deriuatus eſt (l), quem quidem vix breuius deſcribere licet, ac ipſis Auctoris verbis: "Quia per "combinationum Doctrinam (vide Stochaſticen meam Part. II. Cap. 8.) diſcimus, membra "poteſtatis cuiuſuis Multinomii alicuius aliter non exprimi, niſi per coacervationem com-"binationum partis radicis, factarum ſecundum exponentem aequalem poteſtatis indici; "coëffi-

(h) *Boscovichius* ſolutionem ſuam primo pro exponente integro affirmatiuo deduxit, eamque deinceps ad quosuis exponentes extendit, haud tamen addita demonſtratione ſufficienti (*Archiv* IV. 411.).

(i) l. c. pag. 262. §. 21. E coſa facile a vedere, che ove la potenza m ſia un numero determinato poſitivo intero, ſi ſcemerà la fatica di molto; giacche dovranno rigettarſi tutti i modi di comporre il numero n, che abbiano numero di parti maggior di m; mentre in eſſi modi ſi avrebbe nel numeratore m — m = o: poſto poi negli altri per m il ſuo numero, ſi eliderebbo vari numeratori da' denominatori, e il calcolo diverebbe aſſai piu ſemplice. cf. *Gherli* l. c. pag. 364.

(k) *Iac. Bernoullii* Opera T. II. Geneuae MDCCXLIV. N. CIII. Varia Poſthuma. Artic. I. Attollere Infinitinomium ad poteſtatem indefinitam, pag. 993 ſeq.

(l) De ſecundo modo fuſo complicatiore per differentiationes et integrationes procedente, ample expoſuit *Hindenburgius* Infinit. §. XVII. pag. 47-54. oſtenditque eius nexum cum formula recurrente infra (§. XXXV.) demonſtranda.

"coefficientem vero termini cuiusuis exprimi, per numerum permutationum litterarum "illum terminum constituentium; idcirco si series conuergens $a + b + c + d + e + \ldots$ "eleuanda sit ad potestatem m, multiplico a^m per 1; a^{m-1} per singulas reliquarum b, "c, d, e, ...; a^{m-2} per singulas biniones caeterarum bb, bc, bd, be, ... nec "non cc, cd, ce, ...; a^{m-3} per singulas earum terniones b^3, b^2c, b^2d ... "bc^2, bcd, ... etc., et ita consequenter, quousque progredi necesse fuerit; hac ratione

$$
\begin{array}{l}
1 \cdot a^m \\
(b + c + d + e + f + g + \ldots) a^{m-1} \\
\left.\begin{array}{r} bb + bc + bd + be + bf \ldots \\ + cc + cd + ce \ldots \\ + dd \ldots \end{array}\right\} a^{m-2} \\
\left.\begin{array}{r} b^3 + b^2c + b^2d + b^2e \ldots \\ bc^2 + bcd \ldots \\ \ldots\ldots \end{array}\right\} a^{m-3} \\
\left.\begin{array}{r} b^4 + b^3c + b^3d \ldots \\ + b^2c^2 \ldots \\ \ldots\ldots \end{array}\right\} a^{m-4} \\
\left.\begin{array}{r} b^5 + b^4c \ldots \\ \ldots \end{array}\right\} a^{m-5} \\
\left.\begin{array}{r} b^6 \ldots \\ \ldots\ldots \end{array}\right\} a^{m-6} \\
\ldots\ldots\ldots
\end{array}
$$

"Coëfficiens cuiusque termini inuenitur, considerando numerum permutationum litterarum quotcunque $a^p b^q c^r d^s \ldots$ (sumto $p + q + r + s \ldots = m$) generaliter esse
" $\frac{1 \cdot 2 \cdot 3 \cdot 4 \ldots m}{1 \cdot 2 \ldots p \cdot 1 \cdot 2 \ldots q \cdot 1 \cdot 2 \ldots r \cdot 1 \cdot 2 \ldots s}$, hoc est, facta diuisione per $1 \cdot 2 \ldots p$,
" $= \frac{m(m-1)(m-2) \ldots p+1}{1 \cdot 2 \ldots q \cdot 1 \cdot 2 \ldots r \cdot 1 \cdot 2 \ldots s}$."

2) Quae post hanc solutionem Bernoullianam desideranda adhuc essent, ostendit HINDENBVRGIVS (m), eaque ex omni parte suppleuit. Distinctius nimirum docuit, quomodo combinationes, conternationes, conquaternationes ... literarum Polynomii, siue etiam

(m) *Infinit.* §. VIII. pag. 14.

etiam species singulae pro classi indefinita nta actu sint euoluendae (n); deinde cum BERNOVLLII demonstratio omnis deriuata sit ex consideratione multiplicationis repetitae, et numeri permutationum, indeque pro exponentibus tantum integris affirmatiuis stricte valeat, HINDENBVRGIVS ope theorematis binomialis rigorose ostendit (o), eandem formulam (resoluendo tantum formulam coëfficientis polynomialis in coëfficientem binomialem et polynomialem) ad exponentem quemuis indeterminatum extendi posse.

Problema.

§. XXXV. Seriei polynomialis $y = a + bx + cx^2 + dx^3 + ex^4 + \ldots$ potestatem quamcunque mtam per formulas recurrentes euoluere.

Solutio.

(n) l. c. §. XI. pag. 17 sq. In *Arte Coniect.*, ad quam ablegat lectorem *Bernoullius*, occurrit p. 113. (cf. p. 83.) regula inueniendi combinationes, quae quoad essentiam consentit cum regula supra (§. XXXI. 3) commemorata.

(o) §. XV. pag. 39 seq. In eo adsentiri vix possum *Hindenburgio* (pag. 40. p. VIII. praefat.), "quod *Bernoullius* de exponentibus fractis et negatiuis haud cogitauerit, nec cogitare potuerit "ob coëfficientes polynomiales ex permutationum varietate deductos eamque ob caussam numeris integris positiuis vnice adstrictos". Namque *Bernoullius* ipse formulam, suam sic esse restringendam haud monet, verum potius repetita vice exponentem *indefinitum* appellat, atque etiam coëfficientis polynomialis talem expressionem praebet, in qua exponens non necessario tanquam numerus integer positiuus considerandus est; nouimus praeterea *Moivreum* itidem quae primo pro exponente integro affirmatiuo demonstrauerat, disertis verbis ad alios quoscunque exponentes extendisse. Concedendum sane est, hosce Analystas formularum suarum extensionem haud rigorose demonstrasse, id quod inde maxime explicandum videtur, quod transitus ab exponente polynomii potestatum integro affirmatiuo ad indeterminatum fiat per theorema binomiale, cuius ipsius demonstrationem vniuersalem rigorosam omnibusque numeris absolutam, qui dederit ante *Kaestnerum*, equidem noui neminem. Aliunde etiam constat, eundem Geometram aliis quoque occasionibus Analystas de non nimium fidendo inductioni sed firmius adstruendo propositiones vniuersales, iure suo admonuisse, ipsumque seuerius, quam fuerat mos, de rigore demonstrationum methodique concinnitate sollicitum fuisse. — Ceterum sicuti coëfficientes binomiales etiam pro exponentibus fractis et negatiuis accipiuntur, ita nec repugnare videtur, coëfficientium polynomialium notionem extendere, et pro exponente quouis m etiam non integro positiuo, producti $a^{m-p} b^{\beta} c^{\gamma} d^{\delta} \ldots$ (denotantibus literis $\beta, \gamma, \delta, \ldots$ et $p = \beta + \gamma + \delta \ldots$ semper numeros integros) coëfficientem polynomialem appellare numerum $\frac{m(m-1)\ldots(m-p+1)}{1\ldots\beta \cdot 1\ldots\gamma \cdot 1\ldots\delta\ldots}$, quin is resoluendus sit in coëfficientem binomialem et polynomialem strictius sic dictum: quamquam probe sciam, hanc resolutionem alio respectu in formulis vere combinatoriis necessariam esse. Quare apta omnino esse videtur formulae coëfficientis polynomialis ea demonstratio, quam ex theoremate binomiali, nullo respectu habito numeri permutationum, tradiderunt *Castillioneus* (l. c. p. 30 sq.) et *Schoenbergius* (*Infinit.* §. XIII. p. 29 seq.). Resolutionem hac via inuentam coëfficientis polynomialis in mere binomiales ad ipsam etiam computationem in numeris maxime idoneam esse, monuit *Hindenburgius* (*Inf.* p. 35.), cui vsui apprime inseruit tabula numerorum figuratorum ab eodem exhibita (Tab. III. l. c. p. 162-165. *Toepfer* l. c. p. 22. 155.).

Solutio.

1) Coëfficientes feriei datae, a, b, c, d, ... exprimantur per $yk1$, $yk2$, $yk3$, $yk4$, ..., quilibet $r+1^{tus}$, qui igitur in x^r ductus est, per $yk(r+1)$: fic vt habeatur $y = yk1 + yk2 . x + yk3 . x^2 + yk4 . x^3 + ... + yk(r+1)x^r + ... + yk(n+1) . x^n + ...$

Denotentur porro coëfficientes feriei quaefitae fiue affumtae, in quam euoluitur poteftas m^{ta} feriei datae, fimilem in modum, quo fit

$$y^m = y^m k1 + y^m k2 . x + y^m k3 . x^2 + y^m k4 . x^3 + ... + y^m k(n-r+1)x^{n-r} + ... + y^m k(n+1)x^n + ...$$

Tum folutio problematis eo redit, vt pro horum coëfficientium affumtorum valoribus formulae inueniantur, quarum ope quemlibet ex praecedentibus definire liceat.

2) Quem in finem fumatur differentiale τȣ y^m, eritque, diuidendo per dx,

$$\frac{my^{m-1}dy}{dx} = y^m k2 + 2y^m k3 . x + 3y^m k4 . x^2 + ... + (n-r)y^m k(n-r+1) . x^{n-r-1} + ... + ny^m k(n+1)x^{n-1} + ...$$

fiue multiplicando per feriem y,

$$(yk1 + yk2 . x + yk3 . x^2 + ... + yk(r+1) . x^r ... + ykn . x^{n-1} + ...)$$
$$(y^m k2 + 2y^m k3 . x + 3y^m k4 . x^2 + ... + (n-r)y^m k(n-r+1) . x^{n-r-1} + ... + ny^m k(n+1) . x^{n-1} + ...)$$

$$= \frac{mdy}{dx} . y^m$$

$$= m(yk2 + 2yk3 . x + 3yk4 . x^2 + ... + ryk(r+1) . x^{r-1} + ... + nyk(n+1) . x^{n-1} + ...)$$
$$(y^m k1 + y^m k2 . x + y^m k3 . x^2 + ... + y^m k(n-r+1) . x^{n-r} + ... + y^m kn . x^{n-1} + ...)$$

3) Productis, inter quae nunc aequatio eft inuenta, per multiplicationem euolutis, coëfficientes earundem dignitatum τȣ x vtrinque aequandi funt. Iam vero membrum alterutrius producti illud, quod poteftatem quamuis $n-1^{tam}$ τȣ x inuoluit, conflatur, multiplicando terminos vnius factoris dignitatibus x^0; x^1; x^2; x^3; ... x^{r-1} vel x^r; ... x^{n-1}; affectos, fuo quemuis ordine, in terminos alterius factoris cum dignitatibus

bus x^{n-1}; x^{n-2}; x^{n-3}; x^{n-4}; ... x^{n-r} vel x^{n-r-1}; ... x^0. Hinc aequatis membrorum coëfficientibus prodit:

$$\begin{aligned} yk1 \,.\, ny^m k(n+1) + yk2 \,.\, (n-1)y^m kn + yk3 \,.\, (n-2)y^m k(n-1) + \ldots \\ + yk(r+1) \,.\, (n-r)\, y^m k(n-r+1) + \ldots \\ + yk(n-1) \,.\, 2y^m k3 + ykn \,.\, y^m k2 \\ = myk2 \,.\, y^m kn + 2myk3 \,.\, y^m k(n-1) + 3myk4 \,.\, y^m k(n-2) + \ldots \\ + rm \,.\, yk(r+1) \,.\, y^m k(n-r+1) + \ldots + nm \,.\, yk(n+1) y^m k1 \end{aligned}$$

vndè fit $y^m k(n+1) =$,

$$\frac{(m-n+1)yk2 \,.\, y^m kn + (2m-n+2)yk3 \,.\, y^m k(n-1) + (3m-n+3)yk4 \,.\, y^m k(n-2) + \ldots + (rm-n+r)yk(r+1) \,.\, y^m k(n-r+1) + \ldots + nmyk(n+1) y^m k1}{nyk1}$$

Hac formula coëfficiens quilibet $n+1$tus poteſtatis mtae polynomii definitur per coëfficientes datos polynomii ipſius, ſimulque per coëfficientes praecedentes eiusdem poteſtatis cunctos, a primo inde, $p^m k1$, vsque ad ntum, $p^m kn$: ob quem *recurſum* ad coëfficientes antecedentes formula *recurrens* vocatur. Sic itaque coëfficientes quotlibet ſucceſſiue, ſuo quemuis ordine, modo ſimplici et regulari eruere licet.

Scholion.

§. XXXVI. Legis modo expoſitae, qua coëfficientes progrediuntur, et quisque eorum per praecedentes definitur, primus, quantum equidem ſciam, mentionem fecit GEORG CHEYNAEVS (p). Eandem legem tradit W. JONES (q), haud vero addita demonſtratione, nec formula generali, inductione tantum vſus ex ſex prioribus coëfficientibus. Reperitur praeterea huius legis inueſtigatio ex calculo differentiali, adhibitis differentialibus logarithmicis, apud EVLERVM (r). Primus autem, ſicuti theorematis binomialis, ita ſimiliter omnino theorematis etiam infinitinomialis, formula iſta expreſſi, demonſtrationem vniuerſalem ab omni exceptione liberam exhibuit KAESTNERVS (s). Demonſtratio §pho praecedenti tradita cum Kaeſtneriana quoad eſſentiam conuenit, forma tantum

(p) *Fluxionum Methodus inuerſa* Londini 1703. p. 57, 58. Refert de hoc libro *Hindenburgius* (Infinit. Dign. pag. 54.)

(q) *Synopſis* etc. pag. 173.

(r) Inſtitut. Calc. Different. P. II. Cap. VIII. §. 202.

(s) In Programmate Goettingae 1759 edito: Infinitinomii ad poteſtatem indefinitam eleuati formula; cf. *Eiusdem Analyſis des Unendlichen*, §. 56. pag. 41. ſeq.

tantum differt; vfu praefertim literae k coëfficientes defignantis. Similem demonftrationem nuper propofuit KLÜGELIVS (t), hoc faltem difcrimine, quod is loco *rationum differentialium* differentias *finitas* τ y^m (§. XXXV.) fumat, tumque diuidendo per Δx partes vtrinque a Δx liberas fibi inuicem aequales ftatuat, *non* quia Δx fiat $= 0$ (vti nonnulli differentialia explicant), *fed* quoniam valor huius differentiae *arbitrarius* eft nec cum ceteris quantitatibus vllo nexu iungitur, indeque ordinando aequationem fecundum illius poteftates pro fingulis his poteftatibus peculiaris aequatio locum habere debet. Idem auctor (u) vere iudicat cum KAESTNERO (v), formulam recurrentem perquam vtilem effe ad computationem *numericam* coëfficientium, qui quippe plerumque ex *ordine* quaeruntur; eandemque etiam ob progreffum fimplicem ac perfpicuum effe memorabilem.

Cum fecundum formulam recurrentem, refpiciendo tantum ad literas et abftrahendo a factoribus numericis ($m - n + 1$, $2m - n + 2$, ... nm), pro obtinendo $y^m k(n+1)$, ducatur $y^m kn$ in $\frac{yk2}{yk1} = \frac{b}{a}$, $y^m k(n-1)$ in $\frac{yk3}{yk1} = \frac{c}{a}$, $y^m k(n-2)$ in $\frac{yk4}{yk1} = \frac{d}{a}$, et fic porro; facile perfpicitur formulae illius analogia cum regula Moivreana (§. XXXII.), ex qua itidem producta literalia terminorum praecedentium ducuntur ex ordine in $\frac{b}{a}$, $\frac{c}{a}$, $\frac{d}{a}$ ... (w).

Ceterum vti formula Moivreana et ceterae combinatoriae ad formam feriei generaliorem $y = ax^\alpha + bx^{\alpha+\beta} + cx^{\alpha+2\beta} + \ldots$, patent, ita quoque formula recurrens eandem extenfionem admittit (§. XXVI, 5.) (x).

ARTI-

(t) *Polyn. Lehrf.* p. 77 feq.
(u) l. c. §. 23. p. 75.
(v) *Analyf. des Unendl.* p. 45. XIV.
(w) Quem confenfum memorat *Hindenburgius* (Infinitin. pag. 54. nro. 11.). Obferuandum infuper videtur, fecundum Moivreum non *omnia* producta literaria coëfficientium praecedentium multiplicari, fed per limitationem fupra (§. XXXII.) additam caueri, ne producta *identica* occurrant. Formula contra recurrens producta etiam identica fuppeditat, (quae deinceps in vnam fummam funt colligenda), indeque *fuperflua* admixta habet. Quod quidem cum in fingulorum terminorum coëfficientibus iterata vice accidat, oriuntur inde ambages, quibus occurrit methodus combinatoria. Adhibita praefertim *inuolutione directa lexicographica* Moivreo-Hindenburgiana, eae ipfae operationes, quibus producta literalia coëfficientis cuiusuis extra ordinem reperiuntur, fponte fimul offerunt producta coëfficientibus antecedentibus, fuo cuiuis ordine, debita. Quare hac coëfficientes definiendi ratione commode etiam ac compendiofe vti licebit, fi ii *ex ordine* quaerantur: maxime dum coëfficientes polynomii non *numerice*, fed *fpeciofe*, i. e. per literas dentur.
(x) cf. *Kaeftneri* Anal. Inf. p. 45. XV.

ARTICVLVS TERTIVS.

De Reuersione serierum.

PROBLEMA.

§. XXXVII. Proposita serie (*reuertenda*) hac: $z = ax^{\varrho} + bx^{\varrho+\delta} + cx^{\varrho+2\delta} + dx^{\varrho+3\delta} + \dots$ exprimere quamuis potestatem σ^{tam} τ8 x per seriem (*reuersam*) secundum dignitates τ8 z progredientem, cuius coëfficientes formulis *independentibus* exhibeantur.

Solutio.

1) Ex supra (§. XIII. 5.) demonstratis habetur

$$x^{\sigma} = \frac{\sigma}{\varrho} z^{\frac{-\sigma}{\varrho}} k1.z^{\frac{\sigma}{\varrho}} + \frac{\sigma}{\sigma+\delta} z^{\frac{-\sigma-\delta}{\varrho}} k2.z^{\frac{\sigma+\delta}{\varrho}} + \frac{\sigma}{\sigma+2\delta} z^{\frac{-\sigma-2\delta}{\varrho}} k3.z^{\frac{\sigma+2\delta}{\varrho}} + \dots$$

siue $x^{\sigma} k(n+1) = \frac{\sigma}{\sigma+n\delta} z^{\frac{-\sigma-n\delta}{\varrho}} k(n+1)$. Breuitatis gratia exponentes $\frac{\sigma}{\varrho}$; $\frac{\sigma+\delta}{\varrho}$, $\frac{\sigma+2\delta}{\varrho}$, ... $\frac{\sigma+n\delta}{\varrho}$, ... ad quos series data z eleuanda est, designentur literis ${}^{0}m$, ${}^{1}m$, ${}^{2}m$, ... ${}^{n}m$, ... vt fit $\frac{\sigma}{\sigma+n\delta} = \frac{{}^{0}m}{{}^{n}m}$; et $x^{\sigma} k(n+1) = \frac{{}^{0}m}{{}^{n}m} z^{-{}^{n}m} k(n+1)$; tum adhibita formula (§. XXVI. B. 8.) posito illic loco n, $-{}^{n}m$, et considerando a^{m} siue $a^{-{}^{n}m}$ tanquam factorem communem, sponte prodit

$$x^{\sigma} k(n+1) = \frac{{}^{0}m}{{}^{n}m} a^{-{}^{n}m} \left\{ {}^{-{}^{n}m}\mathfrak{A} \frac{a^{n}A}{a} + {}^{-{}^{n}m}\mathfrak{B} \frac{b^{n}B}{a^{2}} + {}^{-{}^{n}m}\mathfrak{C} \frac{c^{n}C}{a^{3}} + \dots + {}^{-{}^{n}m}\mathfrak{N} \frac{n^{n}N}{a^{n}} \right\}.$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots \\ b, & c, & d, & e, & \dots \end{pmatrix}$$

2) Per formulas pro polynomii potestatibus itidem supra expositas (§. XXVIII. 3.), formula reuersoria sic quoque exhiberi potest:

$x^{\sigma} k(n+1)$

$$x^{\sigma} k(n+1) = \frac{{}^{o}m}{{}^{n}m} \cdot \left[-{}^{n}m\, {}^{*}\!-\!1\, \mathfrak{A}\; a \right]_{a} - {}^{n}m - {}^{*}n\, 1 = \frac{{}^{o}m}{{}^{n}m} \left[-{}^{n}m\, {}^{*}\!-\!1\, \mathfrak{A}\; a \right]_{a} - {}^{n}m - {}^{*}n\, J$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ b, & c, & d, & e, & \ldots \end{pmatrix}$$

$$= \frac{{}^{o}m}{{}^{n}m} \cdot \frac{1}{a^{{}^{n}m+n}} : -{}^{n}m \frac{\overset{*}{\mathfrak{A}}}{{}^{n}\overset{o}{\mathfrak{A}}}\, n^{2n} N = \frac{{}^{o}m}{{}^{n}m} \cdot \frac{1}{a^{{}^{n}m+n}} \cdot -{}^{n}m\, \mathfrak{N} \cdot \frac{n^{2n} N}{-{}^{n}m - {}^{*}\overline{\mathfrak{N}}^{*}} .$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ a, & b, & c, & d, & \ldots \end{pmatrix}$$

Corollarium.

§. XXXVIII. 1) Propofita aequatione inter binas feries hac:
$\alpha z^{R} + \beta z^{R+D} + \gamma z^{R+2D} + \delta z^{R+3D} + \ldots = a x^{\varrho} + b x^{\varrho+\delta} + c x^{\varrho+2\delta} + d x^{\varrho+3\delta} + \ldots$ eius folutio, fiue expreffio vnius variabilis x, eiusue poteftatis x^{σ}, per alteram z, reduci poteft ad folutionem problematis praecedentis.

Ponatur feries fecundum x, $a x^{\varrho} + b x^{\varrho+\delta} + c x^{\varrho+2\delta} + \ldots = q$,
feries altera fecundum z, $\alpha z^{R} + \beta z^{R+D} + \gamma z^{R+2D} + \ldots = p$.
Ex illa fit, exprimendo x^{σ} per feriem fecundum q progredientem,

$$x^{\sigma} = \frac{\sigma}{\sigma} q^{\frac{-\sigma}{\varrho}} k1 . q^{\frac{\sigma}{\varrho}} + \frac{\sigma}{\sigma+\delta} q^{\frac{-\sigma-\delta}{\varrho}} k2 . q^{\frac{\sigma+\delta}{\varrho}} + \ldots$$

$$+ \frac{\varrho}{\sigma+n\delta} q^{\frac{-\sigma-n\delta}{\varrho}} k(n+1) . q^{\frac{\sigma+n\delta}{\varrho}} + \ldots$$

vbi coëfficiens quilibet hanc feriem ingrediens, velnti $q^{\frac{-\sigma-n\delta}{\varrho}} k(n+1)$, pro fcala $q(a, b, c, d, \ldots)$ definiendus eft. Iam cum vi aequationis propofitae fit $q = p$, poteftates τῶ q fecundum z euolui poffunt: eft nimirum

$$q^{\frac{\sigma+n\delta}{\varrho}} = p^{\frac{\sigma+n\delta}{\varrho}} k1 . z^{R\left(\frac{\sigma+n\delta}{\varrho}\right)} + p^{\frac{\sigma+n\delta}{\varrho}} k2 . z^{R\left(\frac{\sigma+n\delta}{\varrho}\right)+D}$$

$$+ p^{\frac{\sigma+n\delta}{\varrho}} k3 . z^{R\left(\frac{\sigma+n\delta}{\varrho}\right)+2D} + \ldots$$

in

in qua serie coëfficientes primus, secundus, ... potestatis $p^{\frac{\sigma+n\delta}{\varrho}}$ ad scalam p (α, β, γ, δ, ...) referuntur. Substituendo nunc potestates τȣ q in serie pro x^σ, obtinetur huius potestatis expressio desiderata secundum z, haec:

$$x^\sigma = \frac{\sigma}{\sigma} q^{-\frac{\sigma}{\varrho}} k1 . \left[p^{\frac{\sigma}{\varrho}} k1 . z^{\frac{R}{\varrho}\sigma} + p^{\frac{\sigma}{\varrho}} k2 . z^{\frac{R}{\varrho}\sigma+D} + p^{\frac{\sigma}{\varrho}} k3 . z^{\frac{R}{\varrho}\sigma+2D} + \ldots \right]$$

$$+ \frac{\sigma}{\sigma+\delta} q^{-\frac{\sigma-\delta}{\varrho}} k2 . \left[p^{\frac{\sigma+\delta}{\varrho}} k1 . z^{R\left(\frac{\sigma+\delta}{\varrho}\right)} + p^{\frac{\sigma+\delta}{\varrho}} k2 . z^{R\left(\frac{\sigma+\delta}{\varrho}\right)+D} + p^{\frac{\sigma+\delta}{\varrho}} k3 . z^{R\left(\frac{\sigma+\delta}{\varrho}\right)+2D} + \ldots \right]$$

$$+ \frac{\sigma}{\sigma+2\delta} q^{-\frac{\sigma-2\delta}{\varrho}} k3 . \left[p^{\frac{\sigma+2\delta}{\varrho}} k1 . z^{R\left(\frac{\sigma+2\delta}{\varrho}\right)} + p^{\frac{\sigma+2\delta}{\varrho}} k2 . z^{R\left(\frac{\sigma+2\delta}{\varrho}\right)+D} + \ldots \right]$$

$$+ \text{etc.} \qquad\qquad + \text{etc.}$$

in qua expressione coëfficientes potestatum serierum q et p, ex theoremate polynomiali ad scalas supra dictas definiendi, pro cognitis haberi possunt.

2) Proposita aequatione generaliori

$$(\alpha z^R + \beta z^{R+D} + \gamma z^{R+2D} + \ldots)^\Phi = (a x^\varrho + b x^{\varrho+\delta} + c x^{\varrho+2\delta} + \ldots)^\psi$$

habetur, radicem ψtam vtrinque extrahendo, et ponendo $\frac{\Phi}{\psi} = \omega$,

$$(\alpha z^R + \beta z^{R+D} + \gamma z^{R+2D} + \ldots)^\omega = a x^\varrho + b x^{\varrho+\delta} + c x^{\varrho+2\delta} + \ldots$$

Est autem potestas ωta seriei

$$\alpha z^R + \beta z^{R+D} + \gamma z^{R+2D} + \ldots = p^\omega k1 . z^{R\omega} + p^\omega k2 . z^{R\omega+D} + p^\omega k3 . z^{R\omega+2D} + \ldots$$

Quare aequatio illa abit in hanc:

$$p^\omega k1 . z^{R\omega} + p^\omega k2 . z^{R\omega+D} + p^\omega k3 . z^{R\omega+2D} + \ldots = a x^\varrho + b x^{\varrho+\delta} + c x^{\varrho+2\delta} + \ldots$$

indeque ea reuocata est ad formam (1), quippe coëfficientes $p^\omega k1$, $p^\omega k2$, $p^\omega k3$, ... per scalam p (α, β, γ, δ, ...) cogniti sunt.

PROBLEMA.

§. XXXIX. Proposita serie hac: $z = ax + bx^2 + cx^3 + dx^4 + \ldots$ exprimere x per seriem secundum potestates τȣ z progredientem, cuius coëfficientes formulis *recurrentibus* exhibeantur.

Solutio.

Solutio.

Methodus solutionis in eo consistit, vt *assumatur* series pro x, cui haec tribuenda est forma (y): $x = \dot{\mathfrak{A}}z + \dot{\mathfrak{B}}z^2 + \dot{\mathfrak{C}}z^3 + \dot{\mathfrak{D}}z^4 + \ldots$, vbi coëfficientes *ficti* (z) seu *assumti*, $\dot{\mathfrak{A}}$, $\dot{\mathfrak{B}}$, $\dot{\mathfrak{C}}$, ... more *Hindenburgiano* punctis supra scriptis notantur; tum haec series assumta rite coniungatur cum serie data, quo aequationes simplices determinationi coëfficientium istorum inseruientes reperiantur. Quod quidem duplici ratione assequi licet.

1) *Series assumta substituatur in serie data*, i. e. series illa euehatur ad potestatem secundam, tertiam, quartam, et sic porro, eaeque potestates supponantur in serie altera. Qua ratione fit, in auxilium adhibitis formulis theorematis polynomialis combinatoriis (§. XXVI.)

$$\begin{array}{lllllll}
z = ax & = a\dot{\mathfrak{A}}z & + a\dot{\mathfrak{B}}z^2 & + a\dot{\mathfrak{C}}z^3 & + a\dot{\mathfrak{D}}z^4 & + a\dot{\mathfrak{E}}z^5 & + \ldots \\
\quad + bx^2 & = & + b\mathfrak{b}^2 B & + b\mathfrak{b}^3 B & + b\mathfrak{b}^4 B & + b\mathfrak{b}^5 B & + \ldots \\
\quad + cx^3 & = & & + c\mathfrak{c}^3 C & + c\mathfrak{c}^4 C & + c\mathfrak{c}^5 C & + \ldots \\
\quad + dx^4 & = & & & + d\mathfrak{d}^4 D & + d\mathfrak{d}^5 D & + \ldots \\
\quad + \ldots & & & & & + \ldots &
\end{array}$$

Vnde aequationem ad 0 reducendo, ob coëfficientes singulorum terminorum seorsim euanescentes, hae prodeunt aequationes simplices:

$$\dot{\mathfrak{A}} = \frac{1}{a}$$

$$\dot{\mathfrak{B}} = -\frac{b\mathfrak{b}^2 B}{a}$$

$$\dot{\mathfrak{C}} = -\left(\frac{b\mathfrak{b}^3 B + c\mathfrak{c}^3 C}{a}\right)$$

$$\dot{\mathfrak{D}} = -\left(\frac{b\mathfrak{b}^4 B + c\mathfrak{c}^4 C + d\mathfrak{d}^4 D}{a}\right)$$

$$\dot{\mathfrak{E}} = -\left(\frac{b\mathfrak{b}^5 B + c\mathfrak{c}^5 C + d\mathfrak{d}^5 D + e\mathfrak{e}^5 E}{a}\right)$$

. . . .

quarum lex progressus satis manifesta est. Classes combinatoriae referuntur ad indicem $\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ \dot{\mathfrak{A}}, & \dot{\mathfrak{B}}, & \dot{\mathfrak{C}}, & \dot{\mathfrak{D}} & \ldots \end{pmatrix}$. Ex harum indole facile intelligitur, expressionem coëfficientis cuiusuis ingredi tantum coëfficientes praecedentes; hinc singulos coëfficientes ex ordine successiue definire licet.

2) *Alter*

(y) *Kaestner Analysis endlicher Grössen*, §. 690. p. 476 sq. edit. 3.

(z) cf. de hac denominatione Leibnitiana, *Polyn. Lehrs.* p. 63.

2) *Alter* modus procedendi est is, vt vice versa *series data substituatur in serie assumta*. Quare formando potestates seriei datae, primam, secundam, tertiam ..., easque respectiue in $\dot{\mathfrak{A}}$, $\dot{\mathfrak{B}}$, $\dot{\mathfrak{C}}$, ... ducendo, tumque aggregatum aequando variabili x, obtinetur:

$$\begin{array}{lllllll} x = \dot{\mathfrak{A}}z & = \dot{\mathfrak{A}}ax & + \dot{\mathfrak{A}}bx^2 & + \dot{\mathfrak{A}}cx^3 & + \dot{\mathfrak{A}}dx^4 & + \dot{\mathfrak{A}}ex^5 & + \ldots \\ + \dot{\mathfrak{B}}z^2 & = & + \dot{\mathfrak{B}}\mathfrak{b}^2B & + \dot{\mathfrak{B}}\mathfrak{b}^3B & + \dot{\mathfrak{B}}\mathfrak{b}^4B & + \dot{\mathfrak{B}}\mathfrak{b}^5B & + \ldots \\ + \dot{\mathfrak{C}}z^3 & = & & + \dot{\mathfrak{C}}\mathfrak{c}^3C & + \dot{\mathfrak{C}}\mathfrak{c}^4C & + \dot{\mathfrak{C}}\mathfrak{c}^5C & + \ldots \\ + \dot{\mathfrak{D}}z^4 & = & & & + \dot{\mathfrak{D}}\mathfrak{b}^4D & + \dot{\mathfrak{D}}\mathfrak{b}^5D & + \ldots \\ + \ldots & & & & + \ldots & & \end{array}$$

Hinc ex principiis notis sequentes prodeunt formulae:

$$\dot{\mathfrak{A}} = \frac{1}{a};\quad \dot{\mathfrak{B}} = -\frac{\dot{\mathfrak{A}}b}{a^2};\quad \dot{\mathfrak{C}} = -\frac{(\dot{\mathfrak{A}}c + \dot{\mathfrak{B}}\mathfrak{b}^3B)}{a^3};$$

$$\dot{\mathfrak{D}} = -\frac{(\dot{\mathfrak{A}}d + \dot{\mathfrak{B}}\mathfrak{b}^4B + \dot{\mathfrak{C}}\mathfrak{c}^4C)}{a^4};\quad \dot{\mathfrak{E}} = -\frac{(\dot{\mathfrak{A}}e + \dot{\mathfrak{B}}\mathfrak{b}^5B + \dot{\mathfrak{C}}\mathfrak{c}^5C + \dot{\mathfrak{D}}\mathfrak{b}^5D)}{a^5};\ \text{etc.}$$

vbi classes combinatoriae referuntur ad indicem $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots \\ a, & b, & c, & d, & e, & \ldots \end{pmatrix}$; quare pro $\mathfrak{b}^2B$, $\mathfrak{c}^3C$, $\mathfrak{b}^4D$, ... ponere licebat a^2, a^3, a^4, ... Hic alter modus, coëfficientes seriei reuersae definiendi, quoad vsum commodum omnino praeferendus est priori (1): maxime quod in exhibendis classibus combinatoriis ad elementa tantum simplicia eaque data a, b, c, d, ... respiciendum sit, in prioribus contra formulis (1) classes referantur ad coëfficientes assumtos $\dot{\mathfrak{A}}$, $\dot{\mathfrak{B}}$, $\dot{\mathfrak{C}}$, ... ceu elementa, quae et ipsa iam sunt composita (a).

Corollarium 1.

§. XL. 1) Aequatio inter duas series infinitas:

$$\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \ldots = ax + bx^2 + cx^3 + dx^4 + \ldots$$

simili modo resolui poterit. Assumatur nimirum pro variabili x series haec:

$x = \dot{\mathfrak{A}}z + \dot{\mathfrak{B}}z^2 + \dot{\mathfrak{C}}z^3 + \dot{\mathfrak{D}}z^4 + \ldots$, tum substituendo eam eiusque potestates in aequatione data, modo priori (1), §pho praecedenti exposito, sequentes prodeunt aequationes simplices:

$$\begin{aligned} \alpha &= a\dot{\mathfrak{A}} \\ \beta &= a\dot{\mathfrak{B}} + b\mathfrak{b}^2B \\ \gamma &= a\dot{\mathfrak{C}} + b\mathfrak{b}^3B + c\mathfrak{c}^3C \\ \delta &= a\dot{\mathfrak{D}} + b\mathfrak{b}^4B + c\mathfrak{c}^4C + d\mathfrak{b}^4D \\ \cdot & \quad \cdot \quad \cdot \quad \cdot \\ \overset{n-1}{\alpha} &= a\overset{n}{\mathfrak{a}}A + b\overset{n}{\mathfrak{b}}B + c\overset{n}{\mathfrak{c}}C + \ldots + \overset{n-1}{a}\,\overset{n}{\mathfrak{n}}N \end{aligned}$$

ex

(a) *Polyn. Lehrs.* pag. 297. *Nov. Syst. Permut.* p. XXXI. *Toepfer combinatorische Analytik*, p. 131.

ex quibus, respiciendo ad indicem $\begin{pmatrix} 1 & 2 & 3 & \dots \\ \dot{\mathfrak{A}}, & \dot{\mathfrak{B}}, & \dot{\mathfrak{C}}, & \dots \end{pmatrix}$ singulos coëfficientes assumtos successiue definire licet. Aequatio pro $\overset{n-1}{\alpha}$ etiam sic concinne exprimi potest: $\overset{n-1}{\alpha} = \overset{*}{a}\, j^n I$, vbi asteriscus exponentem distantiae denotat, cuius valor numericus = exponenti classis, vnitate minuto (b).

2) Aequatio generalior haec:

$$\alpha z^m + \beta z^{m+1} + \gamma z^{m+2} + \delta z^{m+3} + \dots = a x^m + b x^{m+1} + c x^{m+2} + d x^{m+3} + \dots$$

vbi primus serierum datarum exponens m, est numerus quilibet integer positiuus, simili omnino ratione tractanda est; vnde hae oriuntur aequationes:

$$\alpha = a m^m M$$

$$\beta = a m^{m+1} M + b \overset{1}{m}{}^{m+1} \overset{1}{M}$$

$$\gamma = a m^{m+2} M + b \overset{1}{m}{}^{m+2} \overset{1}{M} + c \overset{2}{m}{}^{m+2} \overset{2}{M}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\overset{n}{\alpha} = a m^{m+n} M + b \overset{1}{m}{}^{m+n} \overset{1}{M} + c \overset{2}{m}{}^{m+n} \overset{2}{M} + \dots + \overset{n}{a} \overset{n}{m}{}^{m+n} \overset{n}{M}.$$

Signum $\overset{r}{m}{}^{m+n}\overset{r}{M}$ exprimit classem summae $m+n$ eam, quae est r^{ta} post classem m^{tam}, i. e. classem $m+r^{tam}$, singulis complexionibus ductis in coëfficientes suos polynomiales.

Pro indice sumitur, vt in casu praecedente (1), $\begin{pmatrix} 1 & 2 & 3 & 4 & \dots \\ \dot{\mathfrak{A}}, & \dot{\mathfrak{B}}, & \dot{\mathfrak{C}}, & \dot{\mathfrak{D}} & \dots \end{pmatrix}$. Iam facile apparet, elementum $n+1^{tum}$ huius indicis simpliciter tantum in classi ${}^{m+n}M$ occurrere, quippe $n+1$ cum $(m-1).1$ facit iam summam $m+n$, illudque in classibus altioribus deficere. Quare aequationes istae inseruiunt singulis coëfficientibus successiue determinandis.

Ceterum obseruandum insuper est, ad solutionem harum aequationum inter duas series infinitas modum *alterum* (§. XXXIX. 2.) immediate haud applicari posse: quare modus *prior* latius patere videtur, dum seriem assumtam semper in quauis aequatione data substituere licet.

Corol-

(b) Sic etiam indices, literis $\overset{n-1}{\alpha}$ et $\overset{n-1}{a}$ superscripti, sunt exponentes distantiae a primis coëfficientibus α et a, notantque ii coëfficientes suo vtrumque ordine n^{tum}.

Corollarium 2.

§. XLI. 1) Ad reuertendam seriem generaliorem (c)

$$z = ax^{\varrho} + bx^{\varrho+\delta} + cx^{\varrho+2\delta} + \ldots$$

ex eaque eruendam quamuis poteſtatem variabilis x, ſimiliter formulas recurrentes adhibere licet. Aſſumta nimirum ſerie (d)

$$x^{\sigma} = \dot{\mathfrak{A}} z^{\frac{\sigma}{\varrho}} + \dot{\mathfrak{B}} z^{\frac{\sigma+\delta}{\varrho}} + \dot{\mathfrak{C}} z^{\frac{\sigma+2\delta}{\varrho}} + \ldots$$

ope ſubſtitutionis (§. XXXIX. 2.) prodeunt hae aequationes:

$$\dot{\mathfrak{A}} = \frac{1}{p^{\frac{\sigma}{\varrho}} k1}; \quad \dot{\mathfrak{B}} = -\frac{\dot{\mathfrak{A}} p^{\frac{\sigma}{\varrho}} k2}{p^{\frac{\sigma+\delta}{\varrho}} k1}; \quad \dot{\mathfrak{C}} = -\frac{\left(\dot{\mathfrak{A}} p^{\frac{\sigma}{\varrho}} k3 + \dot{\mathfrak{B}} p^{\frac{\sigma+\delta}{\varrho}} k2\right)}{p^{\frac{\sigma+2\delta}{\varrho}} k1};$$

$$\dot{\mathfrak{D}} = -\frac{\left(\dot{\mathfrak{A}} p^{\frac{\sigma}{\varrho}} k4 + \dot{\mathfrak{B}} p^{\frac{\sigma+\delta}{\varrho}} k3 + \dot{\mathfrak{C}} p^{\frac{\sigma+2\delta}{\varrho}} k2\right)}{p^{\frac{\sigma+3\delta}{\varrho}} k1};$$

et ſic porro, vbi coëfficientes poteſtatum ſeriei p, per theorema polynomiale definiendi, referuntur ad ſcalam p (a, b, c, d, ...).

2) Aequatio generalior

$$\alpha z^{R} + \beta z^{R+D} + \gamma z^{R+2D} + \ldots = ax^{\varrho} + bx^{\varrho+\delta} + cx^{\varrho+2\delta} + \ldots$$

reducitur ad praecedentem (1), modo iam antea expoſito. Introducta nimirum noua variabili $Z = ax^{\varrho} + bx^{\varrho+\delta} + cx^{\varrho+2\delta} + \ldots$ exprimatur x^{σ} per Z, eiusque poteſtates: quae deinceps ob $Z = \alpha z^{R} + \beta z^{R+D} + \gamma z^{R+2D} + \ldots$ ſecundum dignitates variabilis z euoluuntur (e).

Scholion.

Notitiae hiſtoricae de reuerſione ſerierum; in primis de MOIVREI et HINDENBVRGII ſolutionibus combinatoriis.

§. XLII. Methodus hactenus adhibita, per ſuppoſitionem ſeriei quaeſitae tanquam inuentae rite factis *ſubſtitutionibus* coëfficientes fictos ſeu aſſumtos determinandi, a NEV-

TONO

(c) *Nov. Syſt. Permut.* p. XXIX. *Toepfer* l. c. p. 127.

(d) De *forma* ſeriei cf. *Kaeſtner* l. c.

(e) Eundem modum, inueniendi x, docet *Kaeſtnerus* l. c. §. 692.

TONO et LEIBNITIO (f) primum fuit adhibita. Ille hac via duo theoremata sequentia pro reuersione serierum inuenit: posito nimirum *primo* $z = ay + by^2 + cy^3 + dy^4 + ey^5 + \ldots$ fore vicissim $y = \frac{z}{a} - \frac{b}{a^3}z^2 + \frac{2b^2 - ac}{a^5}z^3 + \frac{5abc - 5b^3 - a^2d}{a^7}z^4 + \frac{3a^2c^2 - 21ab^2c + 6a^2bd + 14b^4 - a^3e}{a^9}z^5 + \text{etc.}$; posito *deinde* $z = ay + by^3 + cy^5 + dy^7 + ey^9 + \ldots$, fore $y = \frac{z}{a} - \frac{b}{a^4}z^3 + \frac{3b^2 - ac}{a^7}z^5 + \frac{8abc - a^2d - 12b^3}{a^{10}}z^7 + \frac{55b^4 - 55ab^2c + 10a^2bd + 5a^2c^2 - a^3e}{a^{13}}z^9 + \text{etc.}$ Quin NEVTONVS adhibuit iam modum reuersionis talem, qui similis omnino est alteri modo (§. XXXIX. 2.), substituendi seriem datam ipsam eiusque potestates: quique, vt verbis Neutonianis vtar, "intelligi potest per hoc Exemplum. Proponatur aequatio ad aream "Hyperbolae $z = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \ldots$ Et partibus eius multipli"catis in se, emerget $z^2 = x^2 + x^3 + \frac{11}{12}x^4 + \frac{5}{6}x^5 + \ldots$, $z^3 = x^3 + \frac{3}{2}x^4 + \frac{7}{4}x^5$ "$+ \ldots$, $z^4 = x^4 + 2x^5 + \ldots$, $z^5 = x^5 + \ldots$ Iam de z aufero $\frac{1}{2}z^2$, et re"stat $z - \frac{1}{2}z^2 = x - \frac{1}{6}x^3 - \frac{5}{24}x^4 - \frac{13}{60}x^5 \ldots$ Huic addo $\frac{1}{6}z^3$, et fit $z - \frac{1}{2}z^2$ "$+ \frac{1}{6}z^3 = x + \frac{1}{24}x^4 + \frac{3}{40}x^5 \ldots$ Aufero $\frac{1}{24}z^4$, et restat $z - \frac{1}{2}z^2 + \frac{1}{6}z^3$ "$- \frac{1}{24}z^4 = x - \frac{1}{120}x^5$ etc. Addo $\frac{1}{120}z^5$, et fit $z - \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{24}z^4$ "$+ \frac{1}{120}z^5 = x$, quam proxime; siue $x = z - \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{24}z^4 + \frac{1}{120}z^5$ etc. (g)." Cum

(f) Vere obseruat *Hindenburgius* (Infinit. Dign. Praef. p. X. Not.), Leibnitium hanc methodum primum cum eruditis communicasse, in Schediasmate inscripto: *Supplementum Geometriae practicae sese ad Problemata transcendentia extendens, ope nouae Methodi generalissimae per series infinitas* (*Acta Erud. Lips.* a. 1693. April. pag. 178. cf. *Opp. Leibnit.* ex edit. L. *Dutens*, Tom. III. pag. 279.); in cuius fine de applicatione methodi ad reuersionem serierum siue potius ad solutionem aequationum breuiter sic loquitur Leibnitius: (pag. 281.) "Eadem methodo etiam *aequationum* "vtcunque assurgentium *radices* obtineri posse, manifestius est, quam vt explicari hoc loco sit opus." Attamen commemorandum videtur, hanc Methodum *Neutono* prius iam, anno nimirum 1676, cognitam ac vsitatam fuisse: etiamsi concedendum omnino sit, Leibnitium in eandem proprio Marte incidisse (Commercium Epistolicum D. *Ioh. Collins* et aliorum de Analysi promota — editio altera. Londini 1722. 8. pag. 186, 188.) Excerpta ex Neutoni chartis huc facientia primus publicauit *Wallisius* in Opp. Tom. II., qui prodiit 1693 (pag. 393. seq.). De problemate vniuersaliori reuersionis agit *Leibnitius* Opp. III. 366., cf. quae acute de eodem obseruauit *Rothius* (*Probl. de Serier. reuers.* pag. 26.) cf. *Hindenburg. Infinitin.* XV. *Nou. System.* XIII.

(g) *Commerc. Epist.* p. 187. cf. *Analysis per Quantitatum series, fluxiones, ac differentias*: cum enumeratione linearum tertii ordinis. Amstelodami 1723. 4 pag. 34, 35. Eodem modo *Stewartus*, Neutoni commentator, ex serie pro Arcu z per tangentem t, $z = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 \ldots$ quaerit seriem pro tangente per Arcum; idemque distinctius exprimit, quomodo euoluendo potestates seriei datae (z), secundam, tertiam, quartam, ... hae deinceps potestates in debitos factores (coëfficientes) sint multiplicandae, ita quidem, vt ex altera parte aequationis omnes termini, praeter primum (t), sese inuicem tollant. Qui processus quoad rationem atque effectum omnino coincidit cum modo altero supra (§. XXXIX, 2.) exposito (cf. *Is. Newtons* two treatises of the

 Qua-

Cum vero in reuerſione ſemper ſeries vel data vel aſſumta ad varias poteſtates ſit euehenda, MOIVREVS poſtquam theorema polynomiale tractauerat, alterum etiam problema de reuerſione ſerierum vlterius promouit (h). Cuius ſolutio, ob artificia combinatoria ab auctore primo adhibita, breuiter hoc loco commemoranda videtur. Pro aequatione:

$$az + bz^2 + cz^3 + dz^4 + ez^5 + fz^6 + \ldots = gy + hy^2 + iy^3 + ky^4 + ly^5 + my^6 + \ldots$$

hanc reperit ſeriem:

$$z = \frac{g}{a} + \frac{h - bAA}{a}y^2 + \frac{i - 2bAB - cA^3}{a}y^3 +$$

$$\frac{k - bBB - 2bAC - 3cAAB - dA^4}{a}y^4 + \frac{l - 2bBC - 2bAD - 3cABB - 3cAAC - 4dA^3B - eA^5}{a}y^5$$

$+$ etc. (i); vbi literis maioribus (*Capital Letters*) A, B, C, ... denotantur coëfficientes primus, ſecundus, tertius etc. ſeriei ipſius quaeſitae z. Iam obſeruat MOIVREVS, in expreſſione cuiusuis coëfficientis denominatorem conſtanter eſſe a; numeratoris partem primam eſſe coëfficientem ex ordine ſeriei $gy + hy^2 + iy^3 + \ldots$; in ceteris autem partibus literarum maiorum *coëfficientes* omnes *praecedentes* referentium *exponentes* conficere vbique ſummam aequalem exponenti τȣ y, indeque regulam fluere (k), haec producta literarum maiorum determinandi, his deinceps ſingulis iungendum eſſe ex coëfficientibus ſeriei $ax + bx^2 + cx^3 + \ldots$ eum, cuius index ſit aequalis multitudini iſtarum literarum; *vncias* tandem numericas exprimere multitudinem permutationum, quas literae maiores cuiusuis producti admittant. Quorum praeceptorum demonſtrationem condit MOIVREVS, dum aſſumit ſeriem $z = Ay + By^2 + Cy^3 + Dy^4 + \ldots$ eamque cum eiusdem poteſtatibus ad formulam ſuam polynomialem euolutis in aequatione data ſubſtituit (l). Ex

Quadrature of Curves and Analyſis by Equations of an infinite Number of terms, explained by *I. Stewart* Profeſſor of Math. in the Maryhal College and Univerſity of Aberdeen, London 1745. 4. p. 456.) Praeterea Analyſta ſupra iam laudatus *Gherli* modum, ſeriem datam ſubſtituendi in ſerie aſſumta, diſerte ſic enunciat: (l. c. pag 374. §. 902) "Se dalle data l'equazione $y = ax + $ "$bx^2 + \ldots$ con queſto di trovare il valore di x dato per y; cio ſi otterrebbe operando nel mo- "do ſteſſo praticato al num. 898. cioè fare cioè la x ugnale a una ſerie indeterminata, come $Ay +$ "$By^2 + \ldots$, *nella quale devonſi ſoſtituire tutte le potenze di y preſe dall' equazione data con alzarla ſucceſſivamente a queſte potenze*, indi mediante il paragone de' termini ricavare i valori di A, B, "C, ... dalla di cui ſoſtituzione nella ſerie indeterminata, ſi avra la ſerie cercata." Ex haſtenus commemoratis limitandum videtur id, quod *Toepferus* l. c. pag. 132. aſſerit, modum alterum reuerſionis ante Hindenburgium nusquam fuiſſe adhibitum. Erat ſane modus prior, qui a Moivrei inde temporibus maxime inualuit, (vnde is a Toepfero p. 131. vocatur Moivreanus, ſicuti alter Hindenburgianus), longe frequentior: de alterius autem vſu commodiore ſtrictius demum monuit Hindenburgius, quae quidem commoda tum praeſertim cernuntur, cum poteſtates ſerierum combinatorie euoluantur.

(h) Philoſoph. Tranſact. Vol. XX. for 1698. p. 190. ſeq.

(i) Membrum ſextum, a Moivreo inſuper adiectum, breuitatis gratia hic omiſſum eſt.

(k) Combine the Capital Letters as often as you can make the ſum of their Exponents equal to the Index of the power to which they belong. (l. c. pag. 191.)

(l) Addit *Moivreus*, ſimili modo ſe etiam aequationem generaliorem

$$az^m + bz^{m+1} + cz^{m+2} + \ldots = gy^m + hy^{m+1} + \ldots$$ reſoluiſſe (cf. §. XL, 2.).

Eandem

Ex hactenus expositis satis manifestum est, hanc Moivraei solutionem prorsus coincidere cum solutione supra (§. XL. 1.) vel pro casu speciali (§. XXXIX. 1.) tradita: nisi quod auctor illam *verbis* tantum, nec etiam *signis* expresserit (cf. *Töpfer* l. c. p. 125.) Talibus signis analytico-combinatoriis, iisque ad rem ipsam illustrandam aeque ac ad vsum omnino aptis exhibuit solutionem Moivreanam HINDENBVRGIVS (m); idemque adhibito modo substitutionis altero (XXXIX, 2.) addidit formulam similem, vsu commodiorem.

Scholion 2.

Continuatio;

de formulis reuersioni serierum inseruientibus, combinatoria §. XXXVII. et locali §. XIII.

§. XLIII. 1) Cum formulae pro reuersione serierum a MOIVREO et HINDENBVRGIO exhibitae essent *recurrentes*, ita vt inde quilibet coëfficiens seriei quaesitae per coëfficientes omnes praecedentes definiretur: insigniter meritus est de hoc problemate ESCHENBACHIVS, Hindenburgii discipulus, dum nouam formulam *independentem* exhibuit, cuius ope quemlibet coëfficientem extra ordinem independenter a praecedentibus reperire liceat. Sequendo nimirum ad Hindenburgii exemplum *priorem* reuersionis modum (XXXIX. 1.), atque exprimendo potestates seriei datae per formulam combinatoriam (§. XXVI.), feliciter inuenit ESCHENBACHIVS (n), proposita serie $z = ax^{\varrho} + bx^{\varrho+\delta} + cx^{\varrho+2\delta} + dx^{\varrho+3\delta} + \ldots$ et assumta pro x^{σ} serie altera hac:

$$x^{\sigma} = \dot{A}z^{\frac{\sigma}{\varrho}} + \dot{B}z^{\frac{\sigma+\delta}{\varrho}} + \dot{C}z^{\frac{\sigma+2\delta}{\varrho}} + \ldots,$$

fore huius seriei reuersae coëfficientem quemlibet $n+1^{\text{tum}}$, siue

$$\dot{x}^{\sigma}k(n+1) = -{}^{\circ}m.\left\{\frac{\mathfrak{A}^{n}A}{a} - {}^{n}m+1\,\frac{\mathfrak{A}\mathfrak{b}^{n}B}{2a^{2}} + {}^{n}m+2\,\frac{\mathfrak{B}\mathfrak{c}^{n}C}{3a^{3}} - {}^{n}m+3\,\frac{\mathfrak{C}\mathfrak{d}^{n}D}{4a^{4}} + \ldots \pm {}^{n}m+n-1\,\frac{\overset{-1}{\mathfrak{N}}\mathfrak{n}^{n}N}{na^{n}}\right\},$$

pro indice $\begin{pmatrix} 1 & 2 & 3 & 4 \ldots \\ b, & c, & d, & e, \ldots \end{pmatrix}$, posito ${}^{\circ}m = \frac{\sigma}{\varrho}$, ${}^{n}m = \frac{\sigma+n\delta}{\varrho}$.

3) HIN-

Eandem formam contemplatus est Illustr. *Tempelhoffius* (*Anfangsgründe der Analysis endlicher Größen*, pag. 605. seq.), vsus quidem modo reuersionis priori (XXXVII, 1), adhibita insuper formula recurrente pro polynomii dignitate m^{ta}, et substitutionibus iteratis.

Regula *Moivrei* pro reuersione serierum reperitur etiam apud *Hausenium* (*Element. Mathes.* pag. 178.), et in *Synopsi W. Jones* pag. 188.

(m) *Nov. Syst.* pag XXX, XXXI. cf. *Polyn. Lehrs.* pag. 296.

(n) De Serierum reuersione formulis analytico-combinatoriis exhibita — Lipsiae 1789. pag. 24.

2) HINDENBVRGIVS (o) accuratius contemplando formulam Eſchenbachianam videns, literas *diuerſi nominis* eſſe inuicem iunctas (coëfficientem nimirum binomialem primum $\mathfrak{A}$ cum polynomiali ſecundo $\mathfrak{b}$ et claſſi itidem ſecunda B, porro $\mathfrak{B}$ cum $\mathfrak{c}$C, $\mathfrak{C}$ cum $\mathfrak{d}$D ...) cogitauit de tollenda hac aſſymmetria; ſicque adhibita ſimplici omnino transformatione coëfficientium binomialium (ex qua eſt ${}^{m+1}\mathfrak{A} = \frac{2}{m} {}^{-m}\mathfrak{B}$, ${}^{m+2}\mathfrak{C} = -\frac{3}{m} {}^{-m}\mathfrak{C}$, ${}^{m+3}\mathfrak{C} = +\frac{4}{m} {}^{-m}\mathfrak{D} \ldots$), formulam Eſchenbachianam transmutauit in hanc, concinnam magis atque regularem:

$$x^{\sigma} k(n+1) = \frac{{}^{o}m}{{}^{n}m} \cdot \left(\frac{-{}^{n}m\mathfrak{A}\, a^{n} A}{a} + \frac{-{}^{n}m\mathfrak{B}\, b^{n} B}{a^{2}} + \frac{-{}^{n}m\mathfrak{C}\, c^{n} C}{a^{3}} + \ldots \right).$$

Quam porro comparando cum formula pro polynomio ad poteſtatem indeterminatam eleuando, ſponte deduxit formulam localem, ſingulari breuitate conſpicuam, hanc:

$$x^{\sigma} k(n+1) = \frac{\sigma}{\sigma + n\delta} z^{-\left(\frac{\sigma + n\delta}{\varrho}\right)} k(n+1),$$

per quam itaque formulaj regreſſoria ad infinitinomii dignitates eſt reducta. Ad eandem formulam reductam peruenit etiam ROTHIVS, tollendo ex formula Eſchenbachiana diuiſores 2, 3, 4, . . ., idemque praeterea addidit formulae ab ESCHENBACHIO ex inductione tantum petitae ſiue potius formulae ſimplicis reductae demonſtrationem rigoroſam atque directam, egregium omnino acuminis et ſtudii analytici ſpecimen (p).

3) ESCHENBACHIVS (p. 30.) et ROTHIVS (p. 23.) oſtenderunt inſuper applicationem formularum ſuarum, combinatoriae alter, alter localis, ad aequationem inter duas ſeries infinitas, huius formae:

$az^{\pi\varrho}$

(o) Hindenburgius articulum *Noui Syſtematis Permut.* etc. de Tabula IX. ſerierum reuerſioni deſtinata, quam tunc totam iam abſoluerat (p. XXXI.), his verbis concludit (propoſitis antea formulis recurrentibus, iisdem quae ſupra § XXXVIII. traditae ſunt): "Exhiberi etiam *ſtatim* poſſunt coëfficien-"tes terminorum feriei per *datos coëfficientes* a, b, c, d, . . . ſed non opus eſt his diutius immo-"rari." Quibus verbis formulas *independentes* innui monet *Toepferus*. (p. 127. Not. op.) refertque, Hindenburgium in exemplis *Newtoni* (§. XLI.) animaduertiſſe, quod producta literalia ſingula, quae occurrunt in cuiusuis termini coëfficientibus (ſiue potius in horum numeratoribus), praebeant ſummas conſtantes, ſucceſſiue creſcentes 2, 4, 6, 8, ... tribuendo literis a, b, c, . . . exponentes numericos 1, 2, 3, . . . ; hac porro obſeruatione innumeris aliis exemplis comprobata permotum Eſchenbachium, ſpe proſperi ſucceſſus fretum, reductionem laboris plenam formularum recurrentium ad independentes aggreſſum fuiſſe, ſicque (vt ait Hindenburgius, p. IX. *Paralipom.*) formulam *incognitam* diligenter quaerendo feliciter reperiſſe.

(p) Formulae de ſerierum reuerſione demonſtratio vniuerſalis ſignis localibus combinatorio-analyticorum vicariis exhibita. Diſſertatio academica, auctore M. Henrico Auguſto Rothe, Lipſiae (1793.)

$\alpha z^{\pi\varrho} + \beta z^{\pi(\varrho+\delta)} + \gamma z^{\pi(\varrho+2\delta)} + \ldots = a x^{\varrho} + b x^{\varrho+\delta} + c x^{\varrho+2\delta} + \ldots,$ quam quidem aequationem ita vterque resoluit, vt assumta noua quantitate incognita y, x primo per y, deinde y per z exprimi debeat (§. XLI. 2.). ROTHIVS porro adnotauit (p. 24.), ad hanc formam specialiorem reduci posse aequationem generalem: $\alpha z^{f} + \beta z^{f+g} + \gamma z^{f+2g} + \ldots = a x^{F} + b x^{F+G} + c x^{F+2G} + \ldots,$ dum fuerit $\frac{Fg}{fG}$ numerus *rationalis positiuus*. Cuius asserti ratio ab auctore haud diserte expressa haec est: Sit $\frac{Fg}{fG} = \frac{\mu}{\nu}$, vbi μ et ν denotant numeros quosuis integros, tunc posito $F = \varrho$, $G = \nu\delta$, series $a x^{F} + b x^{F+G} + c x^{F+2G} + \ldots$ redit ad formam $a x^{\varrho} + . x^{\varrho+\delta} + . x^{\varrho+2\delta} + \ldots$, dum in serie illa interpolati concipiantur inter terminum primum et secundum, nec non inter terminos quosuis sibi inuicem proximos, $\nu - 1$ termini cum coëfficientibus euanescentibus; deinde sumto $f = \pi\varrho$, erit $g = \mu\pi\delta$, hinc altera etiam series $\alpha z^{f} + \beta z^{f+g} + \gamma z^{f+2g} + \ldots$, post similem interpolationem terminorum $\mu - 1$, conueniet cum forma $\alpha z^{\pi\varrho} + . z^{\pi(\varrho+\delta)} + . z^{\pi(\varrho+2\delta)} + \ldots$

4) Quibus tandem disquisitionibus de reuersione serierum coronam imposuit HINDENBVRGIVS, dum aequationis latissime patentis, supra (§. XXXVIII. 1.) commemoratae, $\alpha z^{R} + \beta z^{R+D} + \gamma z^{R+2D} + \ldots = a x^{\varrho} + b x^{\varrho+\delta} + c x^{\varrho+2\delta} + \ldots$ vel alterius adhuc generalioris (§. XXXVIII. 2.) exhibuit solutionem formulis localibus et signis concinnis aptissime expressam; idemque simul dilucide ostendit (p. XV, XVI, Schol. I. II.), ex applicatione formae *Eschenbachio - Rothianae* (3) ad ea exempla, ad quae *directe* illa applicari nequeat, per interpolationem terminorum euanescentium enasci ambages atque difficultates, quae adhibita demum forma generaliore feliciter tolluntur; indeque huius formae vtilitatem ac necessitatem haud dubiam esse (q).

CAP. III.

(q) Aequatio conditionalis a *Rothio* expressa, $\frac{Fg}{fG} = \frac{\mu}{\nu}$, pro forma etiam Hindenburgiana locum habere debet, si quidem requiritur, vt potestates variabilis z in serie reuersa secundum exponentes arithmetice crescentes progrediantur: quam serierum formam communiter supponi satis constat. De modo colligendi in forma Hindenburgiana coëfficientes, qui ad easdem variabilis z dignitates pertinent, cf. quae monet auctor (*Paralip.* XIV. Exempl. 4.)

CAP. III.

PROBLEMATA GENERALIORA, AD REVERSIONEM SERIERVM SIVE SOLVTIONEM AEQVATIONVM PER SERIES SPECTANTIA.

PROBLEMA.

§. XLIV. Propofita aequatione $y = x - z \cdot \phi(x,y)$, exprimere $\psi(x,y)$ per feriem fecundum poteftates variabilis z progredientem: denotantibus $\phi(x,y)$, et $\psi(x,y)$ quasuis functiones τῶν x et y.

Solutio.

1) Confideremus duas functiones variabilis x, quae quantitatem arbitrariam conftantem α inuoluunt, tanquam functiones τῶν x et α, easque fic exprimamus $\phi(x,\alpha)$, $\psi(x,\alpha)$, tum fi ftatuatur $y = x - z\,\phi(x,\alpha)$, erit $\psi(x,\alpha) =$

$$\psi(x,\alpha) + z \cdot \phi(x,\alpha) \cdot \frac{d\psi(x,\alpha)}{dx} + \frac{z^2\, d\left(\phi(x,\alpha)^2 \cdot \frac{d\psi(x,\alpha)}{dx}\right)}{1 \cdot 2\, dx} + \ldots$$

pofito a parte dextra aequationis poft differentiationes loco x, y.

2) Cum in hac aequatione quantitati α quiuis valor tribui queat, cumque y ab x non pendeat, feu y refpectu τῦ x tanquam quantitas conftans fpectari queat, fic vt tantum x et z pro variabilibus habeantur, quarum vna eft functio alterius, loco α ponere etiam licet y, eritque tum $\psi(x,y) =$

$$\psi(x,y) + z\phi(x,y) \cdot \frac{d\psi(x,y)}{dx} + \frac{z^2\, d\left(\phi(x,y)^2 \cdot \frac{d\psi(x,y)}{dx}\right)}{1 \cdot 2\, dx}$$
$$+ \frac{z^3\, d^2\left(\phi(x,y)^3 \cdot \frac{d\psi(x,y)}{dx}\right)}{1 \cdot 2 \cdot 3\, dx^2} + \ldots$$

pro refolutione aequationis $y = x - z\phi(x,y)$, vbi differentiationes ita funt inftituendae, vt tantum x pro variabili habeatur, y pro conftanti: tumque peractis differentiationibus loco x ftatuatur y.

Scholion.

§. XLV. Aequatio generalior $f(y, t, u, \ldots) = x - z\,\phi(x, y, t, u, \ldots)$, quae plures variabiles t, u, ... earumque functiones arbitrarias fignis f et ϕ expreffas inuoluit, fimili omnino ratione tractatur. Eft nimirum $\psi(x, y, t, u, \ldots) =$

$\psi(x,$

$$\psi(x, y, t, u, \ldots) + z\varphi(x, y, t, u, \ldots) \cdot \frac{d\psi(x, y, t, u, \ldots)}{dx} + \frac{z^2 d\left(\varphi(x, y, t, u, \ldots)^2 \cdot \frac{d\psi(x, y, t, u, \ldots)}{dx}\right)}{dx} + \ldots$$

vbi a parte dextra aequationis in differentiatione tantum x pro variabili habetur, tumque loco x ponitur $f(y, t, u, \ldots)$. Confiderare nimirum licet $t, u, \ldots$ tanquam quantitates conftantes refpectu variabilium x, et z, atque etiam refpectu y, hinc $f(y, t, u, \ldots) = w$ ceu functionem τῦ y, vnde etiam y fit functio τῦ w, et $\varphi(x, y, t, u, \ldots)$, $\psi(x, y, t, u, \ldots)$ abeunt in functiones τῶν x et w. Sic haec forma redit ad priorem (§. XLV.). Sub hac forma PAOLI (r) aequationem contemplatus eft, alteram §pho praecedenti expofitam refoluit LEXELLIVS (s).

Problema.

§. XLVI. Propofita aequatione $u = o$, denotante u functionem quamuis quantitatis x, definire aliam functionem X, eiusdem quantitatis x.

Solutio.

1) Cum fit u functio τῦ x, vice verfa erit x functio τῦ u, hincque etiam X ceu talis functio confiderari poterit. Iam cum defideretur valor τῦ X pro $u = o$, prodibit is ex functione X, ponendo pro u, o feu $u - u$: quare valor quaefitus erit $=$

$$X - \frac{u\,dX}{du} + \frac{u^2 d^2 X}{1.2\,du^2} - \frac{u^3 d^3 X}{1.2.3\,du^3} + \ldots$$

2) In hac expreffione differentialia ita funt transformanda, vt loco du, dx pro differentiali conftanti habeatur. Quod quidem ita obtinetur. Ponatur $\frac{dX}{du} = p$, $\frac{d^2 X}{du^2} = \frac{dp}{du} = q$, $\frac{d^3 X}{du^3} = \frac{dq}{du} = r$, $\frac{d^4 X}{du^4} = \frac{dr}{du} = s \ldots$ Pofito nunc porro $1 : \frac{du}{dx} = z$, habetur

$$p = \frac{dX}{dx} : \frac{du}{dx} = \frac{z\,dX}{dx};\quad q = \frac{dp}{du} = \frac{z\,dp}{dx} = \frac{zd\left(\frac{z\,dX}{dx}\right)}{dx};\quad r = \frac{z\,dq}{dx} = \frac{zd\left(zd\left(\frac{z\,dX}{dx}\right)\right)}{dx^2};$$

$$s = \frac{z\,dr}{dx} = \frac{zd\left(zd\left(zd\left(\frac{z\,dX}{dx}\right)\right)\right)}{dx^3};\quad t = \frac{z\,ds}{dx} = \frac{zd\left(zd\left(zd\left(zd\left(\frac{z\,dX}{dx}\right)\right)\right)\right)}{dx^4} \text{ etc.}$$

Inde fit valor quaefitus functionis X, refpondens aequationi $u = o$, $= X$

(r) Memorie di Matematica e Fifica della Societa Italiana, T. IV. Verona MDCCLXXXVIII, p. 438.
(s) Nov. Comment. Petrop. Tom. XVI. pag. 292.

$$= X - u\frac{zdX}{dx} + \frac{u^2}{1.2}\frac{zd\left(\frac{zdX}{dx}\right)}{dx} - \frac{u^3}{1.2.3}\frac{zd\left(zd\left(\frac{zdX}{dx}\right)\right)}{dx^2}$$
$$+ \frac{u^4}{1.2.3.4}\frac{zd\left(zd\left(zd\left(\frac{zdX}{dx}\right)\right)\right)}{dx^3} - \ldots$$

vbi x arbitrarie sumere licet. Est autem $z = \frac{1}{\frac{du}{dx}} =$ functioni cognitae τȣ x.

Corollarium.

§. XLVII. Quanquam in praecedenti problemate u sumatur ceu functio solius x, eadem tamen alias etiam quantitates variabiles ab x haud pendentes, y, t, s, ... inuoluere potest, quippe quae ipsae respectu x constantum locum sustinent. Hinc proposita aequatione $u = \varphi(x, y, t, s, \ldots) = o$, erit quaeuis functio X τῶν x, y, t, s, ...

$$= X - u\frac{zdX}{dx} + \frac{u^2}{1.2}\frac{zd\left(\frac{zdX}{dx}\right)}{dx} - \frac{u^3}{1.2.3}\frac{zd\left(zd\left(\frac{zdX}{dx}\right)\right)}{dx^2}$$
$$+ \frac{u^4}{1 \ldots 4}\frac{zd\left(zd\left(zd\left(\frac{zdX}{dx}\right)\right)\right)}{dx^3} - \ldots$$

in qua expressione differentiatio ita est instituenda, vt tantum x pro variabili habeatur, tumque huic quantitati valor arbitrarius siue constans siue = functioni τῶν y, t, s, ... tribuatur. Sub hac forma expressionem inuenit PAOLI (t). Ad analysin §pho praecedenti expositam perueni, considerando ea quae L. EVLERVS de resolutione aequationum ope Calculi differentialis tradidit (u), vnde primo hunc valorem τȣ x, respondentem aequationi $u = o$, deduxi:

$$x = x - uz + \frac{u^2}{1.2}\frac{zdz}{dx} - \frac{u^3}{1.2.3}\frac{zd\left(\frac{zdz}{dx}\right)}{dx}$$
$$+ \frac{u^4}{1.2.3.4}\frac{zd\left(zd\left(\frac{zdz}{dx}\right)\right)}{dx^2} - \ldots$$

Lecta deinceps expressione *Paoliana*, nostrae quoad formam simili, mox vidi hanc etiam ex iisdem principiis elici posse. Ceterum

(t) l. c. pag. 433. Auctor exinde deducit theorema *La Grangianum* (p. 437, 438.). Verum tamen hic demonstrandi modus mihi non satis euidens, ac difficile omnino esse videtur, legem generalem sic satis solide adstruere.

(u) Institt. Calc. Diff. P. II. Cap. IX. §. 234.

Ceterum haec expressio occasionem mihi suppeditauit, summandi sequentem seriem generaliorem, quae forma sua memorabilis esse videtur.

PROBLEMA.

§. XLVIII. Summare seriem infinitam: $W =$

$$X + yz\frac{dX}{dx} + \frac{y^2}{1.2} z d\left(\frac{z dX}{dx}\right) + \frac{y^3}{1.2.3} \frac{z d\left(z d\left(\frac{z dX}{dx}\right)\right)}{dx} + \frac{y^4}{1.2.3.4} \frac{z d\left(z d\left(z d\left(\frac{z dX}{dx}\right)\right)\right)}{dx^2} + \ldots$$

vbi denotant X et z functiones variabilis x, estque y quantitas ab x haud pendens.

Solutio.

1) Differentiando seriem secundum x habetur $\left(\frac{dW}{dx}\right) =$

$\frac{dX}{dx} + y d\left(\frac{z dX}{dx}\right) + \frac{y^2}{1.2} \frac{d\left(z d\left(\frac{z dX}{dx}\right)\right)}{dx} + \ldots$ Eandem differentiationem secundum y instituendo, quoniam tum x hinc etiam z et X pro constantibus habentur, prodit

$\left(\frac{dW}{dy}\right) = z\frac{dX}{dx} + yz d\left(\frac{z dX}{dx}\right) + \frac{y^2}{1.2} \frac{z d\left(z d\left(\frac{z dX}{dx}\right)\right)}{dx} + \ldots$ Quare erit $\left(\frac{dW}{dy}\right) = z\left(\frac{dW}{dx}\right)$.

2) Haec aequatio *differentias partiales* inuoluens sic resoluitur: Cum sit W functio τῶν x et y, ponatur $dW = p\,dx + q\,dy$, tum erit $q = \left(\frac{dW}{dy}\right) = z\left(\frac{dW}{dx}\right) = zp$, inde $dW = q\left(dy + \frac{dx}{z}\right)$. Hic sponte sequitur, esse debere W functionem summae $y + \int\frac{dx}{z}$. Iam pro $y = 0$ est $W = X$. Inde pro W eiusmodi functio summae $y + \int\frac{dx}{z}$ sumenda est, qualis est X quantitatis integralis $\int\frac{dx}{z}$. Cum igitur sit z functio cognita τȣ̃ x, posito $\int\frac{dx}{z} = \zeta$, determinari poterit x ceu functio τȣ̃ ζ, inde etiam X ceu talis functio exhibebitur $= f\zeta$. Qua inuenta, erit W similis functio summae $y + \zeta$, siue $W = f(y + \zeta)$.

PROBLEMA.

§. XLIX. Sit proposita aequatio: $x = P(x, y, z)$, denotante $P(x, y, z)$ talem functionem τῶν x, y et z, quae pro $z = 0$ abeat in functionem solius y. Definienda est functio variabilis x, ψ x, per seriem secundum z progredientem,

 Solutio.

Solutio.

1) Cum functio $P(x,y,z) = U$ pro $z = 0$ abeat in functionem τῦ y, $= Fy$, ea secundum z ordinata sic exprimi poterit: $U = Fy + z\varphi^{I}(x,y) + z^2\varphi^{II}(x,y) + z^3\varphi^{III}(x,y) + \ldots$ vbi est $\varphi^{I}(x,y) = \left[\frac{\partial U}{dz}\right]$, $\varphi^{II}(x,y) = \frac{1}{1.2}\left[\frac{\partial^2 U}{d^2 z}\right]$, $\varphi^{III}(x,y) = \frac{1}{1.2.3}\left[\frac{\partial^3 U}{d^3 z}\right], \ldots$ differentiando functionem U ita, vt tantum z pro variabili habeatur, ac deinde ponatur $z = 0$. Inde istae functiones pro cognitis sunt habendae.

2) Iam aequatio proposita $Fy = x - z\varphi^{I}(x,y) - z^2\varphi^{II}(x,y) - z^3\varphi^{III}(x,y) - \ldots$ secundum formulam LAGRANGII resoluta praebet: $\psi x =$

$$\psi x + (z\varphi^{I}(x,y) + z^2\varphi^{II}(x,y) + \ldots)\frac{d\psi x}{dx} + \frac{d\left\{(z\varphi^{I}(x,y) + z^2\varphi^{II}(x,y) + \ldots)^2\frac{d\psi x}{dx}\right\}}{1.2\,dx}$$

$$+ \frac{d^2\left\{(z\varphi^{I}(x,y) + z^2\varphi^{II}(x,y) + \ldots)^3\frac{d\psi x}{dx}\right\}}{1.2.3\,dx^2} + \ldots$$ differentiando tantum secundum x et post differentiationem ponendo pro x, Fy.

3) Est autem secundum formulam polynomialem (§. XXVI.)

$$(z\varphi^{I}(x,y) + z^2\varphi^{II}(x,y) + z^3\varphi^{III}(x,y) + \ldots)^2 = \mathfrak{b}^2Bz^2 + \mathfrak{b}^3Bz^3 + \mathfrak{b}^4Bz^4 + \ldots;$$
$$(z\varphi^{I}(x,y) + z^2\varphi^{II}(x,y) + z^3\varphi^{III}(x,y) + \ldots)^3 = \mathfrak{c}^3Cz^3 + \mathfrak{c}^4Cz^4 + \mathfrak{c}^5Cz^5 + \ldots;$$
$$(z\varphi^{I}(x,y) + z^2\varphi^{II}(x,y) + z^3\varphi^{III}(x,y) + \ldots)^4 = \mathfrak{d}^4Dz^4 + \mathfrak{d}^5Dz^5 + \mathfrak{d}^6Dz^6 + \ldots$$
etc. etc.

assumto indice $\left\{\begin{matrix} 1, & 2, & 3, & 4, & \ldots \\ \varphi^{I}(x,y), & \varphi^{II}(x,y), & \varphi^{III}(x,y), & \varphi^{IV}(x,y), & \ldots \\ \text{siue } \left[\frac{\partial U}{dz}\right], & \frac{1}{1.2}\left[\frac{\partial^2 U}{dz^2}\right], & \frac{1}{1.2.3}\left[\frac{\partial^3 U}{dz^3}\right], & \frac{1}{1.2.3.4}\left[\frac{\partial^4 U}{dz^4}\right], & \ldots \end{matrix}\right\}$. Hinc colligendo terminos ad easdem potestates τῦ z pertinentes, et ponendo $\frac{d\psi x}{dx} = \psi' x$, obtinetur functio quaesita.

$$\psi x = \psi x + z\mathfrak{a}^1 A.\psi' x + z^2\left(\mathfrak{a}^2 A\psi' x + \frac{1}{1.2}\frac{d(\mathfrak{b}^2 B\psi' x)}{dx}\right)$$
$$+ z^3\left(\mathfrak{a}^3 A\psi' x + \frac{1}{1.2}\frac{d(\mathfrak{b}^3 B\psi' x)}{dx} + \frac{1}{1.2.3}\frac{d^2(\mathfrak{c}^3 C\psi' x)}{dx^2}\right)$$
$$+ z^4\left(\mathfrak{a}^4 A\psi' x + \frac{1}{1.2}\frac{d(\mathfrak{b}^4 B\psi' x)}{dx} + \frac{1}{1.2.3}\frac{d^2(\mathfrak{c}^4 C\psi' x)}{dx^2} + \frac{1}{1.2.3.4}\frac{d^3(\mathfrak{d}^4 D\psi' x)}{dx^3}\right.$$
$$+ \text{etc.} \qquad\qquad + \text{etc.}$$

Quantitates combinatoriae $\mathfrak{a}^2A$, $\mathfrak{b}^2B$; $\mathfrak{a}^3A$, $\mathfrak{b}^3B$, $\mathfrak{c}^3C$; ... exprimuntur per x et y (1); in differentiando x tantum pro variabili habetur, y pro constante, tumque post differentiationem pro x ponitur Fy. *Scho-*

Scholion.

Solutio COUSINII cum supplemento.

§. I. Aliam huius problematis solutionem exhibuit COUSIN (v). Quam magis illustratam hoc loco proponere haud superfluum videtur, maxime cum euolutio legis generalis coëfficientium seriei ab auctore fuerit omissa.

1) Ex theoremate *Tayloriano* constat, ψx tanquam functionem variabilis z sequenti serie secundum potestates τῦ z progrediente exprimi posse:

$$\psi x = \psi x + z \cdot \frac{d\psi x}{dz} + z^2 \frac{d^2\psi x}{1.2\; dz^2} + z^3 \frac{d^3\psi x}{1.2.3\; dz^3} + \ldots + z^n \frac{d^n\psi x}{1.2\ldots n\; dz^n} + \ldots$$

dum in ea post differentiationes, sola z variabili assumta, pro z ponatur o, vnde primum membrum seriei, ψx, ex hypothesi abit in functionem cognitam τῦ y. Iam vero quaeritur, quomodo differentialia functionis ψx secundum z sint euoluenda, quae ipsa deinceps per y exprimere oportet. Ad quod obtinendum sequentia praemittenda sunt.

2) Functionis U et cuiusuis alius functionis τῶν x, y et z, $= Q$, differentiale completum duplici ratione considerari potest: primo, quatenus eae functiones sunt expressiones, continentes tres quantitates x, y et z, deinde quatenus eaedem sunt functiones duarum variabilium y et z, quippe per quas ipsas vi aequationis determinatur tertia x. Priori sensu dQ exprimetur tali formula: $q\,dx + q^{I}dy + q^{II}dz$, altero sensu hac: $q\,dy + q^{I}dz$. In illa quantitates q, q^{I}, q^{II}, sponte innotescunt, dum Q ceu functio trium variabilium modo consueto differentietur, primo secundum x, deinde secundum y, tertio secundum z: in altera formula concipitur quantitas x iam expressa esse per y et z: quod priusquam factum fuerit, quantitates q, q^{I}, haud pro cognitis haberi possunt.

Ad indicandum hoc discrimen peculiare signum δ adhibere conuenit, pro differentialibus sensu priori acceptis, ita vt sit $dU = \left(\frac{\delta U}{dx}\right)dx + \left(\frac{\delta U}{dy}\right)dy + \left(\frac{\delta U}{dz}\right)dz$

$$dQ = \left(\frac{\delta Q}{dx}\right)dx + \left(\frac{\delta Q}{dy}\right)dy + \left(\frac{\delta Q}{dz}\right)dz.$$

Ex modo dictis satis clarum est, quomodo differant inter se: $\left(\frac{\delta U}{dy}\right)$ et $\left(\frac{dU}{dy}\right)$; $\left(\frac{\delta U}{dz}\right)$ et $\left(\frac{dU}{dz}\right)$; $\left(\frac{\delta Q}{dy}\right)$ et $\left(\frac{dQ}{dy}\right)$; $\left(\frac{\delta Q}{dz}\right)$ et $\left(\frac{dQ}{dz}\right)$. Cum per aequationem assumtam sit $U = x$, ex priori formula, ob $dU = dx$, sequitur: $dx\left(1 - \left(\frac{\delta U}{dx}\right)\right) = \left(\frac{\delta U}{dy}\right)dy$

+

(v) Introduction à l'etude de l'Astronomie physique, pag. 307. seq.

$+\left(\frac{\delta U}{dz}\right)dz$. Hinc prodit $\left(\frac{dx}{dz}\right)=\left(\frac{\delta U}{dz}\right):1-\left(\frac{\delta U}{dx}\right)$, $\left(\frac{dx}{dy}\right)=\left(\frac{\delta U}{dy}\right)$ $:1-\left(\frac{\delta U}{dx}\right)$, et $\left(\frac{dx}{dz}\right)=\left(\frac{dx}{dy}\right)\cdot\left(\frac{\delta U}{dz}\right):\left(\frac{\delta U}{dy}\right)=V^{I}\left(\frac{dx}{dy}\right)$, exprimendo signo V^{I} quantitatem $\left(\frac{\delta U}{dz}\right):\left(\frac{\delta U}{dy}\right)$, quae per differentiationem determinari indeque pro cognita haberi potest. Hinc porro differentiale cuiusuis functionis τῆς x, $=\psi x$, secundum z, per differentiale secundum y exprimere licet. Sit nimirum $d\psi x=dx.\psi^{I}x$, erit $\left(\frac{d\psi x}{dz}\right)=\left(\frac{dx}{dz}\right)\psi^{I}x=\psi^{I}x\left(\frac{dx}{dy}\right).V^{I}=V^{I}\left(\frac{d\psi x}{dy}\right)$.

3) Quod functionem quamuis τῶν x, y et z attinet, ponebatur (2): $dQ=$ $\left(\frac{\delta Q}{dz}\right)dz+\left(\frac{\delta Q}{dy}\right)dy+\left(\frac{\delta Q}{dx}\right)dx$. Substituendo pro dx valorem inuentum (2), habetur differentiale τῆς Q, tanquam functionis τῶν y et z, $dQ=$

$$\left(\frac{\delta Q}{dz}\right)dz+\left(\frac{\delta Q}{dy}\right)dy+\left(\frac{\delta Q}{dx}\right)\left(\frac{\left(\frac{\delta U}{dy}\right)dy+\left(\frac{\delta U}{dz}\right)dz}{1-\left(\frac{\delta U}{dx}\right)}\right).$$ Inde fit $\left(\frac{dQ}{dy}\right)=\left(\frac{\delta Q}{dy}\right)$

$$+\frac{\left(\frac{\delta Q}{dx}\right)\left(\frac{\delta U}{dy}\right)}{1-\left(\frac{\delta U}{dx}\right)},\quad \left(\frac{dQ}{dz}\right)=\left(\frac{\delta Q}{dz}\right)+\frac{\left(\frac{\delta Q}{dx}\right)\left(\frac{\delta U}{dz}\right)}{1-\left(\frac{\delta U}{dx}\right)};$$ hinc ob $\left(\frac{\delta U}{dz}\right)=V^{I}\left(\frac{\delta U}{dy}\right)$, erit

$\left(\frac{dQ}{dz}\right)-V^{I}\left(\frac{dQ}{dy}\right)=\left(\frac{\delta Q}{dz}\right)-V^{I}\left(\frac{\delta Q}{dy}\right)$, vbi quantitas a parte dextra aequationis, quam differentiatio consueta suppeditat, tanquam cognita spectari, sicque $\left(\frac{dQ}{dz}\right)$ per $\left(\frac{dQ}{dy}\right)$ exprimi potest. Introducta igitur sequenti serie quantitatum per differentiationem determinandarum $\left(\frac{\delta V^{I}}{dz}\right)-V^{I}\left(\frac{\delta V^{I}}{dy}\right)=V^{II}$, $\left(\frac{\delta V^{II}}{dz}\right)-V^{I}\left(\frac{\delta V^{II}}{dy}\right)=V^{III}$, $\left(\frac{\delta V^{III}}{dz}\right)$ $-V^{I}\left(\frac{\delta V^{III}}{dy}\right)=V^{IV}$ etc. erit, posito pro Q successiue V^{I}, V^{II}, V^{III}, . . ; $\left(\frac{dV^{I}}{dz}\right)$ $=V^{I}\left(\frac{dV^{I}}{dy}\right)+V^{II}$, $\left(\frac{dV^{II}}{dz}\right)=V^{I}\left(\frac{dV^{II}}{dy}\right)+V^{III}$, $\left(\frac{dV^{III}}{dz}\right)=V^{I}\left(\frac{dV^{III}}{dy}\right)+V^{IV}$, etc.

4) His

4) His praemissis differentialia altiora τῶ ψx secundum z sequenti ratione in differentialia secundum y transformantur.

Primo est $\left(\frac{d^2\psi x}{dz^2}\right) = \frac{d\left(V^I \frac{d\psi x}{dy}\right)}{dz} = V^I \frac{d^2\psi x}{dz\,dy} + \left(\frac{dV^I}{dz}\right) \frac{d\psi x}{dy} = V^I d \frac{\left(\frac{d\psi x}{dz}\right)}{dy} +$
$\left(V^I \frac{dV^I}{dy} + V^{II}\right) \frac{d\psi x}{dy} = V^I \frac{dV^I}{dy} \frac{d\psi x}{dy} + V^I V^I \frac{d^2\psi x}{dy^2} + V^I \frac{dV^I}{dy} \frac{d\psi x}{dy} + V^{II} \frac{d\psi x}{dy}$
$= \frac{d\left(V^I V^I \frac{d\psi x}{dy}\right)}{dy} + V^{II} \frac{d\psi x}{dy}$. Hinc fit porro $\left(\frac{d^3\psi x}{dz^3}\right) = \frac{d\left(\frac{d^2\psi x}{dz^2}\right)}{dz} =$
$\frac{d\left(\frac{d\left(V^I V^I \frac{d\psi x}{dy}\right)}{dy}\right)}{dz} + \frac{d\left(V^{II} \frac{d\psi x}{dy}\right)}{dz}$; vbi pars prima est $= \frac{d\left(\frac{d\left(V^I V^I \frac{d\psi x}{dy}\right)}{dz}\right)}{dy} =$
$\frac{d\left((V^I)^2 \frac{d^2\psi x}{dz\,dy} + 2V^I \frac{dV^I}{dz} \frac{d\psi x}{dy}\right)}{dy} = \frac{d\left(\frac{(V^I)^2\, d\left(V^I \frac{d\psi x}{dy}\right)}{dy} + 2V^I\left(V^{II} + V^I \frac{dV^I}{dy}\right)\frac{d\psi x}{dy}\right)}{dy}$
$= \frac{d^2\left((V^I)^3 \frac{d\psi x}{dy}\right)}{dy^2} + \frac{2d\left(V^I V^{II} \frac{d\psi x}{dy}\right)}{dy}$; pars altera $= V^{II} \frac{d^2\psi x}{dz\,dy} + \frac{dV^{II}}{dz} \frac{d\psi x}{dy} =$
$V^{II} \frac{d\left(V^I \frac{d\psi x}{dy}\right)}{dy} + \left(V^{III} + V^I \frac{dV^{II}}{dy}\right) \frac{d\psi x}{dy} = \frac{d\left(V^I V^{II} \frac{d\psi x}{dy}\right)}{dy} + V^{III} \frac{d\psi x}{dy}$; quas partes
iungendo obtinetur $\left(\frac{d^3\psi x}{dz^3}\right) = \frac{d^2\left((V^I)^3 \frac{d\psi x}{dy}\right)}{dy^2} + \frac{3d\left(V^I V^{II} \frac{d\psi x}{dy}\right)}{dy} + V^{III} \frac{d\psi x}{dy}$.

Similiter

Similiter operando prodit

$$\left[\frac{d^4\psi x}{dz^4}\right]=\frac{d^3\left[(V^I)^4\frac{d\psi x}{dy}\right]}{dy^3}+\frac{6d^2\left[(V^I)^2V^{II}\frac{d\psi x}{dy}\right]}{dy^2}+\frac{4d\left[V^IV^{III}\frac{d\psi x}{dy}\right]}{dy}+\frac{3d\left[(V^{II})^2\frac{d\psi x}{dy}\right]}{dy}+V^{IV}\frac{d\psi x}{dy};$$

$$\left[\frac{d^5\psi x}{dz^5}\right]=\frac{d^4\left[(V^I)^5\frac{d\psi x}{dy}\right]}{dy^4}+\frac{10d^3\left[(V^I)^3V^{II}\frac{d\psi x}{dy}\right]}{dy^3}+\frac{15d^2\left[V^I(V^{II})^2\frac{d\psi x}{dy}\right]}{dy^2}+\frac{10d^2\left[(V^I)^2V^{III}\frac{d\psi x}{dy}\right]}{dy^2}+\frac{5d\left[V^IV^{IV}\frac{d\psi x}{dy}\right]}{dy}+V^V\frac{d\psi x}{dy}.$$

5) Haec differentialia vsque ad quartum exhibentur a COUSINIO. Exinde tamen neutiquam *lex progressus* perspicitur, cuius inuentio cum difficultate haud careat, operae pretium esse videtur, illam accuratius inuestigare, simulque formulam generalem pro $\frac{d^n\psi x}{dz^n}$ euoluere. Omnis difficultas in eo cernitur, vt lex quantitatum V^I, V^{II}, V^{III},... detegatur, seu ostendatur, qua ratione hae quantitates in expressionibus differentialium τȣ ψx inuoluantur. Iam expressio pro $\frac{d^n\psi x}{dz^n}$ complectitur plura differentialia ex ordine sibi succedentia ab $n-1^{to}$ vsque ad 0^{tum}. Sub quouis huiusmodi signo differentiali p^{to} occurrunt producta $p+1$ factorum, $V^{\pi^I}\colon V^{\pi^{II}}.$ $V^{\pi^{III}}\ldots V^{\pi^{p+1}}$, quorum indices summam n conficiunt $=\pi^I+\pi^{II}\ldots+\pi^{p+1}$. Repraesentet igitur $\frac{d^p\left(V^{\pi^I}V^{\pi^{II}}V^{\pi^{III}}\ldots V^{\pi^{p+1}}\frac{d\psi x}{dy}\right)}{dy^p}$ partem quamcunque τȣ $\frac{d^n\psi x}{dz^n}$, vbi abstrahitur a coëfficiente numerico, quippe hic plures partes aequiualentes cen

ceu coniunctas sistit; quas nunc singulas seorsim consideremus. Iam progressu facto ad differentiale τῦ ψx proximum, est $\frac{d^{n+1}\psi x}{dz^{n+1}} = \frac{d\left(\frac{d^n\psi x}{dz^n}\right)}{dz}$. Inde ex parte illa assumta oritur

$$d\frac{\frac{d^p\left(V^{\pi^I}V^{\pi^{II}}\dots V^{\pi^{p+1}}\frac{d\psi x}{dy}\right)}{dy^p}}{dz} = (\S.\ V.)\ d^p\frac{\frac{d\left(V^{\pi^I}V^{\pi^{II}}\dots V^{\pi^{p+1}}\frac{d\psi x}{dy}\right)}{dz}}{dy^p}.$$

Est autem $\frac{d\left(V^{\pi^I}V^{\pi^{II}}\dots V^{\pi^{p+1}}\frac{d\psi x}{dy}\right)}{dz} = V^{\pi^I}V^{\pi^{II}}\dots V^{\pi^{p+1}}\frac{d^2\psi x}{dz\,dy} +$

$$\frac{d\psi x}{dy}\frac{d\left(V^{\pi^I}V^{\pi^{II}}\dots V^{\pi^{p+1}}\right)}{dz} = V^{\pi^I}V^{\pi^{II}}\dots V^{\pi^{p+1}}\frac{d\left(V^I\frac{d\psi x}{dy}\right)}{dy} +$$

$$\frac{d\psi x}{dy}\left\{\begin{array}{l} V^{\pi^{II}}V^{\pi^{III}}\dots V^{\pi^{p+1}}\left(V^{\pi^I+1}+V^I\frac{dV^{\pi^I}}{dy}\right) \\ +V^{\pi^I}V^{\pi^{III}}\dots V^{\pi^{p+1}}\left(V^{\pi^{II}+1}+V^I\frac{dV^{\pi^{II}}}{dy}\right) \\ + \dots\dots\dots\dots \end{array}\right\} =$$

$$\frac{d\left(V^IV^{\pi^I}V^{\pi^{II}}\dots V^{\pi^{p+1}}\frac{d\psi x}{dy}\right)}{dy} + \left\{\begin{array}{l} V^{\pi^I+1}V^{\pi^{II}}V^{\pi^{III}}\dots V^{\pi^{p+1}} \\ +V^{\pi^I}V^{\pi^{II}+1}V^{\pi^{III}}\dots V^{\pi^{p+1}} \\ +V^{\pi^I}V^{\pi^{II}}V^{\pi^{III}+1}\dots V^{\pi^{p+1}} \\ + \dots\dots\dots \end{array}\right\}\frac{d\psi x}{dy}.$$

Quare ex parte $\frac{d^p\left(V^{\pi^I}V^{\pi^{II}}\dots V^{\pi^{p+1}}\frac{d\psi x}{dy}\right)}{dy^p}$ obtinentur partes sequentes differentiale proximum $\frac{d^{n+1}\psi x}{dz^{n+1}}$ ingredientes:

Yy d^{p+1}

$$\frac{d^{p+1}\left(V^{I}V^{\pi^{I}}V^{\pi^{II}}\ldots V^{\pi^{p+1}}\frac{d\psi x}{dy}\right)}{dy^{p+1}}+\frac{d^{p}\left(V^{\pi^{I}+1}V^{\pi^{II}}V^{\pi^{III}}\ldots V^{\pi^{p+1}}\frac{d\psi x}{dy}\right)}{dy^{p}}$$

$$+\frac{d^{p}\left(V^{\pi^{I}}V^{\pi^{II}+1}V^{\pi^{III}}\ldots V^{\pi^{p+1}}\frac{d\psi x}{dy}\right)}{dy^{p}}+\frac{d^{p}\left(V^{\pi^{I}}V^{\pi^{II}}V^{\pi^{III}+1}\ldots V^{\pi^{p+1}}\frac{d\psi x}{dy}\right)}{dy^{p}}$$

$+$

Inde ex productis quantitatum $V^{I}, V^{II}, V^{III}, \ldots$ quae in differentiali nto τῶ ψx occurrunt, producta pro differentiali $n+1$to assignare licet, dum 1) illis singulis V^{I} praemittitur, 2) in quolibet producto index singulorum factorum successiue vnitate augetur. Singulis productis iungendum est $\frac{d\psi x}{dy}$, tumque numero factorum existente $p+1$ praemittendum d^{p}, signum differentiationis ptae et quidem secundum y.

6) Sic igitur *inuolutio* inuenta est, cuius ope ad differentialia altiora lege satis simplici progredi licet. Cuius autem constructio cum haud parum operosa sit, praestat eam reuocare ad *inuolutionem* vsitatam ac simpliciorem, quae *combinationes summae datae* sistit. Iam satis manifestum est, $\frac{d^{n}\psi x}{dx^{n}}$ complecti cunctas combinationes elementorum $V^{I}, V^{II}, V^{III}, \ldots$ quorum indices conficiunt summam n. Repraesentet $(V^{I})^{\alpha}(V^{II})^{\beta}(V^{III})^{\gamma}\ldots$ quodlibet huiusmodi productum, seu complexionem rite ordinatam elementorum $V^{I}, V^{II}, V^{III}, \ldots$ vbi summa indicum $\alpha.1+\beta.2+\gamma.3+\ldots=n$, et numeri $\alpha, \beta, \gamma, \ldots$ denotant *exponentes repetitionis* eiusdem elementi, inter quos etiam euanescentes admittuntur, deficiente aliquo elemento. Cum vero ex lege inuolutionis nostrae plures complexiones aequiualentes occurrant, assignandus insuper est coëfficiens numericus producti $(V^{I})^{\alpha}(V^{II})^{\beta}(V^{III})^{\gamma}\ldots$ Pro quo quidem coëfficiente hanc formulam inueni: (w)

$$\frac{1.2.3\ldots(\alpha+2\beta+3\gamma\ldots)}{1.2\ldots\alpha.1.2\ldots\beta.1.2\ldots\gamma\ldots1^{\alpha}.(1.2)^{\beta}.(1.2.3)^{\gamma}\ldots}.$$

Existente igitur multitudine elementorum iunctorum $\alpha+\beta+\gamma+\ldots=p$, erit ob $\alpha+2\beta+3\gamma+\ldots=n$, τῶ $\frac{d^{n}\psi x}{1.2.3\ldots n.dx^{n}}$ pars quaeuis $=$

$$\frac{\dfrac{d^{p-1}\left\{\ldots(V^{I})^{\alpha}(V^{II})^{\beta}(V^{III})^{\gamma}\ldots\frac{d\psi x}{dy}+\ldots\right\}}{1.2\ldots\alpha.1.2\ldots\beta.1.2\ldots\gamma\ldots(1)^{\alpha}.(1.2)^{\beta}.(1.2.3)^{\gamma}\ldots}}{dy^{p-1}}$$

vbi

(w) cf. §. seq.

vbi pro p sumuntur numeri successiue ab n vsque ad 1: deinde sub quouis differentiali $p-1^{to}$ plura producta p factorum comprehenduntur, ea nimirum singula, quae prodeunt ex combinationibus classis p^{tae} elementorum V^{I}, V^{II}, V^{III}, ... ad summam n. Cum porro constet, producti $(V^{I})^{\alpha}(V^{II})^{\beta}(V^{III})^{\gamma}$.... coëfficientem polynomialem seu numerum indicantem eorundem elementorum permutationem esse $= \frac{1.2.3....(\alpha+\beta+\gamma+...)}{1.2..\alpha.1.2..\beta.1.2...\gamma}$, differentialis $\frac{d^{n}\psi x}{1.2...n dz^{n}}$ pars praedicta signo differentiationis $p-1^{tae}$ affecta satis concinne sic exprimi poterit: $\frac{1}{1.2..p}\frac{d^{p-1}\left(\overset{n}{p}P\frac{d\psi x}{dy}\right)}{dy^{p-1}}$, dum assumatur index: $\left[\begin{matrix}1, & 2, & 3, & 4, & ...\\ V^{I}, & \frac{V^{II}}{1.2}, & \frac{V^{III}}{1.2.3}, & \frac{V^{IV}}{1.2.3.4}, & ...\end{matrix}\right]$. Inde formulam pro $\left[\frac{d^{n}\psi x}{dz^{n}}\right]$ in hanc transformare licet: $\left[\frac{d^{n}\psi x}{1.2..n dz^{n}}\right] =$

$$\overset{n}{a}A\frac{d\psi x}{dy}+\frac{1}{1.2}\frac{d\left(\overset{n}{b}B\frac{d\psi x}{dy}\right)}{dy}+\frac{1}{1.2.3}\frac{d^{2}\left(\overset{n}{c}C\frac{d\psi x}{dy}\right)}{dy^{2}}+\frac{1}{1.2.3.4}\frac{d^{3}\left(\overset{n}{b}D\frac{d\psi x}{dy}\right)}{dy^{3}}$$

$$+\ldots+\frac{1}{1..p}\frac{d^{p-1}\left(\overset{n}{p}P\frac{d\psi x}{dy}\right)}{dy^{p-1}}+\ldots+\frac{1}{1...n}\frac{d^{n-1}\left(\overset{n}{n}N\frac{d\psi x}{dy}\right)}{dy^{n-1}}.$$

Quae formula sic breuius exprimitur: $\left[\frac{d^{n}\psi x}{1.2..n\, dz^{n}}\right] = \frac{1}{1...*}\frac{d^{*-1}\left(\overset{n}{j}I\frac{d\psi x}{dy}\right)}{dy^{*-1}}$, vbi $\overset{n}{j}I$ denotat inuolutionem combinationum summae n, quauis complexione ducta in coëfficientem debitum polynomialem. Asteriscus varios valores recipit, qui nimirum aequantur multitudini elementorum in singulis complexionibus.

7) Hinc tandem sequens obtinetur series (1): $\psi x =$

$$\psi x + z a^1 A \frac{d\psi x}{dy}$$

$$+ z^2 \left[a^2 A \frac{d\psi x}{dy} + \frac{1}{1.2} \frac{d\left(b^2 B \frac{d\psi x}{dy}\right)}{dy} \right]$$

$$+ z^3 \left[a^3 A \frac{d\psi x}{dy} + \frac{1}{1.2} \frac{d\left(b^3 B \frac{d\psi x}{dy}\right)}{dy} + \frac{1}{1.2.3} \frac{d^2\left(c^3 C \frac{d\psi x}{dy}\right)}{dy^2} \right]$$

$$+ z^4 \left[a^4 A \frac{d\psi x}{dy} + \frac{1}{1.2} \frac{d\left(b^4 B \frac{d\psi x}{dy}\right)}{dy} + \frac{1}{1.2.3} \frac{d^2\left(c^4 C \frac{d\psi x}{dy}\right)}{dy^2} + \frac{1}{1\ldots4} \frac{d^3\left(b^4 D \frac{d\psi x}{dy}\right)}{dy^3} \right]$$

$$+ \text{etc.}$$

$$+ z^n \left[a^n A \frac{d\psi x}{dy} + \frac{1}{1.2} \frac{d\left(b^n B \frac{d\psi x}{dy}\right)}{dy} + \frac{1}{1.2.3} \frac{d^2\left(c^n C \frac{d\psi x}{dy}\right)}{dy^2} + \ldots + \frac{1}{1.2..p} \frac{d^{p-1}\left(p^n P \frac{d\psi x}{dy}\right)}{dy^{p-1}} + \ldots + \frac{1}{1.2..n} \frac{d^{n-1}\left(n^n N \frac{d\psi x}{dy}\right)}{dy^{n-1}} \right]$$

$$+ \text{etc.} \qquad + \text{etc.}$$

assumto pro combinationibus indice hoc: $\begin{bmatrix} 1, & 2, & 3, & 4, & 5, & \ldots \\ V^{I}, & \frac{V^{II}}{1\,2}, & \frac{V^{III}}{1.2.3}, & \frac{V^{IV}}{1\ldots4}, & \frac{V^{V}}{1\ldots5}, & \ldots \end{bmatrix}$,

vbi functiones V^{I}, V^{II}, V^{III}, . . . ex aequationibus supra expositis (2. 3) innotescunt. Quoad differentiationes instituendas obseruandum est, pro z ponendum esse o. indeque pro x ex hypothesi functionem quampiam τȣ y; hinc in differentialibus, quae inuoluit series pro ψx, solius variabilis y ratio erit habenda.

Scholion.

§. LI. 1) Ad formulam (§. L. 6.) $\frac{1.2.3\ldots(\alpha+2\beta+3\gamma+\ldots)}{1.2\ldots\alpha \cdot 1.2\ldots\beta \cdot 1.2\ldots\gamma\ldots 1^{\alpha}.(1.2)^{\beta}.(1.2.3)^{\gamma}\ldots}$ pro

pro coëfficiente numerico producti $(V^{I})^{\alpha}(V^{II})^{\beta}(V^{III})^{\gamma}\ldots$ via indirecta perductus sum, considerando nimirum aequationem *specialiorem* $x = y + P(x, z) = U$: tum erit ob $\left(\frac{U}{dy}\right) = 1$, $V^{I} = \left(\frac{dU}{dz}\right) : \left(\frac{dU}{dy}\right) = \frac{\delta P(x, z)}{dz}$, porro ob $\left(\frac{\delta V^{I}}{dy}\right) = 0$, $V^{II} = \left(\frac{\delta V^{I}}{dz}\right) = \left(\frac{\delta^{2} P(x, z)}{dz^{2}}\right)$; $V^{III} = \left(\frac{\delta^{3} P(x, z)}{dz^{3}}\right)$; etc. Hinc apparet, quantitates $V^{I}, V^{II}, V^{III}, \ldots$ conuenire cum quantitatibus $\left(\frac{\delta U}{dz}\right)$, $\left(\frac{\delta^{2} U}{dz^{2}}\right)$, $\left(\frac{\delta^{3} U}{dz^{3}}\right)$, ... quae in serie solutionis prioris (§. XLIX.) occurrunt. Quam seriem cum altera §pho praecedenti inuenta comparando, formula praedicta pro coëfficiente numerico producti $(V^{I})^{\alpha}(V^{II})^{\beta}(V^{III})^{\gamma}\ldots$ sponte se offerebat: quae quanquam ex casu speciali esset deriuata, mox tamen apparebat, eandem latius patere ac ad aequationem generalem extendi.

2) Ceterum inuolutio singularis, quae complexiones elementorum signo V expressorum sistit, etiam per se abstrahendo ab hoc problemate, attentione Analystarum haud indigna esse videtur. Notentur elementa ex ordine numeris 1, 2, 3, 4, ... tum lex inuolutionis haec est: progressus a complexionibus summae n ad complexiones summae proxime maioris fit, dum 1) illis singulis praemittitur elementum 1, 2) singularum elementa singula ex ordine vnitate augentur. Differentia huius inuolutionis ab inuolutione *combinationum* et *variationum* ex sequenti schemate patebit:

1) *Inuolutio combinationum pro summa* 1 *vsque ad* 5.

1	1	1	1	1
1	1	1	2	
1	1	3		
1	2	2		
1	4			
2	3			
5				

2) *Inuolutio variationum.*

1	1	1	1	1
1	1	1	2	
1	1	2	1	
1	1	3		
1	2	1	1	
1	2	2		
1	3	1		
1	4			
2	1	1	1	
2	1	2		
2	2	1		
2	3			
3	1	1		
3	2			
4	1			
5				

3) *Inuolutio noua.*

1	1	1	1	1
1	1	1	2	
1	1	2	1	
1	1	1	2	
1	1	3		
1	2	1	1	
1	1	2	1	
1	1	1	2	
1	2	2		
1	1	3		
1	3	1		
1	2	2		
1	2	2		
1	1	3		
1	4			
2	1	1	1	
1	2	1	1	
1	1	2	1	
1	1	1	2	
2	1	2		
1	2	2		
1	1	3		
2	2	1		
1	3	1		
1	2	2		
2	1	2		
1	2	2		
1	1	3		
2	3			
1	4			
3	1	1		
2	2	1		
2	1	2		
2	2	1		
1	3	1		
1	2	2		
2	1	2		
1	2	2		
1	1	3		
3	2			
2	3			
2	3			
1	4			
4	1			
3	2			
3	2			
2	3			
3	2			
2	3			
2	3			
1	4			
5				

In inuolutionibus 2) et 3) plures complexiones occurrunt, quae situ tantum seu ordine elementorum differunt: quarum sumta repraesentatrice ea, quae rite est ordinata, quamque inuolutio 1) exhibet, veluti $1^{\alpha}\, 2^{\beta}\, 3^{\gamma}\, 4^{\delta} \ldots$, erit huius coëfficiens numericus, repetitionem seu multitudinem complexionum aequiualentium indicans; pro inuolutione (2) $= \frac{1.2.3\ldots(\alpha+\beta+\gamma+\delta+\ldots)}{1.2\ldots\alpha.\,1.2\ldots\beta.\,1.2\ldots\gamma.\,1.2\ldots\delta\ldots}$, pro inuolutione 3) $=$

$$\frac{1.2.3\ldots(\alpha+2\beta+3\gamma+4\delta+\ldots)}{1.2\ldots\alpha.\,1.2\ldots\beta.\,1.2\ldots\gamma.\,1.2\ldots\delta\ldots 1^{\alpha}(1.2)^{\beta}(1.2.3)^{\gamma}(1.2.3.4)^{\delta}\ldots}$$

Illa formula exprimi *numerum permutationum*, satis constat. Quod alteram attinet, ponamus eam veram esse pro complexionibus summae $n-1$, tum eadem obtinebit pro complexionibus summae n. Sit nimirum $1^{\alpha}2^{\beta}3^{\gamma}4^{\delta}5^{\varepsilon}\ldots$ quaeuis complexio huius summae, tum ea secundum legem inuolutionis orietur ex complexionibus summae praecedentis his: 1) ex $1^{\alpha-1}2^{\beta}3^{\gamma}4^{\delta}5^{\varepsilon}\ldots$, praemittendo 1; 2) ex $1^{\alpha-1}2^{\beta}3^{\gamma}4^{\delta}5^{\varepsilon}\ldots$ augendo elementum 1 vnitate, quod quidem $\alpha+1$ vicibus fieri potest, quoniam adsunt $\alpha+1$ elementa $=1$, quorum quodlibet seorsim augendum est; 3); 4); 5); ... ex complexionibus $1^{\alpha}2^{\beta+1}3^{\gamma-1}4^{\delta}5^{\varepsilon}\ldots$; $1^{\alpha}2^{\beta}3^{\gamma+1}4^{\delta-1}5^{\varepsilon}\ldots$; $1^{\alpha}2^{\beta}3^{\gamma}4^{\delta+1}5^{\varepsilon-1}\ldots$; ... augendo elementa 2; 3; 4; ..., quoduis vnitate, id quod fieri potest $\beta+1$; $\gamma+1$; $\delta+1$; ... vicibus. Exprimendo nunc ex hypothesi harum complexionum summae $n-1$ coëfficientes numericos per formulam praedictam, prodibit coëfficiens numericus complexionis $1^{\alpha}2^{\beta}3^{\gamma}4^{\delta}5^{\varepsilon}\ldots =$

$$1.2\ldots(n-1).\left\{\begin{array}{l}\dfrac{1}{1.2\ldots(\alpha-1).\,1\ldots\beta.\,1\ldots\gamma\ldots(1)^{\alpha-1}.(1.2)^{\beta}.(1.2.3)^{\gamma}\ldots}\\ +\dfrac{\alpha+1}{1.2\ldots(\alpha+1).\,1\ldots(\beta-1).\,1\ldots\gamma.(1)^{\alpha+1}.(1.2)^{\beta-1}.(1.2.3)^{\gamma}\ldots}\\ +\dfrac{\beta+1}{1.2\ldots\alpha.\,1\ldots(\beta+1).\,1\ldots(\gamma-1)\ldots 1^{\alpha}.(1.2)^{\beta+1}.(1.2.3)^{\gamma-1}\ldots}\\ +\ \text{etc.}\qquad +\ \text{etc.}\end{array}\right\}$$

$$=\frac{1.2\ldots n-1}{1.2\ldots\alpha.\,1\ldots\beta.\,1\ldots\gamma.\,1^{\alpha}(1.2)^{\beta}.(1.2.3)^{\gamma}\ldots}\left\{\alpha+\beta.\frac{1.2}{1}+\gamma.\frac{1.2.3}{1.2}+\delta.\frac{1\ldots4}{1.2.3}+\ldots\right\}=\frac{1.2\ldots(\alpha+2\beta+3\gamma+4\delta+\ldots)}{1.2\ldots\alpha.\,1\ldots\beta.\,1\ldots\gamma\ldots 1^{\alpha}(1.2)^{\beta}(1.2.3)^{\gamma}\ldots},$$ id quod ipsum cum formula assumta consentit.

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